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The Large-Scale Geometry of Some Metabelian Groups

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1. Introduction

In this paper we consider quasi-isometries of the upper triangular subgroup Γ_n of $\mathrm{PSL}_2(\mathbb{Z}[1/n])$. These groups arise in a geometric way because they are subgroups of both $\mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{PSL}_2(\mathbb{Q}_p)$ for all p dividing n. The group $\mathrm{PSL}_2(\mathbb{Q}_p)$ acts on its Bruhat–Tits building, a regular $(p+1)$-valent tree, and $\mathrm{PSL}_2(\mathbb{R})$ acts on $\mathbb{H}^2$. Then $G = \mathrm{PSL}_2(\mathbb{R}) \times \prod_{p_i|n} \mathrm{PSL}_2(\mathbb{Q}_{p_i})$ has an induced action on $\mathbb{H}^2 \times \prod_{i=1}^{k} T_i$, where T_i is the Bruhat–Tits building of $\mathrm{PSL}_2(\mathbb{Q}_{p_i})$. The restriction to Γ_n gives a properly discontinuous action with infinite volume quotient. However, the induced action of Γ_n on the product of trees is cocompact; the quotient is a k-torus. The stabilizer of any point is an infinite cyclic group that acts parabolically on $\mathbb{H}^2$. Thus Γ_n has a decomposition as a k-dimensional complex of groups [BH].

The upper triangular subgroup Γ_n arises naturally as the stabilizer of a point at infinity under the action of $\mathrm{PSL}_2(\mathbb{Z}[1/n])$ on $\mathbb{H}^2 \times \prod_{i=1}^{k} T_i$. For n prime, this group of upper triangular matrices is isomorphic to the solvable Baumslag–Solitar group $\mathrm{BS}(1, p^2) = \langle a, b \mid aba^{-1} = b^{p^2}\rangle$, and our results on quasi-isometries and rigidity generalize the results of [FM1]. In this case, the rigidity of the groups Γ_n should be useful for understanding the groups $\mathrm{PSL}_2(\mathbb{Z}[1/n])$, analogously to how the results of Farb and Mosher are used in [T].

The upper triangular groups Γ_n are also basic examples of metabelian groups fitting into the short exact sequence

$$1 \to \mathbb{Z}[1/n] \to \Gamma \to \mathbb{Z}^k \to 1.$$

In the sections that follow, we describe geometric models for these groups as warped products of $\mathbb{R}$ with the product of trees on which Γ_n acts. This identifies Γ_n as a cocompact lattice in the isometry group $\mathbb{R} \rtimes (\mathrm{Sim}(\mathbb{Q}_{m_1}) \times \cdots \times \mathrm{Sim}(\mathbb{Q}_{m_k}))$ of this model space, where $\mathrm{Sim}(\mathbb{Q}_m)$ is the group of similarities of the m-adic rationals and the m_i are determined by the prime factors of n. We also describe the group of all self–quasi-isometries of Γ_n and classify them up to quasi-isometry.

Our results rely on the technology available for groups acting on trees. However, products of trees are substantially more complicated than trees. For example, a group that acts freely on a tree is free whereas groups that act freely on a product

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of trees need not be products of free groups. Such groups can, in fact, be simple (see [BM]).

Our results generalize immediately to a larger class of groups that do not arise as nicely in a geometric context but are interesting nonetheless. This larger class of groups generalizes the solvable Baumslag–Solitar groups $\mathrm{BS}(1,n) = \langle a, b \mid aba^{-1} = b^n \rangle$. Let $S = \{n_1, n_2, \dots, n_k\}$, where $(n_i, n_j) = 1$ when $i \neq j$, and define $\Gamma = \Gamma(S)$ by

$$\Gamma = \Gamma(S) = \langle a_1, \dots, a_k, b \mid a_i^{-1} b a_i = b^{n_i},\ a_i a_j = a_j a_i,\ i \neq j \rangle.$$

These groups are $(k+1)$-dimensional metabelian groups fitting into a short exact sequence

$$1 \to A \to \Gamma \to \mathbb{Z}^k \to 1,$$

where the map onto $\mathbb{Z}^k$ is given by sending the $\{a_i\}$ to a basis and sending b to 0. The kernel, A, is normally generated by b and is an infinitely generated abelian group. Hence these groups provide natural examples of finite-type solvable groups that are not polycyclic.

The groups Γ_n are also of this same form. Namely, let $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, where the p_i are distinct primes. Then Γ_n is isomorphic to $\Gamma(p_1^{2e_1}, \dots, p_k^{2e_k})$, where the isomorphism is given by

$$a_i \mapsto \begin{pmatrix} p_i^{e_i} & 0 \\ 0 & p_i^{-e_i} \end{pmatrix},$$

$$b \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The decomposition of Γ_n into a k-dimensional complex of groups can be generalized to the groups $\Gamma(S)$. Indeed, the presentation given is that of a k-torus of infinite cyclic groups, generalizing the fact that all the Baumslag–Solitar groups are HNN extensions of $\mathbb{Z}$. This decomposition is fundamental to our study of the geometry of these groups. The groups $\Gamma(S)$ have geometric models analogous to those of the Γ_n. As a result, our quasi-isometry classification and rigidity results immediately generalize to this larger class of groups. We are able to identify $\Gamma(S)$ as a cocompact lattice in the isometry group of the model space, describe its quasi-isometry group, and classify these groups up to quasi-isometry.

1.1. Statement of Results

Let Γ_n be the upper triangular subgroup of $\mathrm{PSL}_2(\mathbb{Z}[1/n])$, and let X_n be the model space for Γ_n that is quasi-isometric to Γ_n and constructed in Section 3.

THEOREM 1.1 (Quasi-isometry classification). *Let Γ_n be the upper triangular subgroup of* $\mathrm{PSL}_2(\mathbb{Z}[1/n])$ *and Γ_m the upper triangular subgroup of* $\mathrm{PSL}_2(\mathbb{Z}[1/m])$. *If $n = p_1^{e_1} \cdots p_k^{e_k}$ and $m = q_1^{f_1} \cdots q_l^{f_l}$, where $\{p_i\}$ and $\{q_j\}$ are sets of distinct primes, then Γ_n and Γ_m are quasi-isometric if and only if $k = l$ and for $i = 1, 2, \dots, k$, after possibly re-ordering, $p_i = q_i$.*

THEOREM 1.2 (Quasi-isometry group). *Let $n = p_1^{e_1} \cdots p_k^{e_k}$, where all p_i are distinct primes. The quasi-isometry group,* $\mathrm{QI}(\Gamma_n)$, *is isomorphic to the product*

$$\mathrm{Bilip}(\mathbb{R}) \times \mathrm{Bilip}(\mathbb{Q}_{p_1}) \times \cdots \times \mathrm{Bilip}(\mathbb{Q}_{p_k}).$$

THEOREM 1.3 (Cusp group rigidity). *If Γ' is a finitely generated group quasi-isometric to Γ_n, then there is a finite normal subgroup F of Γ' such that Γ'/F is commensurable to Γ_n, meaning that Γ'/F and Γ_n have isomorphic subgroups of finite index.*

When we replace Γ_n by the more general group $\Gamma(S)$ defined previously, where all elements in S are pairwise relatively prime, we obtain the following generalizations of the previous theorems.

THEOREM 1.4. *Consider the sets $S_1 = \{n_1, n_2, \ldots, n_k\}$ and $S_2 = \{m_1, m_2, \ldots, m_l\}$ with $(n_i, n_j) = (m_i, m_j) = 1$ for $i \neq j$. Define $\Gamma_1 = \Gamma(S_1)$ and $\Gamma_2 = \Gamma(S_2)$. The groups Γ_1 and Γ_2 are quasi-isometric if and only if $k = l$ and for $i = 1, 2, \ldots, k$, after possibly re-ordering, each n_i is a rational power of m_i.*

THEOREM 1.5. *Let $S = \{n_1, n_2, \ldots, n_k\}$. Then the quasi-isometry group* $\mathrm{QI}(\Gamma(S))$ *is isomorphic to the product*

$$\mathrm{Bilip}(\mathbb{R}) \times \mathrm{Bilip}(\mathbb{Q}_{n_1}) \times \cdots \times \mathrm{Bilip}(\mathbb{Q}_{n_k}).$$

THEOREM 1.6. *Let Γ' be any finitely generated group quasi-isometric to $\Gamma(S)$. Then there exist integers $m_1, m_2, \ldots, m_k$, with each m_i a rational power of n_i, as well as a finite normal subgroup F of Γ' such that Γ'/F is isomorphic to a cocompact lattice in* $\mathrm{Iso}(X(m_1, \ldots, m_k)) = \mathbb{R} \rtimes (\mathrm{Sim}(\mathbb{Q}_{m_1}) \times \cdots \times \mathrm{Sim}(\mathbb{Q}_{m_k}))$.

1.2. Outline of the Proofs

The key to all our results is understanding the self–quasi-isometries of the model space $X = X_n$ for Γ_n, and in general for $\Gamma(S)$, constructed in Section 3. This model space is the warped product of $\mathbb{R}$ and a product of trees $\prod_{i=1}^k T_i$. We begin with a definition crucial to understanding the following outline of the proofs and refer the reader to Section 2 for additional definitions. Throughout, let $f: X \to X$ be any quasi-isometry.

When considering points in $\prod_{i=1}^k T_i$, it is important to define a notion of *height* on each tree T_i. Fix a basepoint $(t_1, t_2, \ldots, t_k) \in \prod_{i=1}^k T_i$. The height of a vertex $t \in T_i$ is the height change between t and the ith coordinate t_i of the basepoint. Extend this notion to a height function h_i on each tree T_i through linear interpolation along the edges. The metric on X is then given by a warped product of $\mathbb{R}$ and $\prod_{i=1}^k T_i$, where on each tree T_i the warping function is given by e^{-h_i}.

In the following outline, as in most of the paper, we consider only the groups Γ_n. The similarities in the construction of model spaces for the groups Γ_n and $\Gamma(S)$ ensure that generalizations of the proofs are immediate. Let $n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$, where the p_i are distinct primes.

1. *Warped product structure is preserved.* We first show that any quasi-isometry preserves, up to bounded distance, the *horocycles,* that is, the subsets of the form $\mathbb{R} \times (t_1, \ldots, t_k)$. In other words, there is a quasi-isometry $\bar{f}$ of the product of trees $T_1 \times \cdots \times T_k$ such that f and $\bar{f}$ commute with the projection $X \to T_1 \times \cdots \times T_k$. Results of [KL] imply that, up to permuting the factors, $\bar{f}$ splits as a product of quasi-isometries $\bar{f}_i$ of the trees T_i.

2. *Quasi-isometries are almost height translations on the tree factors.* The geometry of the space X restricts the quasi-isometries $\bar{f}_i$. The warping function can be reconstructed as the (logarithm of the) amount of stretching induced by closest point projection between the horocycles. This splits as a sum of functions, h_i, on each of the trees. The quasi-isometries $\bar{f}_i$ preserve these warping functions in the sense that $h_i(\bar{f}_i(x)) - h_i(\bar{f}_i(y))$ differs from $h_i(x) - h_i(y)$ by a uniformly bounded amount. We call such quasi-isometries *almost height translations.* In [FM1], the group of almost height translations of T_n is identified as $\mathrm{Bilip}(\mathbb{Q}_n)$.

3. *f induces a bilipschitz homeomorphism of $\mathbb{R}$.* This shows that the group of quasi-isometries of $T_1 \times \cdots \times T_k$ that quasi-preserve the warping function is $\mathrm{Bilip}(\mathbb{Q}_{p_1}) \times \cdots \times \mathrm{Bilip}(\mathbb{Q}_{p_r})$. All of these quasi-isometries extend to quasi-isometries of X. The quasi-isometries of X that induce the identity on $T_1 \times \cdots \times T_r$ induce a bilipschitz homeomorphism of $\mathbb{R}$. This allows us to identify the quasi-isometry group of X and so prove Theorem 1.2.

4. *These methods hold for quasi-isometries between Γ_n and Γ_m.* Consider a quasi-isometry $f\colon \Gamma_n \to \Gamma_m$. Using again the methods described here shows that f induces a bilipschitz homeomorphism of $\mathbb{R}$ and a quasi-isometry on the product of trees that is a bounded distance from a product quasi-isometry. Theorem 1.1 now follows by combining results of [FM1] and [W1].

5. *Quasi-actions.* Understanding the quasi-isometries of X enables us to understand groups that are quasi-isometric to Γ_n via the quasi-action principle. Suppose Γ' is quasi-isometric to Γ_n (and hence to X), and let $f\colon \Gamma' \to X$ be a quasi-isometry. For every $\gamma' \in \Gamma'$ we obtain a quasi-isometry of X by $x \mapsto f(\gamma' f^{-1}(x))$. These quasi-isometries all have uniform constants and compose, up to bounded distance, according to the multiplication table of Γ'. In other words, Γ' *quasi-acts on X* and thus gives an almost height translation action on each of the T_i.

6. *Obtaining similarity actions on $\mathbb{Q}_n$.* According to [MSW] these almost height translation actions are equivalent, via a quasi-isometry $T_i \to T_i'$, to a height translation action on trees T_i'. In terms of the p_i-adics, this means that there is some q_i such that the bilipschitz action of Γ' on $\mathbb{Q}_{p_i}$ is bilipschitz equivalent to a similarity action on $\mathbb{Q}_{q_i}$. Similarly, the bilipschitz action on $\mathbb{R}$ is equivalent to an affine action on $\mathbb{R}$. Further, the uniformity of the quasi-isometry constants implies that the expansion factor of the affine action on $\mathbb{R}$ is the inverse of the product of the factors from the similarity actions on the $\mathbb{Q}_{q_i}$. This shows Γ' to be a lattice in the subgroup of $\mathrm{Aff}(\mathbb{R}) \times \mathrm{Sim}(\mathbb{Q}_{q_1}) \times \cdots \times \mathrm{Sim}(\mathbb{Q}_{q_k})$ that satisfies this condition. This subgroup is $\mathbb{R} \rtimes \mathrm{Sim}(\mathbb{Q}_{q_1}) \times \cdots \times \mathrm{Sim}(\mathbb{Q}_{q_k})$ and can be identified as the isometry group of a complex X', proving Theorem 1.3.

2. Preliminaries

2.1. Definitions and Notation

We begin with the definition of a quasi-isometry.

Definition. Let $K \geq 1$ and $C \geq 0$. A (K, C)-*quasi-isometry* between metric spaces (X, d_X) and (Y, d_Y) is a map $f: X \to Y$ that satisfies the following conditions.

(1) $\frac{1}{K} d_X(x_1, x_2) - C \leq d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2) + C$ for all $x_1, x_2 \in X$.
(2) For some constant C', we have $Nbhd_{C'}(f(X)) = Y$.

We will assume that our quasi-isometries have been changed by a bounded amount, using the standard "connect-the-dots" procedure to be continuous (see e.g. [SW]). A quasi-isometry has a *coarse inverse,* that is, a quasi-isometry $g: Y \to X$ such that $f \circ g$ and $g \circ f$ are a bounded distance from the appropriate identity map in the supremum norm. A map satisfying (1) but not (2) in the definition is called a *quasi-isometric embedding.*

We define the *quasi-isometry group* $\mathrm{QI}(X)$ of a space X to be the collection of all self–quasi-isometries of X, identifying those that differ by a bounded amount in the supremum norm.

Given a group G and a metric space X, a *quasi-action* of G on X associates to each $g \in G$ a quasi-isometry of X; that is, $A_g: X \to X$, subject to certain conditions. This map is defined by $A_g(x) = g \cdot x$ and the collection of these maps has uniform quasi-isometry constants, so that $A_{\mathrm{Id}} = \mathrm{Id}_X$ and $d_{\sup}(A_g \circ A_h, A_{gh})$ is bounded independently of g and h.

2.2. Previous Results

The following theorems will be referred to repeatedly in Section 4. We state them here for easy reference.

2.2.1. *Rigidity of Baumslag–Solitar Groups.* Since the geometry of the group Γ_n is so dependent on its various Baumslag–Solitar subgroups, we will often refer to the following classification and rigidity results (for the solvable Baumslag–Solitar groups) due to Farb and Mosher.

Theorem 2.1 [FM1, Thm. 7.1]. *For integers $m, n \geq 2$, the groups $\mathrm{BS}(1, m)$ and $\mathrm{BS}(1, n)$ are quasi-isometric if and only if they are commensurable. This happens if and only if there exist integers $r, j, k > 0$ such that $m = r^j$ and $n = r^k$.*

Theorem 2.2 [FM1, Thm. 8.1]. *The quasi-isometry group of $\mathrm{BS}(1, n)$ is given by the following isomorphism*:

$$\mathrm{QI}(\mathrm{BS}(1, n)) \cong \mathrm{Bilip}(\mathbb{R}) \times \mathrm{Bilip}(\mathbb{Q}_n).$$

2.2.2. *Products of Trees and Groups Acting on Products of Trees.* A major step in the proofs that follow is showing that a quasi-isometry $f\colon \Gamma_1 \to \Gamma_2$ induces a map on the product of trees on which each group acts. Once this is accomplished, we use the following result of Kleiner and Leeb to show that our map is uniformly close to a product of quasi-isometries.

THEOREM 2.3 [KL, Thm. 1.1.2]. *Let T_i and T_i' be irreducible thick Euclidean Tits buildings with cocompact affine Weyl group. Let $X = \mathbb{E}^n \times \prod_{i=1}^{k} T_i$ and $X' = \mathbb{E}^{n'} \times \prod_{i=1}^{k'} T_i'$ be metric products. Then, for all $K, C > 0$, there exist K', C', D' such that the following statement holds: If $f\colon X \to X'$ is a (K, C)-quasi-isometry, then $n = n'$, $k = k'$, and there are (K', C')-quasi-isometries $f_i\colon T_i \to T_j$ such that $d\big(p \circ f, \prod_{i=1}^{k} f_i \circ p\big) \le D'$, where p is the projection map.*

The following result of [MSW] will be needed for the proof of rigidity of the groups Γ_n. It applies to *bushy* trees, meaning that each vertex is a uniformly bounded distance from a vertex having at least three unbounded complementary components. We also require *bounded valence,* meaning that vertices have uniformly finite bounded valence. All of the trees in this paper satisfy these properties.

THEOREM 2.4 [MSW]. *If $G \times T \to T$ is a quasi-action of a group G on a bounded valence and bushy tree T, then there exist a bounded-valence and bushy tree T', an isometric action $G \times T' \to T'$, and a quasi-isometry $f\colon T' \to T$ that intertwines the actions of G on T' and the quasi-action of G on T to within a uniformly bounded distance.*

3. The Geometric Models

To illustrate the geometry of Γ_n and of $\Gamma(S)$ in general, we describe a metric $(k+1)$-complex X quasi-isometric to Γ_n (i.e., on which Γ_n acts properly discontinuously and cocompactly by isometries). We begin with the simplest case of the upper triangular subgroup Γ_n of $\mathrm{PSL}_2(\mathbb{Z}[1/n])$. We then describe the geometry of the more general groups $\Gamma(S)$. For all of these groups, the complex X is a warped product of $\mathbb{R}$ with a product of trees on which the group acts. If the n_i are not relatively prime then the group $\Gamma(S)$ does not act on a product of trees, so we do not consider this case here.

First recall that the Baumslag–Solitar groups $\mathrm{BS}(1, n) = \langle a, b \mid aba^{-1} = b^n \rangle$, for integral $n \ge 2$, act properly discontinuously and cocompactly by isometries on metric 2-complexes. These complexes Y_n are topologically the product $T \times \mathbb{R}$, where each vertex of the tree T has one incoming edge and n outgoing edges. Metrically we define a height function on T so that, if $l \subset T$ is a line on which the height function is strictly increasing, then $l \times \mathbb{R}$ is metrically a hyperbolic plane. See [FM1] and Figure 1 for a more detailed construction of this complex.

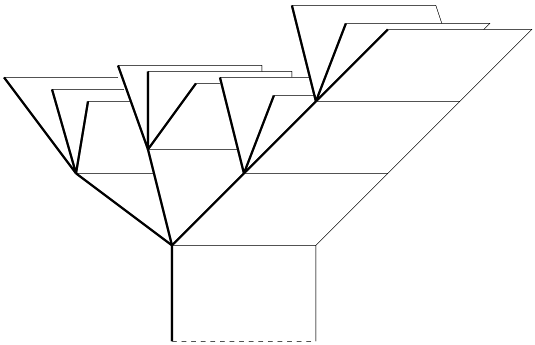

Figure 1 The geometric model of the solvable Baumslag–Solitar group $\mathrm{BS}(1, 3)$, which is topologically a warped product of a tree and $\mathbb{R}$

3.1. The Geometric Model of Γ_n

We give the most comprehensive description of the model space X in this case because the trees on which Γ_n acts are easier to understand than the trees on which $\Gamma(S)$ acts. We present several ways to understand the complex X.

When p is prime, the group $\mathrm{BS}(1, p)$, acts on the Bruhat–Tits tree T_p associated to $\mathrm{PSL}_2(\mathbb{Q}_p)$. This is not true for $\mathrm{BS}(1, n)$ when n is not prime. We will describe the $\mathrm{BS}(1, n)$ tree in Section 3.2.

Assume p is prime, and consider the geometric model Y_p of $\mathrm{BS}(1, p)$. Let (x, y) be the coordinates on the upper half-space model of hyperbolic space, where $y > 0$. One can also view Y_p as being built from the "horobrick" with $0 \leq x \leq n$ and $1 \leq y \leq p$. The vertical sides of this brick have length $\log p$. In the Cayley graph of $\mathrm{BS}(1, p)$, this horobrick has the form given in Figure 2.

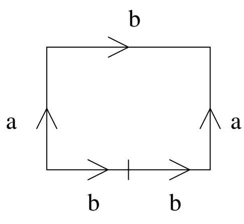

Figure 2 The "horobrick" building block for the geometric model of $\mathrm{BS}(1, 2)$

If $n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$ then Γ_n acts on the product of the trees $\prod_{i=1}^{k} T_i$, where T_i is the tree on which $\mathrm{BS}(1, p_i)$ acts; that is, T_i has one incoming edge at each vertex and p_i outgoing edges. The complex X is the same warped product of $\prod_{i=1}^{k} T_i$ with $\mathbb{R}$ as we saw previously for $\mathrm{BS}(1, p)$.

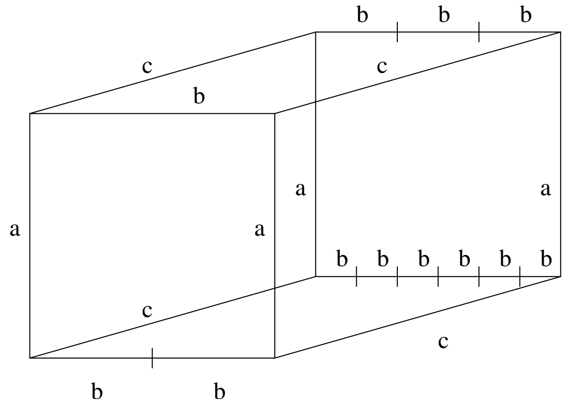

Figure 3 The analogous building block for $\Gamma(2, 3)$

Analogously for Γ_n, there is an $(k+1)$-dimensional building block used to construct the complex X, whose 1-skeleton is the Cayley graph of Γ_n. An example of this block is given in Figure 3 for n a product of two primes. It is not difficult to see that the correct branching occurs when these blocks are arranged so as to form the appropriate Baumslag–Solitar subcomplexes. In general, the $(k+1)$-dimensional building block will be an $(k+1)$-cube with appropriate edge labels in terms of the generators of $\Gamma(S)$. We refer to the horocycles along which the sheets meet as *branching horocycles.*

A second way of understanding the complex X is in terms of some of its special subspaces. Let $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, where the p_i are prime. Then

$$\Gamma_n \cong \langle a_1, \dots, a_k, b \mid a_i^{-1} b a_i = b^{p_i^{2e_i}},\ a_i a_j = a_j a_i,\ i \neq j \rangle.$$

We consider in particular two types of subspaces of X:

(i) Y_{p_i}, corresponding to $\mathrm{BS}(1, p_i^{2e_i})$ and generated by a_i and b in the presentation above; and

(ii) $\mathbb{Z}^l$ for $1 \le l \le k$, generated by l distinct generators a_i in the presentation above.

Notice that the $\mathrm{BS}(1, p_i)$ subgroups of Γ all share the generator b; in X this means that the subcomplexes Y_{p_i} for $i = 1, 2, \dots, k$ are joined along branching horocycles. Namely, consider a subcomplex Y_{p_i} of X. At each branching horocycle of Y_{p_i} there is a copy of Y_{p_j} for all $j \neq i$ attached along that horocycle. The same is true for every branching horocycle of those Y_{p_j}, and the process continues.

For any vertex $x \in X$, there is a Y_{p_i} subspace for each $i = 1, 2, \dots, k$ in X that contains x. For each i, the set $\{a_i^m \cdot x \mid m \in \mathbb{Z}\}$ is the vertex set of a line in the Cayley graph of Γ, which is the 1-skeleton of X. These lines form the axes of an $\mathbb{R}^k$ subspace of X. The orbit of x under the group generated by the entire collection $\{a_i\}$ is a $\mathbb{Z}^k$ subspace of X.

3.2. The Geometric Model of $\Gamma(S)$

When $\Gamma = \Gamma(n_1, n_2, \ldots, n_k)$ and the n_i are relatively prime but not all prime, the product of trees on which Γ acts is not as simple.

We first discuss the tree T^n on which the group $\mathrm{BS}(1, n)$ acts when n is not prime. Suppose that $n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$, and let T_i be the Bruhat–Tits tree associated to $\mathrm{PSL}_2(\mathbb{Q}_{p_i})$. The tree T^n is a subspace of $\prod_{i=1}^r T_i$ whose branching may not be constant and depends on the exponents of the primes.

Define a folding function $F_i\colon T_i \to \mathbb{R}$ as follows. If h_i is the height function defined on T_i, then $F_i(t) = h_i(t)$ for $t \in T_i$. Combining folding functions on the T_i yields a map $F(r)\colon \prod_{i=1}^r T_i \to \mathbb{R}^r$ defined by $F(r) = (F_1, F_2, \ldots, F_r)$. Consider the grid of lines in $\mathbb{R}^r$ of the form $(x_1, x_2, \ldots, x_{j-1}, \mathbb{R}, x_{j+1}, \ldots, x_r)$, where $x_i \in \mathbb{Z}$. So we actually have r families of parallel lines in $\mathbb{R}^r$. View each family as representing folded copies of one of the trees T_i under $F(r)$.

The branching of the tree T^n is determined by the line

$$e_1 x_1 + e_2 x_2 + \cdots + e_r x_r = 0$$

in $\mathbb{R}^r$. When the line crosses a line in the family of parallel lines corresponding to T_i, the tree T^n branches n times. When the line crosses the intersection of two lines, one from the family of T_i and one from the family of T_j, the branching is $i + j$.

Example. Consider the group $\mathrm{BS}(1, 6)$. The tree T^6 on which it acts is a subset of $T_2 \times T_3$ determined by the line $y = -x$ in the plane $\mathbb{R}^2$, since the exponent of each prime is 1. This line only crosses vertices of the grid of lines, so the branching is uniform of valence 6.

Example. Consider the group $\mathrm{BS}(1, 12)$. The tree T^{12} on which it acts is a subset of $T_2 \times T_3$, only now the line in $\mathbb{R}^2$ that determines the branching is $-2x = y$. From the way this line crosses the grid of lines, we see that the branching of T^{12} is not uniform. The vertices alternate between valence 2 and valence 6, where the valence 2 arises from the line crossing only a horizontal grid line and the valence 6 arises when the line crosses a vertex in this grid of lines.

Example. Consider the group $\mathrm{BS}(1, 60)$. The tree T^{60} on which it acts is a subset of $T_2 \times T_3 \times T_5$, and now the folding map $F(3)$ is a map to $\mathbb{R}^3$. The line in $\mathbb{R}^3$ determining the branching of T is $2x + y + z = 0$. Again we see that the amount of branching at each valence varies.

Now consider $\Gamma = \Gamma(n_1, n_2, \ldots, n_k)$ where the n_i are not all prime. Consider any n_i and let $p_1, p_2, \ldots, p_r$ be the list of primes dividing n_i, with T_i the Bruhat–Tits tree of $\mathrm{PSL}_2(\mathbb{Q}_{p_i})$. Then let T^i be the tree on which $\mathrm{BS}(1, n_i)$ acts (described previously) that is a subspace of $\prod_{i=1}^r T_i$. Then Γ acts on $\prod_{i=1}^k T^i$. The complex $X(S)$ is then the warped product of $\prod_{i=1}^k T^i$ with $\mathbb{R}$.

4. The Structure of Quasi-Isometries

The key step in the proofs of the theorems in this paper is understanding the structure of the quasi-isometries of Γ, or equivalently of X. We begin with two groups Γ_1 and Γ_2 and a (K, C)-quasi-isometry between their geometric models, $f\colon X_1 \to X_2$.

Let π be the projection $X \to \prod_{i=1}^k T_i$. Define a *horocycle* of the complex X to be a subset of the form $\pi^{-1}(t_1, t_2, \dots, t_k)$, where $(t_1, t_2, \dots, t_k)$ is a point in $\prod_{i=1}^k T_i$. A *hyperplane* in X is a subcomplex of the form $\pi^{-1}(l_1 \times \cdots \times l_n)$, where each l_i is a geodesic in T_i. The first goal is to show that the quasi-isometry f preserves horocycles and hence induces a quasi-isometry of a product of trees. These arguments are similar to those in [W2].

LEMMA 4.1. *For any (K, C) there is an $R > 0$ such that, for any $f\colon X_1 \to X_2$ (a (K, C)-quasi-isometry) and every hyperplane H of X_1, there exists a subset Y of $\prod_{i=1}^k T_i$ such that the image $f(H)$ is within distance R of $\pi^{-1}(Y)$.*

Proof. Let g be a quasi-inverse of f, so that $g \circ f$ is a bounded distance from the identity map and hence proper. By a standard connect-the-dots argument, we may assume that f and g are continuous. Since the X_i are uniformly contractible, the compositions are homotopic to the identity through homotopies of length at most R_0 (depending only on the X_i and the constants (K, C)). Then the maps f and g are, in particular, proper homotopy equivalences.

Consider the fundamental class $[H]$ in $H_{n+1}^{uf}(X_1)$. The push-forward $f_*([H])$ is thus a nontrivial class in $H_{n+1}^{uf}(X_2)$. Further, this class clearly has a representative c with support contained in the R_0 neighborhood of $f(H)$. The simplicial structure of X_2 forces the coefficients of c to be constant along horocycles. Thus, the support of c is of the form $Y \times \mathbb{R}$ for some subcomplex Y of $\prod_{i=1}^k T_i$. This shows that the R_0 neighborhood of $f(H)$ contains $Y \times \mathbb{R}$.

To complete the proof we must show that a neighborhood of $Y \times \mathbb{R}$ $(= \mathrm{supp}(c))$ contains $f(H)$. If not, then there are arbitrarily large balls in $f(H)$ that are not contained in $Y \times \mathbb{R}$. Applying the inverse map, g, this would give a representative of $[H]$ whose support misses large balls in H. This is impossible, since any representative of the fundamental class has full support. □

LEMMA 4.2 (Horocycles are preserved). *For every (K, C) there exists an R such that, if f is a (K, C)-quasi-isometry of X and h is a horocycle in X, then there is a horocycle h' such that $d_H(f(h), h') \le R$.*

Proof. This is an immediate consequence of Lemma 4.1. For any horocycle h, there are a finite number of hyperplanes $H_1, \dots, H_k$ in X_1 that have coarse intersection at Hausdorff distance at most R from h. The constants k and R depend only on the geometry of X_1. Lemma 4.1 implies that the image of h is Hausdorff equivalent to a complex of the form $Y \times \mathbb{R}$ for some subset Y of X_2. Applying the same argument to the inverse map g and each horocycle in $Y \times \mathbb{R}$, we conclude that Y must be of finite diameter (bounded independently of h). This proves the lemma. □

Corollary 4.3 (Factor preserving). *Consider the groups* $\Gamma_1 = \Gamma(n_1, n_2, \dots, n_k)$ *and* $\Gamma_2 = \Gamma(m_1, m_2, \dots, m_l)$, *where* $(n_i, n_j) = (m_i, m_j) = 1$ *for* $i \neq j$, *and as a quasi-isometry* $f : \Gamma_1 \to \Gamma_2$ *between them. Then*:

(a) $k = l$;
(b) f *induces a quasi-isometry* $f_T : \prod_{i=1}^k T_i \to \prod_{i=1}^k T_i'$; *and*
(c) *there are* (K', C')*-quasi-isometries* $\overline{f_i} : T_i \to T_i'$ (*after possibly re-indexing the tree factors*) *such that* f_T *is a bounded distance from the product quasi-isometry* $\overline{f_1} \times \cdots \times \overline{f_k}$.

Proof. Since every point $(t_1, t_2, \dots, t_n) \in \prod_{i=1}^k T_i$ determines a horocycle in X, it follows from Lemma 4.2 that the quasi-isometry f induces a quasi-isometry on the product of trees: $f_T : \prod_{i=1}^k T_i \to \prod_{i=1}^l T_i$. It now follows from Theorem 2.3 that $k = l$ and hence there exists the same number of parameters in Γ_1 and Γ_2. It then follows from Theorem 2.3 that this map is a bounded distance from a product $\overline{f_1} \times \cdots \times \overline{f_k}$ of quasi-isometries. □

Corollary 4.4 (Bilipschitz maps). *Let* $f : \Gamma_1 \to \Gamma_2$ *be a* (K, C)*-quasi-isometry. Then there exist bilipschitz maps* $g : \mathbb{R} \to \mathbb{R}$ *and* $f_i : T_i \to T_i$ (*after possibly re-indexing the tree factors*) *so that* f *is a bounded distance from* $(g, f_1, \dots, f_k)$.

Proof. Applying Corollary 4.3, we may assume that the quasi-isometry f preserves the individual tree factors. We use the notation of Corollary 4.3 and let f_i denote the induced map on the ith tree factor. It follows that the quasi-isometry f restricts to a map on each Baumslag–Solitar subcomplex $T_i \times \mathbb{R}$ that is also a quasi-isometry. Applying Theorem 2.2, we conclude that f_i is a bounded distance from the product of a bilipschitz map of T_i with a bilipschitz map of $\mathbb{R}$. It is easy to see that we must obtain the same bilipschitz map of $\mathbb{R}$ regardless of which Baumslag–Solitar subspace we restrict to, and the corollary follows. □

We are now able to prove Theorem 1.1.

Proof of Theorem 1.1. Applying Corollary 4.4, we consider our quasi-isometry to be factor-preserving and of the form $(g, f_1, \dots, f_n)$, with each individual map bilipschitz. Then any pair $(g, f_i) : \mathbb{R} \times T_i \to \mathbb{R} \times T_i'$ is a quasi-isometry of $BS(1, p_i)$ to $BS(1, q_i)$ by Theorem 2.2. It follows from Theorem 2.1 that, after re-ordering, $p_i = q_i$. □

Description of the Quasi-Isometry Group

We begin with a lemma that is important for the proof of Theorem 1.2.

Lemma 4.5 ([FM2]; Rubber-band principle). *For all* $L, M > 0$, *there is a constant* C *that satisfies the following property. Let* X *and* Y *be path metric spaces, and let* $f : X \to Y$ *be a map. Suppose there are collections of isometrically embedded subspaces* C_X *of* X *and* C_Y *of* Y *that satisfy the following statements*:

(a) *any two points in* X (*or in* Y) *can be connected by an* M*-quasi-geodesic consisting of a finite number of subpaths, each lying in an element of* C_X (*or* C_Y);
(b) f *induces a one-to-one correspondence between elements of* C_X *and* C_Y; *and*

(c) *f restricts to an L-quasi-isometry between corresponding elements of C_X and C_Y.*

Then $f\colon X \to Y$ is a C-quasi-isometry.

We are now able to prove Theorem 1.2 and describe the quasi-isometry group of Γ.

Proof of Theorem 1.2. It is clear that we have a homomorphism

$$\Phi\colon \mathrm{QI}(X) \to \mathrm{Bilip}(\mathbb{R}) \times \mathrm{Bilip}(\mathbb{Q}_{p_1}) \times \mathrm{Bilip}(\mathbb{Q}_{p_2}) \times \cdots \times \mathrm{Bilip}(\mathbb{Q}_{p_k})$$

given by $\Phi(f) = (f_{\mathbb{R}}, f_1, f_2, \ldots, f_n)$. In addition, we have a homomorphism

$$\Phi_i\colon \mathrm{QI}(X) \to \mathrm{QI}(\mathrm{BS}(1, p_i)) \cong \mathrm{Bilip}(\mathbb{R}) \times \mathrm{Bilip}(\mathbb{Q}_{p_i})$$

for each $i = 1, 2, \ldots, n$ that is given by $\Phi(f) = (f_{\mathbb{R}}, f_i)$. By the reasoning in [FM1] it follows that, for any $f \in \ker(\Phi)$, the quasi-isometry $\Phi_i(f)$ is a bounded distance B_i from the identity map on X_n. Letting $B = \max\{B_i\}$, the rubber-band principle implies that $\Phi(f)$ is a bounded distance B from the identity.

To see that Φ is surjective, we again use the rubber-band principle to piece together quasi-isometries of the X_n subcomplexes. Choose $f_R \in \mathrm{Bilip}(\mathbb{R})$ and maps $f_i \in \mathrm{Bilip}(\mathbb{Q}_{p_i})$. We must show that $f_R \times f_1 \times \cdots \times f_n$ is a quasi-isometry of Γ_1. From [FM1] we know that any pair (f_R, f_i) yields a quasi-isometry of X_i. We can assume the quasi-isometry constants are uniform by taking the largest pair of constants from any of these maps. Since the f_R is common to any two pairs, we obtain a product map f of the entire complex. Thus we have a collection of subspaces and a map $f = f_R \times f_1 \times \cdots \times f_n$ satisfying the rubber-band principle, so f is a quasi-isometry of Γ. □

5. Rigidity

We finish with the proof of Theorem 1.3, which shows that this class of groups is quasi-isometrically rigid.

Proof of Theorem 1.3. Let Γ' be any finitely generated group quasi-isometric to $\Gamma(S)$, with X the model space for $\Gamma(S)$ as before, and let $f\colon \Gamma' \to X$ be a quasi-isometry with g a coarse inverse. We thus have a quasi-action of Γ' on X, where $\gamma' x = f(\gamma' g(x))$. By Lemma 4.2, horocycles are preserved and so we obtain an induced quasi-action of Γ' on the product of trees $\prod_{i=1}^{k} T_i$. By passing to a finite index subgroup of Γ', we may assume that the quasi-action is the diagonal quasi-action of a collection of quasi-actions Γ' on T_i. The maps to the complexes of the Baumslag–Solitar subgroups of $\Gamma(S)$ are Γ' equivariant (to within finite distance) and so quasi-preserve the height function. By [MSW], there exist trees T_i' that are quasi-isometric to the T_i as well as actions of Γ' on T_i that are quasi-conjugate to the quasi-actions on the T_i. Further, each of these trees is homogeneous, with a Γ' invariant orientation where one edge is directed into each vertex. Thus we get an action of Γ' on the product of the T_i', with vertex stabilizers that are orientation-preserving and virtually cyclic, and with finite quotient; in other words, we obtain a description of Γ' as a finite complex of virtual $\mathbb{Z}$s.

Consider the edges in this quotient that come from edges of a T_i'. These are oriented, and as there is exactly one edge oriented toward every vertex in T_i', the same is true in the quotient. Since the quotient is finite, this implies that there is precisely one such edge oriented away from each vertex of the quotient. Thus the collection of these edges consists of a finite union of circles. Furthermore, for any $v \in T_i'$, the action of $\mathrm{stab}(v)$ on edges pointing away from v is transitive. Similarly, fixing any edge in T_i' and looking at two cells coming from $T_i' \times T_j'$, we have exactly two such cells at every edge of the quotient: one oriented toward and one away from this edge. Continuing over higher-dimensional cubes we see that the quotient is a product of oriented circles, where the inclusion of the cube stabilizer to a face stabilizer is an isomorphism if it goes against the orientation. Hence we may collapse such a cube unless its opposite faces are the same in the quotient. Making all such possible collapses yields a complex-of-groups description of Γ' with underlying complex a product of oriented loops, with stabilizers all virtually $\mathbb{Z}$, and with the inclusions isomorphism when going against the orientation. As in [FM1], we may pass to a finite-index subgroup of Γ' that has such a description with all stabilizers $\mathbb{Z}$. Such a complex of groups has a presentation precisely of the form $\Gamma(S')$ for some S'. Thus Γ' is commensurable to $\Gamma(S')$ for some S', as desired. □

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