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Polynomial Solutions of the Complex Homogeneous Monge–Ampère Equation

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1. Introduction

Let P be a homogeneous real polynomial on $\mathbf{C}^n$ of bidegree (k,k), that is, a polynomial such that, for every $z \in \mathbf{C}^n$ and $\lambda \in \mathbf{C}$,

$$P(\lambda z) = |\lambda|^{2k} P(z).$$

It follows that the vector field

$$Z = \frac{1}{k} \sum_{i=1}^{n} z_i \frac{\partial}{\partial z_i}$$

annihilates the complex Hessian of the function $\log P(z)$. Thus it follows that, on $\{z \in \mathbf{C}^n \mid P(z) \neq 0\}$, the function $u = \log P$ satisfies the complex homogeneous Monge–Ampère equation

$$(dd^c u)^n = 0. \tag{1}$$

A natural question is: To what extent does property (1) characterize the homogeneous real polynomial of bidegree (k,k), for instance, among all real homogeneous polynomials on $\mathbf{C}^n$? Of course, one needs to add some reasonable condition to make the question a feasible one. Burns [3] asked if a homogeneous polynomial P on $\mathbf{C}^n$ that is positive (i.e., with $P(z) > 0$ and $P(z) = 0$ iff $z = 0$ and such that $dd^c P \geq 0$ and $(dd^c P)^{n-1} \not\equiv 0$, with $u = \log P$ satisfying (1)) needs to be a homogeneous polynomial of bidegree (k,k). In this paper we prove the following result.

Theorem 1.1. *Let P be a convex positive homogeneous polynomial on $\mathbf{C}^n$ such that $u = \log P$ is plurisubharmonic and satisfies* (1). *Then P is a homogeneous polynomial of bidegree* (k,k).

Of course, the result is much weaker than the statement in Burns's question. Moreover, we employ arguments that rely heavily on properties that hold only in the convex case and cannot be extended to the plurisubharmonic case. On the other hand, our proof points out the main difficulties in establishing a more general statement, and the rather heavy machinery needed gives an idea of the progress that should be made. As Burns indicates in [3], the point of his question is to understand the singularities of Monge–Ampère foliations. The key ingredient in our

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resolution of this question in the convex case is the possibility of reconstructing a Monge–Ampère foliation even for degenerate solutions of (1) in the convex case. Under the hypothesis of convexity, this is achieved by exploiting Lempert's results on the Kobayashi metric of convex domains. In the general case, one needs new insights on the singularities of foliations associated to solutions of (1).

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2. Preliminaries

We recall a few facts about real homogeneous polynomials on $\mathbf{C}^n$ for future reference. Let $P\colon \mathbf{C}^n \to \mathbf{R}$ be a homogeneous polynomial of degree d, that is, such that

$$P(tz) = t^d P(z) \quad \forall t \in \mathbf{R},\ \forall z \in \mathbf{C}^n. \tag{2}$$

Then P admits a unique decomposition

$$P(z) = \sum_{k+h=d} P^{k,h}(z) \quad \text{with } P^{k,h}(z) = \overline{P^{k,h}(z)}, \tag{3}$$

where the $P^{k,h}$ are homogeneous (complex) polynomials of bidegree (k,h):

$$P^{k,h}(\lambda z) = \lambda^k \bar{\lambda}^h P^{k,h}(z) \quad \forall \lambda \in \mathbf{C},\ \forall z \in \mathbf{C}^n. \tag{4}$$

If $z = (z^1, \dots, z^n) \in \mathbf{C}^n$ and Q is a (complex) homogeneous polynomial of bidegree (k,h) then, differentiating the relation $Q(\lambda z) = \lambda^k \bar{\lambda}^h Q(z)$ with respect to λ and taking $\lambda = 1$, one derives the following useful formulas:

$$Q_\mu(z) z^\mu = k Q(z), \qquad Q_{\bar{\nu}}(z) z^{\bar{\nu}} = h Q(z), \tag{5}$$

where subscripts denote differentiation as usual. Obviously, if Q is a (complex) homogeneous polynomial of bidegree (k,h) then Q_μ is a homogeneous polynomial of bidegree $(k-1,h)$ and $Q_{\bar{\nu}}$ is a homogeneous polynomial of bidegree $(k,h-1)$.

For a real homogeneous polynomial P, we say that it is *nonnegative* ($P \ge 0$) if $P(z) \ge 0$ for all $z \in \mathbf{C}^n$ and that it is *positive* ($P > 0$) if $P(z) = 0$ if and only if $z = 0$. With this terminology we have the following lemma.

LEMMA 2.1.

(i) *Let P be a real homogeneous polynomial of bidegree (k,k). If $dd^cP \ge 0$ on $\mathbf{C}^n \setminus \{0\}$ then $P \ge 0$. If furthermore $dd^cP_z > 0$ for some $z \in \mathbf{C}^n \setminus \{0\}$, then $P(z) > 0$.*

(ii) *Let P be a real homogeneous polynomial of degree d. If $P \ge 0$ and $P \not\equiv 0$ then $d = 2k$ and $P^{k,k} \ge 0$. If furthermore $P(z) > 0$ for some $z \in \mathbf{C}^n \setminus \{0\}$, then $P^{k,k} > 0$.*

(iii) *Let P be a real homogeneous polynomial of degree d. If $dd^cP \geq 0$ then $m = 2k$ and $dd^cP^{k,k} \geq 0$. If furthermore $dd^cP_z > 0$ for some $z \in \mathbf{C}^n \setminus \{0\}$, then $dd^cP_z^{k,k} > 0$.*

Proof. (i) This is immediate, since if $z = (z^1, \ldots, z^n) \in \mathbf{C}^n$ then using (5) yields

$$P_{\mu\bar{\nu}}(z)z^{\mu}\bar{z}^{\nu} = kP_{\mu}(z)z^{\mu} = k^2P(z).$$

(ii) If $z \in \mathbf{C}^n \setminus \{0\}$ and $\theta \in \mathbf{R}$, then

$$0 \leq P(e^{i\theta}z) = \sum_{l+h=d} P^{l,h}(e^{i\theta}z) = \sum_{l+h=d} e^{i(l-h)\theta}P^{l,h}(z),$$

so that

$$0 \leq \int_0^{2\pi} P(e^{i\theta}z)\,d\theta = \sum_{l+h=d} P^{l,h}(z)\int_0^{2\pi} e^{i(l-h)\theta}\,d\theta = I.$$

If d is odd then $I = 0$ and, since $P \geq 0$, it follows that $P(e^{i\theta}z) = 0$ for all θ. Since $z \in \mathbf{C}^n \setminus \{0\}$ was arbitrary, this implies that $P \equiv 0$, contradicting the hypothesis. Thus $m = 2k$ and

$$0 \leq \int_0^{2\pi} P(e^{i\theta}z)\,d\theta = \sum_{l+h=d} P^{l,h}(z)\int_0^{2\pi} e^{i(l-h)\theta}\,d\theta = 2\pi P^{k,k}(z).$$

If one assumes the strict inequality $P(z) > 0$ for $z \in \mathbf{C}^n \setminus \{0\}$, then P is positive in a neighborhood of z and the same proof shows that $P^{k,k}(z) > 0$.

(iii) If $z, v \in \mathbf{C}^n \setminus \{0\}$, then

$$L_P(z,v) = \sum P_{\mu\bar{\nu}}(z)v^{\mu}\bar{v}^{\nu} \geq 0.$$

For a fixed v, $L(z) = L_P(z,v)$ is a homogeneous polynomial in z of degree $d-2$ that decomposes as

$$L = \sum_{l+h=d} P^{l,h}_{\mu\bar{\nu}}v^{\mu}\bar{v}^{\nu}.$$

By (ii) we then have $d = 2k$ and $P^{k,k}_{\mu\bar{\nu}}v^{\mu}\bar{v}^{\nu} \geq 0$. Since $v \in \mathbf{C}^n \setminus \{0\}$ was arbitrary, it follows that $dd^cP^{k,k} \geq 0$. Starting again with strict inequality, one obtains strict inequality. □

Of course, the next lemma applies (as a special case) to positive homogeneous polynomials.

LEMMA 2.2. *Let F be a convex real analytic function on $\mathbf{R}^N \setminus \{0\}$. Let*

(i) *$F(X) > 0$ and $F(X) = 0$ iff $X = 0$, and let*

(ii) *there be a positive integer d such that $F(tX) = |t|^dF(X)$ for all $X \in \mathbf{R}^N \setminus \{0\}$ and $t \in \mathbf{R}$.*

Then, for every $r > 0$, there are no line segments on the boundary of $\Omega_r = \{X \mid F(X) < r\}$, which is therefore geometrically strictly convex.

Proof. Because of (i) and (ii), if

$$m = \min_{\|X\|=1} F(X) \quad \text{and} \quad M = \max_{\|X\|=1} F(X)$$

then for all X we have

$$m\|X\|^d \leq F(X) \leq M\|X\|^d,$$

and hence $\partial\Omega_r = \{X \mid F(X) = r\}$ is a compact set for every $r > 0$. On the other hand, if a line segment were contained in $\partial\Omega_r$, then for some $A \in \mathbf{R}^N \setminus \{0\}$ and $B \in \mathbf{R}^N$ and for all $t \in (-\varepsilon, \varepsilon)$ we would have $F(At + B) = r$. But then, by analyticity, $F(At + B) = r$ for all $t \in \mathbf{R}$ so that the compact set $\partial\Omega$ would contain a line. □

3. Proof of Theorem 1.1

Let P be a convex positive homogeneous polynomial on $\mathbf{C}^n$ of degree d such that $u = \log P$ satisfies (1). Then by Lemma 2.1 we have that $d = 2k$ and that, for all $r > 0$, the sublevel set $\Omega_r = \{X \mid P(X) < r\}$ is geometrically strictly convex for any $r > 0$. Lempert's results [7] on the Kobayashi metric of convex domains imply that, for every direction v with $\|v\| = 1$, there exists a unique holomorphic curve $\phi_r(\cdot, v)\colon \Delta \to \Omega_r$ of the unit disk Δ into Ω_r, with $\phi_r(0, v) = 0$ and $\phi_r'(0, v) = sv$ for $s > 0$ real, that realizes the Kobayashi length of v at 0. Furthermore, for v in $\{v \mid \|v\| = 1\}$, the images of maps $\phi_r(\cdot, v)$ define a foliation of $\Omega_r \setminus \{0\}$ parameterized by v (see [7] and, concerning uniqueness for just geometrically strictly convex domains, [5]). In fact, all these holomorphic discs meet only at 0 and, for every $z \in \Omega_r \setminus \{0\}$, there exists a unique v with $\|v\| = 1$ such that $z = \phi_r(t, v)$ with $t > 0$ real; the Kobayashi distance $\gamma_r(0, z)$ from 0 to z is given exactly by

$$\gamma_r(0, z) = \tanh^{-1}(t).$$

The Kobayashi metric of any Ω_r is at least continuous. Hence, for any $r > 0$, the maps $\phi_r(\cdot, v)$ depend continuously on v and, again by the uniqueness of Kobayashi extremal maps, for any $r > 0$ we have

$$\phi_r(\zeta, e^{i\theta} v) = \phi_r(e^{i\theta}\zeta, v).$$

Lempert proves much more. For instance, the holomorphic disks $\phi_r(\cdot, v)$ are in fact complex geodesics for both the Kobayashi and Carathéodory distance and metric, which agree on each Ω_r. Furthermore, if $v_r(z) = \log\tanh(\gamma_r(0, z))$ then the following holds.

PROPOSITION 3.1. *The function v_r is the pluricomplex Green function of Ω_r with pole at zero; that is, v_r is the unique function such that:*

(i) $v_r \in C^0(\overline{\Omega_r} \setminus \{0\}) \cap \mathrm{PSH}(\Omega_r)$;
(ii) $(dd^c v_r)^n \equiv 0$ *on* $\Omega_r \setminus \{0\}$;
(iii) $v_r \equiv 0$ *on* $\partial\Omega_r$;
(iv) $v_r(z) - \log|z| = O(1)$ *as* $z \to 0$.

The proposition is just a compilation of known results. The fact that u_r satisfies (i)–(iv) is due to Lempert [7], while the uniqueness statement can be derived easily from the Beford and Taylor comparison principle (see [2] or [6]).

If the domain Ω_r were strongly convex (i.e., if at each point of the boundary the real Hessian of a defining function were positive definite on the tangent space to the boundary), then Lempert's theory would further show that, in fact, $u_r \in C^\omega(\overline{\Omega_r} \setminus \{0\})$. Here we have just that Ω_r is geometrically strictly convex and—though the notions are close to each other—the regularity argument of Lempert does not apply. On the other hand we have the following.

COROLLARY 3.2. *If $u = \log P$ and if $2k$ is the degree of P, then $u = 2kv_r + \log r$.*

Proof. This is immediate from the hypothesis on P because u is plurisubharmonic, satisfies (1), and has $(1/2k)(u - \log r)$ as the prescribed boundary behavior since P is a positive homogeneous polynomial of degree $2k$. □

As an immediate consequence of Corollary 3.2, it follows that if $0 < r < R$ then Ω_r is a ball of suitable radius centered at the origin for the Kobayashi distance of Ω_R. More precisely, if γ_r and γ_R denote the Kobayashi distances of Ω_r and Ω_R (respectively), then on Ω_r we have

$$\gamma_r = \tanh^{-1}\left[\left(\frac{R}{r}\right)^{1/2k} \tanh \gamma_R\right]. \tag{6}$$

Furthermore, (6) and the uniqueness of Kobayashi extremal disks imply that, for z in the unit disk Δ,

$$\phi_r(\zeta, v) = \phi_R\left(\left(\frac{R}{r}\right)^{1/2k} \zeta, v\right), \tag{7}$$

where $\phi_r(\cdot, v)$ and $\phi_R(\cdot, v)$ are the (unique) Kobayashi extremal disks at the origin in the direction v, $\|v\| = 1$, of Ω_r and Ω_R, respectively. Since P is an exhaustion of $\mathbf{C}^n$, it follows that if $w \neq 0$ then $w \in \Omega_r$ for any $r > P(w)$. Then there exists a unique complex curve L_w, locally parameterized by $\phi_r(\cdot, v)$, that passes through 0 and w. As (7) holds, L_w is well-defined; in fact, if $w \neq 0$ and L_w is in the image of $\phi_r(\cdot, v)$ for some $r > 0$ and v, $\|v\| = 1$, then L_w is the surface such that, for every $R > 0$,

$$L_w \cap \Omega_R = \phi_R(\cdot, v)(\Delta).$$

Furthermore, since $u = \log P$ is an unbounded harmonic function on $L_w \setminus \{0\}$ with logarithmic singularity at the origin, it follows that L_w is a parabolic Riemann surface and hence is biholomorphic to $\mathbf{C}$. Thus we obtain a foliation of $\mathbf{C}^n \setminus \{0\}$ whose leaves all extend through the origin and meet only there. We call this *foliation in extended Kobayashi disks.* Finally, as mentioned before, the Kobayashi metric of any Ω_r is at least continuous, so for any $r > 0$ the maps $\phi_r(\cdot, v)$ depend continuously on v and $\phi_r(z, e^{i\theta}v) = \phi_r(e^{i\theta}z, v)$. Hence the foliation in extended Kobayashi disks of $\mathbf{C}^n \setminus \{0\}$ obtained in this way is continuous, and the leaf space of the foliation in extended Kobayashi disks has $\mathbf{CP}^{n-1}$ as its leaf space.

We can summarize our discussion so far as follows.

LEMMA 3.3. *There exists a continuous foliation of* $\mathbf{C}^n \setminus \{0\}$ *in parabolic Riemann surfaces with leaf space* $\mathbf{CP}^{n-1}$ *and with the property that any leaf L of the foliation extends holomorphically through the origin and, for any* $r > 0$, *the intersection of a leaf with* Ω_r *is the image of a Kobayashi extremal disk through the origin of* Ω_r. *Furthermore, the restriction of* $u = \log P$ *to any leaf of this foliation of* $\mathbf{C}^n \setminus \{0\}$ *is harmonic.*

Because P is a convex positive homogeneous polynomial on $\mathbf{C}^n$ (whose sublevel sets are even strictly geometrically convex), there exists an open dense subset U of $\mathbf{C}^n$ such that P has positive definite real Hessian on U and hence is strictly plurisuharmonic. Thus the complex vector field

$$Z = P^{\mu\bar{\nu}} P_{\bar{\nu}} \frac{\partial}{\partial z^\mu} \tag{8}$$

is defined on U, where we use the usual summation convention, subscripts indicate derivatives, and $P^{\mu\bar{\nu}}$ is the inverse of the Levi matrix $P_{\mu\bar{\nu}}$ of P. It is well known (see e.g. [3]) that, on the set U where P is strictly plurisubharmonic, the vector field Z is tangent to the Monge–Ampère foliation associated to $u = \log P$, which by definition is the foliation in Riemann surfaces generated by the distribution defined for $w \in U$:

$$\operatorname{Ann}_w dd^c u = \{V \in T_w(U) = \mathbf{C}^n \mid dd^c u_w(V, \bar{X}) = 0 \ \forall X \in \mathbf{C}^n\}. \tag{9}$$

The leaf L of the Monge–Ampère foliation on U passing through a given point is exactly the maximal Riemann surface through that point such that the restriction of u to L is harmonic.

It is also useful to recall (see [3] or [8]) that on U the Monge–Ampère equation (1) is equivalent to

$$Z(P) = P^{\mu\bar{\nu}} P_{\bar{\nu}} P_\mu = P. \tag{10}$$

It is also known that, in general, the restriction of the vector field Z to any leaf of the Monge–Ampère foliation is holomorphic. We can actually say more as follows.

LEMMA 3.4. *The vector field* Z *extends to a holomorphic vector field on* $\mathbf{C}^n$. *Furthermore, for* $w = (w^1, \dots, w^n) \in \mathbf{C}^n$ *we have*

$$Z_w = \frac{1}{k} w^\mu \frac{\partial}{\partial z^\mu}. \tag{11}$$

Proof. Again, let U be the open subset of $\mathbf{C}^n$ on which P has positive definite Hessian and hence is strictly plurisuharmonic (notice that $0 \notin U$). On U the vector field Z is defined and has real analytic coefficients. Furthermore, on U the function $u = \log P$ satisfies $(dd^c u)^{n-1} \neq 0$; that is, the Levi form of u has exactly $n-1$ positive eigenvalues (and one equal to 0). As we noticed previously, the restriction of $u = \log P$ to a leaf of the foliation in extended Kobayashi disks is a harmonic function. Thus, for every $r > 0$, the leaves of Monge–Ampère foliation on $U \cap \Omega_r$ coincide with the leaves of the foliation induced by the Kobayashi extremal discs through the origin. Hence, on $U \cap \Omega_r$, for every v by (10) we have

$$dd^c u_{\phi_r(z,v)}(\phi_r'(z,v),\bar{X}) = 0 \quad \forall X \in \mathbf{C}^n,$$

which in terms of P implies that, for all $\bar{\nu}$,

$$P(\phi_r(z,v))P_{\mu\bar{\nu}}(\phi_r(z,v))\phi_r'^{\mu}(z,v) = P_{\mu}(\phi_r(z,v))\phi_r'^{\mu}(z,v)P_{\bar{\nu}}(\phi_r(z,v)). \tag{12}$$

On the other hand, as a consequence of Corollary 3.2 we have

$$P(\phi_r(z,v)) = \exp((2k)\log\tanh(\gamma_r(0,\phi_r(z,v)) + \log r) = r|z|^{2k}. \tag{13}$$

Differentiating gives

$$P_{\mu}(\phi_r(z,v))\phi_r'^{\mu}(z,v) = kr|z|^{2(k-1)}\bar{z}. \tag{14}$$

Then, (12), (13), and (14) together yield

$$zP_{\mu\bar{\nu}}(\phi_r(z,v))\phi_r'^{\mu}(z,v) = kP_{\bar{\nu}}(\phi_r(z,v)),$$

which implies (using definition (8) of the vector field Z) that

$$Z^{\mu}_{\phi_r(z,v)} = z\phi_r'^{\mu}(z,v). \tag{15}$$

From (15) we conclude that the vector field Z has continuous extension on Ω_r for any $r > 0$ and hence on all $\mathbf{C}^n$. Notice that the extension is continuous because the Kobayashi metric is continuous on any Ω_r and hence, for all $r > 0$, the map $\phi_r(z,v)$ depends continuously on v.

In conclusion, since P is a homogeneous polynomial of degree $2k$, the very definition (8) implies that the components of Z are rational functions, homogeneous of degree 1. The components of Z are real analytic on the open set U, where P is strictly plurisubharmonic, and have continuous extensions to all of $\mathbf{C}^n$. We shall now show that Z is holomorphic on U and, as a consequence, on all $\mathbf{C}^n$. The idea is to adapt the proof of [3, Thm. 3.1], as we now indicate. If $T^{(1,0)}M$ is the bundle of $(1,0)$ tangent vectors of M and if $\mathcal{T}$ is the subbundle of vectors tangent to the foliation, then let $\mathcal{N} = T^{(1,0)}M/\mathcal{T}$ be the $(1,0)$ normal bundle to the foliation. The Bedford–Burns twist tensor of the foliation $\mathcal{L}\colon \mathcal{T} \otimes \bar{\mathcal{N}} \to \mathcal{N}$ is defined by

$$\mathcal{L}(V,\bar{W}) = [V,\bar{W}] \operatorname{mod}(\mathcal{T} \oplus T^{(0,1)}M), \tag{16}$$

where the Lie bracket is computed for vector fields that extend the tangent vectors V and W. It is known (see e.g. [1; 3]) that $\mathcal{L}$ vanishes on the open set U if and only if the restriction of the foliation to U is holomorphic which is the case iff the vector field Z is holomorphic on U. Let $u = \log P$. Outside the degeneracy set, $E = \{(dd^c u)^{n-1} = 0\} \subset \mathbf{C}^n \setminus U$, and the form $dd^c u$ naturally induces a metric on the normal bundle $\mathcal{N}$. Let L_0 be a leaf of the foliation in extended Kobayashi disks. Then either $L_0 \cap E$ is a discrete set or $L_0 \subset E$. If $L_0 \cap E$ is a discrete set, then the Ricci form ϕ of the restriction of $\mathcal{N}$ to L_0 vanishes iff the Bedford–Burns twist tensor vanishes along $L_0 \setminus E$ (see [1]). Letting $\operatorname{Ric}(\phi)$ denote the Ricci form of the form ϕ at the points where $\phi > 0$, a word-by-word repetition of the proof of [3, Thm. 3.1] yields, on $L_0 \cap \{\phi > 0\} \setminus E$,

$$\operatorname{Ric}(\phi) \geq \frac{2}{n-1}\phi. \tag{17}$$

Since L_0 is a parabolic Riemann surface, the curvature estimate (17) and Ahlfors's lemma imply that ϕ, and hence the Bedford–Burns twist tensor $\mathcal{L}$, vanishes along L_0. Since this is true for any leaf of the foliation that is not entirely contained in $E = \{(dd^c u)^{n-1} = 0\} \subset \mathbf{C}^n \setminus U$, we can conclude that the vector field Z is holomorphic on U. The set $\hat{E} = \mathbf{C}^n \setminus U$ is the polynomial variety where the determinant of the Levi matrix of P vanishes, so for any compact subset $K \subset \mathbf{C}^n$ it follows that the $(2n-1)$-Hausdorff measure of $\hat{E} \cap K$ is finite. Hence, a standard result on the removability of singularities of a continuous holomorphic function (see e.g. [4, Thm. A1.5]) shows that Z is holomorphic on $\mathbf{C}^n$. On the other hand, since by (8) we know that Z is homogeneous of degree 1 on a dense subset of $\mathbf{C}^n$, it follows that Z is, in fact, linear. In order to obtain the expression (11) for Z, we use (10) and a bidegree argument. Suppose that

$$P = \sum_{l+m=2k} P^{l,m}$$

is the decomposition of P into homogeneous polynomials of bidegree (l,m). Using (10) and differentiating with respect to $\bar{z}^\alpha$, we have

$$\sum_{\substack{l+m=2k \\ m\geq 1}} P^{l,m}_{\bar{\alpha}} = P_{\bar{\alpha}} = (Z^\mu P_\mu)_{\bar{\alpha}} = \sum_{\substack{l+m=2k \\ l,m\geq 1}} Z^\mu P^{l,m}_{\mu\bar{\alpha}}.$$

Comparing bidegrees, we conclude that

$$0 = P^{0,2k}_{\bar{\alpha}} = P^{2k,0}_{\bar{\alpha}}$$

and, for every l,m with $l+m = 2k$ and $l,m \geq 1$,

$$P^{l,m}_{\bar{\alpha}} = Z^\mu P^{l,m}_{\mu\bar{\alpha}};$$

here, as before, we have used the summation convention. Let $w = (w^1,\dots,w^n)$ be such that $dd^c P_w > 0$; then $dd^c P^{k,k}_w > 0$ by Lemma 2.1 and

$$Z^\mu_w = (P^{k,k})^{\bar{\alpha}\mu}(W) P^{k,k}_{\bar{\alpha}}(W) = \frac{1}{k} w^k,$$

where the last equality is immediate from (5) because $P^{k,k}_{\bar{\alpha}}$ is homogeneous of degree $(k,k-1)$ and hence $P^{k,k}_{\mu\bar{\alpha}}(W) w^\mu = k P^{k,k}_{\bar{\alpha}}$. Therefore, by continuity, (11) holds on all $\mathbf{C}^n$. □

To conclude the proof of Theorem 1.1 it is enough to observe that, by Lemma (3.3), the restriction of $\log P$ to any complex line through the origin is harmonic with a logarithmic singularity of weight $2k$ at the origin; hence, for any $0 \neq z \in \mathbf{C}^n$ and $\lambda \in \mathbf{C}$,

$$\log P(\lambda z) = 2k \log|\lambda| + O(1).$$

The restriction of P to any complex line through the origin is therefore homogeneous of bidegree (k,k) and so P is a homogeneous polynomial of bidegree (k,k) on $\mathbf{C}^n$.

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