Noncompact Codimension-1
Real Algebraic Manifolds

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1. Introduction and the Main Theorem

A classical theorem of Seifert asserts that every smooth, closed, codimension-1 submanifold of Euclidean n-space can be smoothly isotoped to a nonsingular real algebraic set [S, Satz 4]. We consider the noncompact analogue of Seifert's theorem.

The main result (Theorem 1) is a necessary and sufficient topological condition for X^n a smooth compact manifold with boundary to have a codimension-1, real algebraic interior. In particular, for such an X^n, there is a smooth proper embedding $X^n \hookrightarrow D^{n+1}$ if and only if the interior of X^n is diffeomorphic to a nonsingular real algebraic subset of $\mathbb{R}^{n+1}$. Moreover, if such an embedding exists, then $\operatorname{int}(X^n)$ is isotopic to a nonsingular real algebraic subset of $\operatorname{int}(D^{n+1}) \approx \mathbb{R}^{n+1}$.

Using Theorem 1 we show (Corollary 1) that the noncompact analogue of Seifert's theorem is intimately related to completions of a pair. This observation yields a complete answer to the noncompact Seifert problem in ambient dimension < 4 and also in the high-dimensional simply connected case (Corollary 2). As a final application of Theorem 1, we show that a real algebraic problem of V. I. Arnŏld concerning exotic $\mathbb{R}^4$s being real algebraic in $\mathbb{R}^5$ is in fact equivalent to an open topological problem.

A guiding problem here is the noncompact analogue of Seifert's classical theorem [S, Satz 4].

PROBLEM 1. Which smooth, proper, codimension-1 submanifolds M^n (not necessarily compact) of $\mathbb{R}^{n+1}$ are isotopic to nonsingular real algebraic sets?

The method of proof employed by Seifert in the compact case does not readily extend to the noncompact case. The main difference is the amount of control needed near infinity. In the compact case, one approximates a suitable smooth function by a polynomial over a large compact set and then, following Seifert, adds an algebraic correction term to keep the polynomial from picking up more zeros outside of the compact set. The only control needed near infinity is that the polynomial should always be greater than (or always less than) zero. Clearly much more control is needed in the proper noncompact case, since the set of zeros extends all the way to infinity. The noncompact case does eventually involve approximating

suitable smooth functions; however, the approximations are much more delicate. The bulk of the work presented here is devoted to proving the following, which is our main theorem.

THEOREM 1. *Let X^n be a smooth compact manifold. Then $\text{int}(X^n)$ is diffeomorphic to a nonsingular real algebraic subset of $\mathbb{R}^{n+1}$ if and only if X^n admits $X^n \hookrightarrow D^{n+1}$ as a smooth proper embedding. If such an embedding exists, then $\text{int}(X^n)$ is isotopic to a nonsingular real algebraic subset W of $\text{int}(D^{n+1}) \approx \mathbb{R}^{n+1}$. In fact, for all balls B_R^{n+1} of sufficiently large radius R, the pair $(B_R^{n+1}, B_R^{n+1} \cap W)$ is diffeomorphic to (D^{n+1}, X^n).*

REMARK 1. A map is *proper* if it maps boundary points to boundary points and is transverse at boundaries and if the inverse image of every compact set is compact. All isotopies will be smooth, proper, and ambient.

The remainder of this paper is organized as follows. Section 2 presents some applications of Theorem 1 to the study of Problem 1. Secton 3 is a brief discussion of algebraic regular neighborhoods. Section 4 constructs the ends of the algebraic set W in Theorem 1, and Section 5 completes the proof of Theorem 1.

2. Applications of the Main Theorem

We first recall the notion of a completion, which we will need only for manifolds without boundary. Let (A, B) be a smooth manifold pair with B a proper submanifold (B may be empty). Assume further that ∂B and ∂A are empty. A completion of the pair (A, B) is a smooth compact manifold pair $(\bar{A}, \bar{B})$ and a smooth embedding $i \colon (A, B) \hookrightarrow (\bar{A}, \bar{B})$ satisfying the following conditions: $i(A, B) = (\text{int}(\bar{A}), \text{int}(\bar{B}))$, $\bar{B}$ intersects $\partial \bar{A}$ in $\partial \bar{B}$, and this intersection is transverse.

Our first corollary to Theorem 1 is a topological condition characterizing those manifolds that answer Problem 1 in the affirmative.

COROLLARY 1. *Let $M^n \subset \mathbb{R}^{n+1}$ be any smooth proper submanifold. Then M^n is isotopic to a nonsingular real algebraic subset of $\mathbb{R}^{n+1}$ if and only if the pair $(\mathbb{R}^{n+1}, M^n)$ admits a completion $(\overline{\mathbb{R}^{n+1}}, \overline{M^n})$ such that $\overline{\mathbb{R}^{n+1}}$ is diffeomorphic to D^{n+1}.*

Proof. First, assume M is isotopic to $p^{-1}(0)$, a nonsingular real algebraic subset of $\mathbb{R}^{n+1}$. The idea is that ∞ is an isolated singularity (or perhaps a nonsingular point) of $p^{-1}(0)$ controlling the topology near ∞. Specifically, let $\theta \colon \mathbb{R}^{n+1} - 0 \to \mathbb{R}^{n+1} - 0$ be inversion through the unit sphere—namely, $\theta(x) = x/|x|^2$. Let $d = \deg p$ and define $q \colon \mathbb{R}^{n+1} \to \mathbb{R}$ to be the polynomial given by $|x|^{2d} p \circ \theta$ with denominators cleared (cf. [AK3, Lemma 2.1.5]). Notice that q vanishes on $\theta(M) \cup 0$ and that 0 is either a nonsingular point or an isolated singularity of $q^{-1}(0)$. In either case, [M3, pp. 16–18] implies that $(\varepsilon D^{n+1}, \varepsilon D^{n+1} \cap q^{-1}(0))$ is homeomorphic (in fact, diffeomorphic except possibly at 0) to the cone $(c(\varepsilon S^n), c(\varepsilon S^n \cap q^{-1}(0)))$ for sufficiently small $\varepsilon > 0$, where $\varepsilon S^n \cap q^{-1}(0)$ is a smooth closed manifold. The

result follows by reinverting through the unit sphere and reparameterizing collars at infinity.

Next, assume $(\mathbb{R}^{n+1}, M)$ admits a completion $\left(\overline{\mathbb{R}^{n+1}}, \overline{M}\right)$ such that $\overline{\mathbb{R}^{n+1}}$ is diffeomorphic to D^n. Recall that, by definition, $\overline{M}$ intersects $\partial\overline{\mathbb{R}^{n+1}}$ in $\partial\overline{M}$ and this intersection is transverse [Si, p. 109] (transverse data imply transverse results). The result then follows immediately from Theorem 1. $\qquad\square$

REMARK 2.   At least for $n \neq 3, 4$, the hypothesis that $\overline{\mathbb{R}^{n+1}}$ is diffeomorphic to D^{n+1} is superfluous. For $n = 2$, use the smooth form of the Alexander–Schoenflies theorem (i.e., any smooth $S^2 \hookrightarrow \mathbb{R}^3$ bounds a smooth 3-disk); for $n \geq 5$, use the h-cobordism theorem.

REMARK 3.   In the compact case, the isotopy in Seifert's theorem can be chosen ε-small. This is not possible in the noncompact case, as the following example shows. We can, however, choose our isotopy to be ε-small over any prescribed compact subset of $\mathbb{R}^{n+1}$, as the proof of Theorem 1 shows.

EXAMPLE 1.   Let $M^n \subset \mathbb{R}^{n+1}$ be any noncompact, smooth, proper, real algebraic submanifold. We will give an isotopy of M^n to a submanifold that cannot be isotoped to a real algebraic set by any bounded isotopy. By the assumptions on M^n, there is a smooth proper ray $[0, \infty)$ in M^n; let p_k ($k \in \mathbb{Z}^+$) be the integer points of this ray. At each p_k send a hair (literally, a closed interval) out of M^n so that the end of the hair is at least k units farther from the origin of $\mathbb{R}^{n+1}$ than the point p_k and the hairs are disjoint. This can be arranged, say, by sending the hairs out following the ray. For each p_k, choose a very small closed n-disk neighborhood D_k of p_k in M^n. Using the hairs as guides, isotope the D_k while fixing the boundaries ∂D_k, so that the points p_k coincide with the far endpoints of the hairs; we denote the isotoped D_k by D'_k, the isotoped p_k by p'_k, and the isotoped M^n by M'. Now, any isotopy of M' bounded by $N > 0$ leaves the boundary of D'_k closer to the origin than p'_k for $k > 2N$. Hence, the squared norm function on $\mathbb{R}^{n+1}$ will have infinitely many critical points on the image of M' under any bounded isotopy. Consequently, the image of M' could not be algebraic.

Next, we obtain basic necessary conditions for a manifold to answer Problem 1 in the affirmative. Assume $M^n \subset \mathbb{R}^{n+1}$ is a smooth proper submanifold that is isotopic to a nonsingular real algebraic set. The implicit function theorem implies that the boundary of M^n is empty (even real algebraic manifolds whose defining polynomials have singularities in their zero sets cannot have boundary points by [Su, Cor. 2]). Corollary 1 implies that M^n is smoothly collared by N^{n-1}, a smooth closed manifold. In particular, N^{n-1} is the disjoint union of at most finitely many connected, smooth closed manifolds, and M^n has finitely many components and ends. We tacitly assume a smooth proper embedding $N^{n-1} \times [0, 1) \hookrightarrow M^n \subset \mathbb{R}^{n+1}$ that collars M^n has been fixed, and we regard $N^{n-1} \times [0, 1)$ as a submanifold of M^n.

These necessary conditions provide immediate counterexamples to Problem 1. The pair $(\mathbb{R} \times \mathbb{R}^n, \mathbb{Z} \times \mathbb{R}^n)$ gives an example in every dimension of a very nice

(smooth, proper, tame, etc.) submanifold that cannot be real algebraic. However, these counterexamples are not fatal and the necessary conditions given here are, in fact, nearly sufficient. Before we make this precise, we describe two additional hypothesis one may impose on M^n. First, the "Ray hypothesis" will mean that there is an isotopy of M^n simultaneously taking one smooth proper ray in each end of M^n to a straight ray in the Euclidean sense. Second, the "End hypothesis" will mean that each end of $\mathbb{R}^{n+1} - T(M^n)$ has finitely presented and stable fundamental group. Here $T(M^n)$ denotes a closed tubular neighborhood of M^n. We refer the reader to [CKS] for a discussion of ends in this context.

COROLLARY 2.   *Let $M^n \subset \mathbb{R}^{n+1}$ satisfy the necessary conditions just listed. Then M^n is isotopic to a nonsingular real algebraic subset of $\mathbb{R}^{n+1}$, provided that (a) $n = 1$ or (b) $n = 2$ and either the Ray or End hypothesis is satisfied or (c) $n \geq 5$ and each component of the collaring manifold N is simply connected.*

Proof. By [CKS] there is a completion of the pair $(\mathbb{R}^{n+1}, M^n)$. The result follows by Corollary 1. ☐

REMARK 4.   Examples in [Si, p. 110] show that a pair $(\mathbb{R}^{n+1}, M^n)$, $n \geq 2$, need not have a completion if the components of N^{n-1} are not assumed to be simply connected. (In these examples, $N^{n-1} = S^1 \times S^{n-2}$.) Thus, the non–simply connected case carries deeper subtleties, as do ambient dimensions 3 and 4.

We close this section with a very special case of Problem 1 that is related to that in [A].

PROBLEM 2 (Arnoĭd).   Does there exist an exotic $\mathbb{R}^4$ that is diffeomorphic to a nonsingular real algebraic subset of $\mathbb{R}^5$?

REMARK 5.   By "exotic" we mean a smooth manifold that is homeomorphic but not diffeomorphic to the standard one. An n-dimensional pseudodisk is a smooth, compact, contractible n-manifold with boundary.

In [MV] it was shown that any exotic $\mathbb{R}^4$, say $\mathcal{R}$, answering Arnoĭd's problem in the affirmative is necessarily collared by a smooth homotopy 3-sphere that smoothly embeds in S^4 (this also follows from Corollary 1 and topological invertible cobordisms). Thus, if such an $\mathcal{R}$ exists, then there is a pseudodisk with interior $\mathcal{R}$ and one of the remaining Poincaré conjectures is false. In fact, the existence of such a pseudodisk is equivalent to Arnoĭd's problem. To show this, we will need a lemma that appears to be new.

LEMMA 1.   *Let X^4 be a pseudodisk with simply connected boundary. Then X^4 admits $X^4 \hookrightarrow D^5$ as a smooth proper embedding if and only if ∂X^4 smoothly embeds in S^4.*

Proof. One direction is obvious, so assume there exists a smooth embedding $\partial X^4 \hookrightarrow S^4$. Let Y^5 be the smooth manifold with boundary obtained from $S^4 \times [0,1]$ by gluing on $X^4 \times [-1,1]$ along a product neighborhood, $N = \partial X^4 \times [-1,1] \subset S^4 \times 0$, of ∂X^4 in the canonical way and smoothing corners. Then

∂Y^5 consists of three connected components, say $S^4 \times 1$, Σ_A^4, and Σ_B^4. Recalling that ∂X^4 is simply connected, standard theorems imply that the $\Sigma_{A,B}^4$ are simply connected $\mathbb{Z}$-homology 4-spheres and hence (by Freedman's theorem [F]) topological 4-spheres. The fourth homotopy sphere cobordism group is trivial; that is, $\Theta_4 = 0$ [KM]. Thus, there exist $W_{A,B}^5$ smooth null h-cobordisms of $\Sigma_{A,B}^4$, respectively (sew a standard 5-disk onto a smooth h-cobordism between $\Sigma_{A,B}^4$ and S^4). Let Z^5 be the smooth manifold with boundary obtained from Y^5 by sewing on $W_{A,B}^5$ along $\Sigma_{A,B}^4$ in the canonical way. Again, standard theorems imply that Z^5 is simply connected and has the integral homology of a point. As $\partial Z^5 = S^4$, we may conclude that Z^5 is diffeomorphic to D^5 [M2, p. 110]. The result follows. $\qquad\square$

COROLLARY 3. *There exists an exotic $\mathbb{R}^4$ diffeomorphic to a nonsingular real algebraic subset of $\mathbb{R}^5$ if and only if there exists an exotic $\mathbb{R}^4$ diffeomorphic to the interior of a compact manifold.*

Proof. First, suppose $\mathcal{R}$ is an exotic $\mathbb{R}^4$ that is a nonsingular real algebraic subset of $\mathbb{R}^5$. By [MV], there exist a homotopy 3-sphere Σ^3 and a neighborhood of infinity in $\mathcal{R}$ that is diffeomorphic to $\Sigma^3 \times [0, 1)$. Let X^4 be the compact manifold obtained from $\mathcal{R}$ by removing $\Sigma^3 \times (0, 1)$, so $\text{int}(X^4)$ is diffeomorphic to $\mathcal{R}$. The result follows.

For the other direction, suppose that X^4 is a smooth compact manifold with boundary and that $\text{int}(X^4)$ is diffeomorphic to $\mathcal{R}$, an exotic $\mathbb{R}^4$. Reparameterizing collars, we see that X^4 is homotopy equivalent to $\mathcal{R}$ and so X^4 is contractible. Also, ∂X^4 is a homotopy 3-sphere [GS, pp. 366, 519]. Therefore, X^4 is a pseudo-disk with simply connected boundary. There are two cases.

Case 1: ∂X^4 smoothly embeds in S^4. Then, Lemma 1 and Theorem 1 imply that $\mathcal{R}$ is diffeomorphic to a real algebraic subset of $\mathbb{R}^5$.

Case 2: ∂X^4 does not smoothly embed in S^4. The punctured double, $2X^4 - \text{pt}$, is a smooth manifold homeomorphic to $\mathbb{R}^4$ because $2X^4$, the double of X^4, is homeomorphic to S^4 by [F]. However, $2X^4 - \text{pt}$ is not diffeomorphic to $\mathbb{R}^4$ since otherwise we have a smooth embedding of ∂X^4 in S^4. Thus, $2X^4 - \text{pt}$ is an exotic $\mathbb{R}^4$ collared at infinity by S^3. Lemma 1 and Theorem 1 then imply that $2X^4 - \text{pt}$ is diffeomorphic to a real algebraic subset of $\mathbb{R}^5$. $\qquad\square$

Thus, Arnol̆d's real algebraic problem is equivalent to a topological one. We remind the reader that all exotic $\mathbb{R}^4$s $\mathcal{R}$ are smooth proper submanifolds of $\mathbb{R}^5$, since $\mathcal{R} \times \mathbb{R} \approx \mathbb{R}^5$ by engulfing or by the smooth proper h-cobordism theorem. Moreover, these smooth embeddings may be chosen to be real analytic. Still, there are only countably many smooth compact manifolds (with or without boundary) yet uncountably many pairwise nondiffeomorphic exotic $\mathbb{R}^4$s [GS, p. 370], so *most* exotic $\mathbb{R}^4$s are neither real algebraic in $\mathbb{R}^5$ nor the interior of a smooth compact manifold. Exotic $\mathbb{R}^4$s are problematic at infinity: all *known* handle decompositions of exotic $\mathbb{R}^4$s are infinite [GS, p. 366]; all *known* exotic $\mathbb{R}^4$s contain a compact subset that cannot be contained in the bounded region formed by any smoothly embedded S^3 ("Property $\star$" in [MV]); and *every* exotic $\mathbb{R}^4$ contains a compact

subset that cannot be contained in any smoothly embedded D^4 (a "weak Property ★" [M1, p. 168; GS, p. 366]). It is unknown whether every exotic $\mathbb{R}^4$ possesses Property ★.

3. Algebraic Regular Neighborhoods

DEFINITION 1.   Suppose $X \subset Y$ are real algebraic sets with X compact. An algebraic regular neighborhood of X in Y is obtained as follows. Pick any proper rational function $p \colon Y \to \mathbb{R}$ with $X = p^{-1}(0)$; then, for small enough $\varepsilon > 0$, the set $p^{-1}([-\varepsilon, \varepsilon])$ is an *algebraic regular neighborhood* of X in Y.

Algebraic regular neighborhoods are explored in [D]. They are unique up to isotopy and are mapping cylinder neighborhoods. In our context, X will always contain the singular points of Y, so $Y - X$ is a smooth manifold. Then ε small enough just means that p has no critical values in $[-\varepsilon, 0) \cup (0, \varepsilon]$, which easily implies independence of ε up to isotopy.

A related notion is an algebraic regular neighborhood of infinity for a real algebraic set Y. This is $p^{-1}((-\infty, -R] \cup [R, \infty))$ for a proper rational function p and large enough R. Algebraic regular neighborhoods of infinity are unique and collar the ends of Y, since they are algebraic regular neighborhoods of points added when compactifying Y. An example of such a neighborhood is the intersection of Y with the complement of a sufficiently large open ball.

4. Constructing the Ends of W

NOTATION 1.   Let x, (x, t), and (x, t, s) denote typical elements of $\mathbb{R}^n$, $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$, and $\mathbb{R}^{n+2} = \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$, respectively. Let $B_r^m(x)$ denote the closed ball of radius r centered at x in $\mathbb{R}^m$, and let B_r^m denote $B_r^m(0)$. We let S_r^{n-1} denote the sphere of radius r about the origin in $\mathbb{R}^n$.

DEFINITION 2.   Let $h \colon \mathbb{R}^n \to \mathbb{R}$ be a polynomial. The *homogenization* of h is the polynomial $h^* \colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ given by $h^*(x, t) = t^{\deg(h)} h(x/t)$. The polynomial h is said to be *overt* if $h^*(x, 0) \neq 0$ for $x \neq 0$.

Essential to the proof of Theorem 1 is the following lemma, whose proof will take up the rest of this section. One way of looking at it is that the pair $(\mathbb{R}^{n+1}, Z)$ has its algebraic regular neighborhood of infinity diffeomorphic to $(S^n, \Sigma) \times [R, \infty)$.

LEMMA 2.   *Let Σ^{n-1} be a closed, smooth, codimension-1 submanifold of S^n. If $n = 1$, assume also that Σ^{n-1} is an even number of points. Then there is an algebraic subset $Z \subset \mathbb{R}^{n+1}$ such that, for all sufficiently large R, the pair $(S_R^n, S_R^n \cap Z)$ is diffeomorphic to (S^n, Σ^{n-1}). In fact, there is a proper embedding $h \colon S^n \times [R, \infty) \to \mathbb{R}^{n+1}$ such that $h^{-1}(Z) = \Sigma^{n-1} \times [R, \infty)$ and $|h(x, t)| = t$ for all (x, t).*

REMARK 6.   The orientable codimension-2 version of Lemma 2 already follows from known results: For an orientable "knot" $M^n \hookrightarrow S^{n+2}$, a generalized Seifert

surface (i.e. a compact, connected, smooth manifold $N^{n+1} \hookrightarrow S^{n+2}$ with trivial normal bundle such that $\partial N^{n+1} = M^n$) always exists (see e.g. the Introduction and Section 27 of [R]); coupling this with [AK1, Thm. 0.2] and inversion through the sphere, the result follows. In codimension ≥ 3, the problem is open (the interested reader may refer to [AK2], for example).

Proof of Lemma 2. The proof may be outlined as follows. First, by deleting a point from S^n we will think of Σ^{n-1} as being a submanifold of $\mathbb{R}^n$. The major effort is then to construct a real algebraic set $V \subset \mathbb{R}^n \times [0, \infty)$ such that, for $t > 0$ small enough, $V \cap \mathbb{R}^n \times t$ is isotopic to $\Sigma^{n-1} \times t$. Now put V in projective space $\mathbb{RP}^{n+1}$ and delete the subspace $t = 0$ to obtain an affine algebraic set Z. Now $Z \cap \mathbb{R}^n \times t$ is isotopic to $\Sigma^{n-1} \times t$ for all sufficiently large $t > 0$. By uniqueness of algebraic regular neighborhoods at infinity, we then see that $(S^n_R, Z \cap S^n_R)$ is diffeomorphic to (S^n, Σ^{n-1}) for large enough radii R. Now for the details.

Let $Y^n \subset \mathbb{R}^n$ be the compact codimension-0 submanifold such that $\partial Y^n = \Sigma^{n-1}$. Following Lemma 3.2 of [AK1] (and its proof), there is a finite collection of smoothly embedded disks $D^n_\alpha, \alpha \in A$, in int(Y^n) such that:

(i) the boundaries $S^{n-1}_\alpha = \partial D^n_\alpha$ ($\alpha \in A$) are in general position; and
(ii) Y^n minus some other finite disjoint collection of disks is a smooth regular neighborhood of $\bigcup_\alpha S^{n-1}_\alpha$ in Y^n.

Essentially, the disks D^n_α ($\alpha \in A$) are obtained from a smooth triangulation of Y^n by choosing one disk D^n_α to engulf each simplex not touching $\Sigma^{n-1} = \partial Y^n$.

We now define a finite abstract graph $\mathcal{G}$. The vertices v_j ($j \in B \subset \mathbb{Z}^+$) of $\mathcal{G}$ are the connected components of $Y^n - \bigcup_\alpha S^n_\alpha$. We order them so that (a) for $j \leq m$, v_j is a connected component of $Y^n - \bigcup_\alpha D^n_\alpha$ and (b) for $j > m$, v_j is a connected component of $\bigcup_\alpha D^n_\alpha - \bigcup_\alpha S^n_\alpha$. We put an edge between two vertices v_i and v_j if and only if the intersection of the closures of their corresponding components contains an $n - 1$ dimensional subset (i.e., iff their corresponding components are nontrivially adjacent). The crucial property here is that we could actually realize the graph as a subset of Y^n, with the vertices as points in their component and with the edges as smooth curves between these points, which are transverse to $\bigcup_\alpha S^n_\alpha$ and intersect it in a single point. We can take a subgraph $\mathcal{F} \subset \mathcal{G}$, the disjoint union of m trees $\mathcal{F}_i$ such that $v_i \in \mathcal{F}_i$ for $i \leq m$, so that $\mathcal{F}$ contains all the vertices of $\mathcal{G}$. Deleting a smooth regular neighborhood of $\mathcal{F}$ from Y gives us a manifold isotopic to Y as long as we chose $v_i \in \partial Y^n$ for $i \leq m$. See Figure 1.
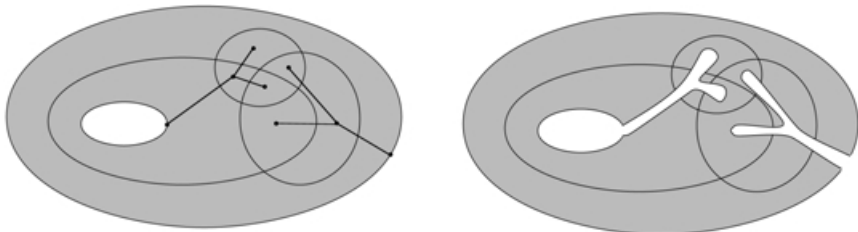

Figure 1 Y^n with $\mathcal{F}$ and with a regular neighborhood of $\mathcal{F}$ deleted

ASSERTION 1.   *After a small isotopy, we can assume $\bigcup_\alpha S^{n-1}_\alpha = p^{-1}(0)$, where $p: \mathbb{R}^n \to \mathbb{R}$ is an overt polynomial.*

Proof. By [AK3, Thm. 2.8.2] we may suppose that each S^{n-1}_α is a nonsingular, real algebraic set and hence that each is $p^{-1}_\alpha(0)$ for some polynomial $p_\alpha: \mathbb{R}^n \to \mathbb{R}$ such that $\nabla p_\alpha \neq 0$ on S^{n-1}_α. We may suppose that $p_\alpha > 0$ outside some compactum. If this polynomial p_α is not overt then we may replace it by $p_\alpha(x) + \varepsilon |x|^{2k}$ (for small $\varepsilon > 0$ and $2k >$ degree of p_α) and it will become overt. Now just let p be the product of all the p_α.   □

Let $\mathcal{E}$ be the set of edges of $\mathcal{F}$ and $|\mathcal{E}|$ the number of edges in $\mathcal{E}$. If $\deg p \leq |\mathcal{E}|$, replace $p(x)$ with $(1 + |x|^2)^k p(x)$ for a large enough k so that $\deg p > |\mathcal{E}|$. For each edge $e \in \mathcal{E}$, let x_e be the point of intersection of the edge with $\bigcup_\alpha S^n_\alpha$. Let $r: \mathbb{R}^n \to \mathbb{R}$ be the polynomial of degree $2|\mathcal{E}|$ given by

$$r(x) = \prod_{e \in \mathcal{E}} |x - x_e|^2.$$

ASSERTION 2.   *We may choose analytic coordinates in a neighborhood U_e of each x_e such that, in these coordinates, $r(x) = |x|^2$ and $p(x) = \alpha_e(x_n)$ for some diffeomorphism $\alpha_e: (\mathbb{R}, 0) \to (\mathbb{R}, 0)$.*

Proof. By induction there exist a $k \leq n$ and analytic coordinates such that $r(x) = \sum_{i=1}^{k-1} x_i^2 + h(x_k, \ldots, x_n)$, $r(0) = 0$, and $p(x) = x_n$. Since the Hessian of r is positive definite, we have $\partial^2 h / \partial x_k^2 \neq 0$ and so, by the implicit function theorem, there is a smooth function $\beta(x_{k+1}, \ldots, x_n)$ such that

$$\partial h / \partial x_k (\beta(x_{k+1}, \ldots, x_n), x_{k+1}, \ldots, x_n) = 0.$$

We replace the coordinate x_k by the new coordinate $u = x_k - \beta$. Then $\partial h / \partial u = \partial h / \partial x_k$ vanishes on $u = 0$, so $h = u^2 h_1(u, x_{k+1}, \ldots, x_n) + h_2(x_{k+1}, \ldots, x_n)$ by Taylor's theorem. Now replace the coordinate u with the coordinate $v = u\sqrt{h_1}$, and the induction step is complete. Note that the coordinate x_n remains unchanged until the very last induction step. In this step $u = x_n$ and we let the germ of α_e be the inverse of the map $x_n \mapsto x_n \sqrt{h_1(x_n)}$.   □

ASSERTION 3.   *Let $g(x,t) = p^2(x) + bt^2 - 2ctr(x)$ with positive constants b and c to be determined. Let $V = g^{-1}(0)$. Then:*

  (i) *$V \cap \mathbb{R}^n \times (0,1] \subset \text{Nonsing } V$;*
 (ii) *the pair $(\mathbb{R}^n \times (0,1], V \cap \mathbb{R}^n \times (0,1])$ is diffeomorphic to $(\mathbb{R}^n, \Sigma^{n-1}) \times (0,1]$;*
(iii) *$V \subset \mathbb{R}^n \times [0, \infty)$.*

Proof. Before giving a careful (but boring and opaque) proof, we'll give a rough idea why this works. For each t, let $V_t = V \cap \mathbb{R}^n \times t$ and let N_t be the set of x so that $g(x,t) \leq 0$; then N_t is compact and $V_t = \partial N_t$. Now N_t satisfies the equation $p^2(x) \leq \beta(x)$, where $\beta(x) = 2ctr(x) - bt^2$. The constants b and c will be small, so β is small. If we are in a region where $\beta > 0$, then locally N_t is given roughly

by $-d \leq p(x) \leq d$ for some small d; hence N_t looks like the regular neighborhood $p^{-1}([-d, d])$ of $p^{-1}(0)$. But $\beta \leq 0$ only where r is small. There we may use the local coordinates given in Assertion 2. In these coordinates, V_t is roughly a hyperboloid $\sum_{i=1}^{n-1} x_i^2 - a x_n^2 = a'$. This has the effect of boring a hole through a regular neighborhood of $p^{-1}(0)$—in other words, deleting a regular neighborhood of an arc going from one edge of the regular neighborhood to the other. Thus, in the end, N_t is obtained from a regular neighborhood of $p^{-1}(0)$ by deleting a regular neighborhood of each edge in $\mathcal{E}$. But a regular neighborhood of $p^{-1}(0)$ is obtained from Y^n by deleting a disc around each vertex v_i with $i > m$. Thus N_t is obtained from Y^n by deleting a regular neighborhood of $\mathcal{F}$, and so N_t is isotopic to Y^n. Consequently, V_t is isotopic to $\Sigma^{n-1} = \partial Y^n$. See Figure 2.
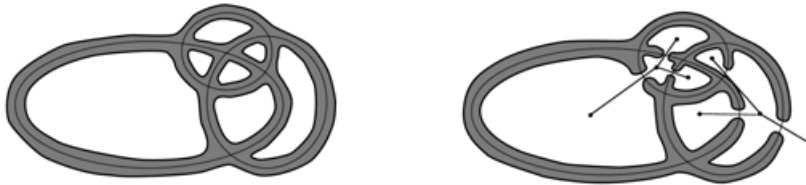

Figure 2 A regular neighborhood of $p^{-1}(0)$ and N_t

Now for the details. Pick $\varepsilon > 0$ so that $r^{-1}([0, 2\varepsilon]) \subset \bigcup_{e \in \mathcal{E}} U_e$, where U_e is as in Assertion 2. Since p is overt, we know it is proper. Let R be the maximum of $|\nabla r|$ on the compact set $p^{-1}([-1, 1])$. Note that $|\nabla p(x)|/|p(x)| \to \infty$ as $p(x) \to 0$. This is because, near a point of $p^{-1}(0)$, there are local coordinates such that $p(x) = \prod_{i=k}^{n} x_i$ and in these coordinates we have $|\nabla p(x)|/|p(x)| = \sqrt{\sum_{i=k}^{n} 1/x_i^2}$. Consequently we may choose a $\delta \in (0, 1)$ such that $|\nabla p(x)|/|p(x)| > R/\varepsilon$ whenever $|p(x)| \leq \delta$.

Now, since p and r are overt and $2 \deg p > \deg r$, we know that $p^2(x)/r(x) \to \infty$ as $x \to \infty$. Consequently we may choose $c \in (0, 1)$ so that $2c < p^2(x)/r(x)$ whenever $|p(x)| \geq \delta$. We also require that $c < \delta^2/\varepsilon$ and $\sqrt{2c} < \gamma_e(t)$ for all $e \in \mathcal{E}$ if $t^2 \leq \varepsilon$, where $\gamma_e(t) = \alpha_e(t)/t$. Now let $b = c\varepsilon$.

The first step is to show that $V \cap \mathbb{R}^n \times (0, 1] \subset \text{Nonsing } V$ and that the coordinate t as a function on $V \cap \mathbb{R}^n \times (0, 1]$ has no critical points. The result is an isotopy $h_t \colon \mathbb{R}^n \to \mathbb{R}^n$, $t \in (0, 1]$, with compact support such that h_1 is the identity and $h_t(V_1) = V_t$. (You can get h_t by integrating a vector field $(v, -1)$ on $\mathbb{R}^n \times (0, 1]$, which is tangent to V.) It suffices to show that, whenever $g(x, t) = 0$ and $0 < t \leq 1$, we have $\nabla_x g(x, t) \neq 0$. Here ∇_x denotes the gradient in the x variables. Note that $g(x, t) = 0$ implies $p^2(x)/r(x) < 2ct$, so $p^2(x) < \delta^2$ by our choice of c. Hence we suppose $\nabla_x g(x, t) = 0$. Then

$$0 = \nabla_x g(x, t) = 2p \nabla p - 2ct \nabla r;$$

therefore,

$$R/\varepsilon < |\nabla p|/|p| = ct |\nabla r|/p^2 \leq ctR/p^2(x)$$

and so we have $p^2(x) < ct\varepsilon$. But then

$$r(x) = (p^2(x) + bt^2)/(2ct) < \varepsilon/2 + bt/(2c) = \varepsilon(1+t)/2 \le \varepsilon.$$

Hence x must be in some U_e. In local coordinates we then have

$$0 = \nabla_x g(x,t) = (-4ctx_1, \ldots, -4ctx_{n-1}, -4ctx_n + 2\alpha_e(x_n)\alpha'_e(x_n)),$$

from which we see that $x_i = 0$ for $i < n$. On the other hand,

$$0 = g(x,t) = \alpha_e(x_n)^2 + bt^2 - 2ctx_n^2.$$

So $2ct \ge \gamma_e^2$, contradicting our choice of c. Thus $\nabla_x g \ne 0$ on $V \cap \mathbb{R}^n \times (0,1]$ as required.

Now, it remains only to show that V_1 is isotopic to Σ^{n-1}. Let $p^{-2}(b)$ denote $p^{-1}(\{-\sqrt{b}, \sqrt{b}\})$, and let $V_1^+ = V_1 \cap \{x \mid p^2(x) \ge b\}$. We will show that V_1^+ is diffeomorphic to $p^{-2}(b)$ with two discs removed for every $e \in \mathcal{E}$; moreover, V_1 is obtained from V_1^+ by gluing a 1-handle between each pair of these discs. But $p^{-2}(b)$ is the boundary of a regular neighborhood of $p^{-1}(0)$, which is Σ disjoint union a collection of spheres. Just as in [AK3], the 1-handles have the effect of connected summing these boundary components, and we end up with V_1 being a manifold isotopic to Σ.

For each $e \in \mathcal{E}$ and $k = \pm 1$, let $D_{ke} = U_e \cap p^{-1}(k\sqrt{b}) \cap r^{-1}([0,\varepsilon])$, which in the local coordinates around x_e is

$$D_{ke} = \left\{ x \,\middle|\, x_n = b_k \text{ and } \sum_{i=1}^{n-1} x_i^2 \le \varepsilon - b_k^2 \right\},$$

where $b_k = \alpha_e^{-1}(k\sqrt{b})$. Note that $\alpha_e(\pm\sqrt{\varepsilon})^2 = \varepsilon\gamma_e(\pm\sqrt{\varepsilon})^2 > 2c\varepsilon = 2b$, so $|b_k| < \sqrt{\varepsilon}$ and hence each D_{ke} is an $n-1$ disc. Now let $E_e = U_e \cap V_1 \cap p^{-1}([-\sqrt{b}, \sqrt{b}])$, which in the local coordinates around x_e is

$$E_e = \left\{ x \,\middle|\, b_{-1} \le x_n \le b_1 \text{ and } \sum_{i=1}^{n-1} x_i^2 = \frac{\varepsilon}{2} + x_n^2\left(\frac{\gamma_e^2(x_n)}{2c} - 1\right) \right\}.$$

Recall that $\gamma_e^2 > 2c$, so each E_e is a 1-handle $[-1,1] \times S^{n-2}$ that is attached to $\partial D_{1e} \cup \partial D_{-1e}$. We claim that V_1^+ is isotopic to $p^{-2}(b) \cap r^{-1}([\varepsilon, \infty))$ rel $p^{-2}(b) \cap r^{-1}(\varepsilon) = \bigcup_{e \in \mathcal{E}} \partial D_{1e} \cup \partial D_{-1e}$. Once we show this, we will know that V_1 is isotopic to $p^{-2}(b)$ with a 1-handle attached near each x_e. But this is isotopic to Σ.

The isotopy from V_1^+ to $p^{-2}(b) \cap r^{-1}([\varepsilon, \infty))$ is obtained by integrating the vector field $-p\nabla p$, which points into the region $\{x \mid p^2(x) \ge b \text{ and } g(x,1) \le 0\}$ on V_1^+ and out on $p^{-2}(b) \cap r^{-1}([\varepsilon, \infty))$. To see it points in on V_1^+, recall that we saw before that $|p(x)| < \delta$ if $g(x,1) = 0$. But this means $|\nabla p(x)|/|p(x)| > R/\varepsilon$ by our choice of δ, so

$$-p\nabla p \cdot \nabla_x g = -2p^2|\nabla p|^2 + 2cp\nabla r \cdot \nabla p$$

$$\le -2cr|\nabla p|^2 + 2c|p||\nabla r||\nabla p| \le -2c|p\nabla p|(\varepsilon|\nabla p|/|p| - R)$$

$$< 0. \qquad \qquad \square$$

There are a number of routes to obtaining the desired Z from V. One route is to use [AK3, Prop. 2.6.1] to algebraically crush V_0 to a point, then invert through the sphere to send this point to infinity. This would correspond to the transformation $(x, t) \mapsto (x, 1)/(t + t|x|^2)$. We'll take another route, corresponding to the transformation $\theta(x, t) = (x/t, 1/t)$.

Let $g^*(x, t, s)$ be the homogenization of g. Let $G(x, t) = g^*(x, 1, t)$ and let $Z = G^{-1}(0)$. Note that $Z - \mathbb{R}^n \times 0 = \theta(V - V_0)$.

We want to show, for large enough radii R, that $(S_R^n, S_R^n \cap Z)$ is diffeomorphic to (S^n, Σ^{n-1}). But this follows from uniqueness of algebraic regular neighborhoods of infinity. Since [D] does not explicitly deal with regular neighborhoods of pairs, we will outline an argument that is similar to arguments in [D]. Consider $D = \{(x, t) \mid 1 \leq t \text{ and } |x|^2 + t^2 \leq R^2\}$. The boundary of D is $D_+ \cup D_-$, where D_- is the disc $\{(x, 1) \mid R^2 - 1 \geq |x|^2\}$ and D_+ is the spherical cap $\{(x, t) \in S_R^n \mid 1 \leq t\}$. For large enough R, there is a vector field $(w, 1)$ on D that is tangent to Z and points outward on D_+ and inward on D_-. Integrating this vector field gives a diffeomorphism between the pairs $(D_-, D_- \cap Z)$ and $(D_+, D_+ \cap Z)$. Note that $D_- \cap Z = V_1 \approx \Sigma$ and $D_+ \cap Z = S_R^n \cap Z$ and, as a result, $(S_R^n, S_R^n \cap Z)$ is diffeomorphic to (S^n, Σ^{n-1}).

This completes the proof of Lemma 2. □

5. Proof of Theorem 1

First, suppose $\text{int}(X^n)$ is diffeomorphic to a nonsingular real algebraic subset of $\mathbb{R}^{n+1}$. Then, for large r, $\partial B_r^{n+1} \pitchfork \text{int}(X^n)$ in a smooth manifold Σ^{n-1} that collars $\text{int}(X^n)$ at infinity (see [M3, Sec. 2] and use stereographic projection to compactify $\mathbb{R}^n$). Let $X_0^n = B_r^{n+1} \cap \text{int}(X^n)$. Now ∂X^n and ∂X_0^n are not necessarily diffeomorphic; however, it is not difficult to see that they are invertibly cobordant (see [St]), say by $(W; \partial X^n, \partial X_0^n)$. By definition, $(W; \partial X^n, \partial X_0^n)$ embeds smoothly in $\partial X_0^n \times [0, 1]$. Using this and the fact that there is a smooth proper embedding $X_0^n \hookrightarrow B_r^{n+1} \approx D^{n+1}$, it follows that there exists a smooth proper embedding $X^n \hookrightarrow D^{n+1}$ as desired.

The other direction follows from Lemma 2 and the following.

LEMMA 3. *Let $V \subset \mathbb{R}^n$ be a codimension-1 real algebraic set with $\text{Sing } V$ compact. Let $M \subset \mathbb{R}^n$ be a proper, smooth, codimension-1 submanifold such that, for some R, $M - B_R^n = V - B_R^n$. Then there exists a nonsingular real algebraic set $W \subset \mathbb{R}^n$ that is properly isotopic to M. In fact, we may suppose there is a smooth isotopy $h_t \colon \mathbb{R}^n \to \mathbb{R}^n$ and a radius R' such that h_0 is the identity, $h_1(M) = W$, and $|h_t(x)| = |x|$ whenever $|x| \geq R'$.*

Proof. Pick a polynomial $p \colon \mathbb{R}^n \to \mathbb{R}$ generating the ideal of polynomials vanishing on V. Hence $p^{-1}(0) = V$, and the only solutions to $p = 0$ and $\nabla p = 0$ are $\text{Sing } V$. Let $r(x) = |x|^2$, and let $q(x) = p^2(x) + |\nabla p|^2 |x|^2 - (\nabla p \cdot x)^2$. Then $q^{-1}(0)$ is the set of points in V where ∇p and x are linearly dependent, so it is the union of $\text{Sing } V$ and the critical points of $r|_{\text{Nonsing } V}$. Thus $q^{-1}(0)$ is compact, which means (by [AK3, Lemma 2.1.5]) that, for some radius R'' and integer

$m \geq 0$, $q(x) \geq 3|x|^{-2m}$ whenever $|x| \geq R''$. Since M separates $\mathbb{R}^n$, we may find a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ and a radius $R' > R''$ such that 0 is a regular value of f, $f^{-1}(0) = M$, and $f(x) = p(x)$ if $|x| \geq R'$. Pick an integer $k > 1 + m + \deg(p)$, and choose $\varepsilon > 0$ so $|\nabla f(x)| > \varepsilon$ whenever $|f(x)| < \varepsilon$ and $|x| \leq R'$. Also make sure that $|\nabla p(x)| < |x|^{2k-2m-2}/\varepsilon$ whenever $|x| \geq R'$ and that $\varepsilon < (R')^{2k-m}$. By [AK3, Lemma 2.8.1] applied to $f - p$, there is an entire rational function $u : \mathbb{R}^n \to \mathbb{R}$ approximating f and so $|f(x) - u(x)| < \varepsilon(1 + |x|^2)^{-k}$ and $|\nabla f(x) - \nabla u(x)| < \varepsilon(1 + |x|^2)^{-k}$ for all $x \in \mathbb{R}^n$. Let $W = u^{-1}(0)$.

Let $F(x, t) = tu(x) + (1 - t)f(x)$. We claim there is a vector field $(v(x, t), 1)$ on $\mathbb{R}^n \times [0, 1]$ that is tangent to $F^{-1}(0)$, so that $v(x, t) \cdot x = 0$ if $|x| \geq R'$. Then, integrating this vector field gives the isotopy h_t.

It suffices to construct v locally, where we may take $v = 0$ if $F \neq 0$. If $F(x, t) = 0$ and $|x| < R'$ then we will locally take $v(x, t) = \alpha(x, t)\nabla f$ for an appropriate α; in particular,

$$\alpha(x, t) = (f(x) - u(x))/(|\nabla f|^2 - t\nabla f \cdot (\nabla f - \nabla u)).$$

If $F(x, t) = 0$ and $|x| \geq R'$ then we will locally take $v(x, t) = \alpha(x, t)v'(x, t)$, where $v'(x, t) = |x|^2 \nabla f - (\nabla f \cdot x)x$ for an appropriate α; in particular,

$$\alpha(x, t) = (f(x) - u(x))/v'(x, t) \cdot ((1 - t)\nabla f + t\nabla u).$$

Note that $p^2(x) = f^2(x) = t^2(f(x) - u(x))^2 \leq \varepsilon^2 |x|^{-4k} < |x|^{-2m}$, so the denominator is nonzero since

$$
\begin{aligned}
v'(x, t) \cdot ((1 - t)\nabla f + t\nabla u) &= q(x) - p^2(x) - tv'(x, t) \cdot (\nabla f - \nabla u) \\
&> 3|x|^{-2m} - |x|^{-2m} - 2|x|^2|\nabla p||\nabla f - \nabla u| \\
&> 2|x|^{-2m} - 2\varepsilon|x|^{2-2k}|\nabla p| > 0. \qquad \square
\end{aligned}
$$

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