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Poles Near the Origin Produce Lower Bounds for Coefficients of Meromorphic Univalent Functions

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1. Introduction

Let D denote the open unit disc. In this paper we consider functions f meromorphic and univalent in D that have a simple pole at the point $p \in D \setminus \{0\}$. For $r \in (0,1)$ fixed, let U_r denote the class of all such functions f, $|p| = r$, that are normalized by $f(0) = 0$ and $f'(0) = 1$. Hence, any function $f \in U_r$ has an expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n(f) z^n, \quad |z| < r,$$

and $f(p) = \infty$.

As usual, we denote by S the class of functions g holomorphic and univalent in D with Taylor coefficients $a_n(g) = g^{(n)}(0)/n!$, where $a_0(g) = 0$ and $a_1(g) = 1$. By de Branges's famous proof [4] of the validity of the Bieberbach conjecture, it is known that the domain of variability of $a_n(g)$ for $g \in S$ and $n \geq 2$ is the whole disc defined by

$$|a_n(g)| \leq n \tag{1}$$

and that equality in (1) is attained if and only if

$$g(z) = \kappa_d(z) := \frac{z}{(1 - \bar{d}z)^2}, \quad |d| = 1. \tag{2}$$

In [9] Goodman conjectured that, for $f \in U_r$ with $r \in (0,1)$, the inequalities

$$|a_n(f)| \leq \frac{1}{r^{n-1}} \sum_{k=0}^{n-1} r^{2k}, \quad n \geq 2, \tag{3}$$

are valid. Jenkins [12] proved that (3) is true for $a_N(f)$ if (1) is valid for $n = 2, \dots, N$. Hence (3) holds for all $n \geq 2$. From Jenkins's proof it is also evident that equality in (3) is attained if and only if

$$f(z) = \kappa_p(z) := \frac{z}{(1 - \bar{p}z)(1 - z/p)}, \quad |p| = r. \tag{4}$$

Concerning the inverse functions g^{-1} of $g \in S$, a classical theorem of Löwner [19] indicates that, for the Taylor coefficients $A_n(g^{-1})$, the inequalities

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$$|A_n(g^{-1})| \leq L_n := \frac{1}{n+1}\binom{2n}{n}, \quad n \geq 2, \tag{5}$$

are valid. For the Taylor coefficients $A_n(f^{-1})$, $n \geq 2$, of the inverse functions of $f \in U_r$ with $r \in (0,1)$, Baernstein and Schober [3] proved that

$$|A_n(f^{-1})| \leq \frac{1}{n}\sum_{k=0}^{n-1}\binom{n}{k+1}\binom{n}{k}r^{n-2k-1}. \tag{6}$$

In (5) and in (6), equality is attained if and only if g^{-1} and f^{-1} are the inverses of the functions defined in (2) and (4), respectively.

This paper is devoted to the perhaps surprising fact that, at least for little $r \in (0,1)$, the domain of variability of $a_n(f)$ and $A_n(f^{-1})$, $f \in U_r$, is *not* a disc but an annulus. For $n = 2$ this is a consequence of some formulas due to Goluzin ([8, Chap. IV.3]; cf. [15]). As far as we know, there has been scant attention given to this fact for $n \geq 3$. We have found only one reference (namely, [6]) concerning this for U_r and some results and conjectures in this direction for subclasses of U_r (see [1; 2; 17; 18; 20]).

In Section 2 we discuss in some detail the domain of variability of $a_2(f)$, $f \in U_r$—especially for little $r \in (0,1)$, where this domain is an annulus. Furthermore, we give some simple examples showing that, for $n \geq 2$ and $r \in (0,1)$ big enough, the domain of variabilty of $a_n(f)$ is a disc.

In Section 3 (Theorem 1) we prove the estimate

$$\left|\frac{1 - p^{n-1}a_n(f)}{1 - n^{-2}p^{n-1}a_n(f)}\right| \leq nr^2, \quad n \geq 2, \;\; f \in U_r, \;\; r \in (0,1), \tag{7}$$

and its consequence, the asymptotically sharp formula

$$p^{n-1}a_n(f) = 1 + O(r^2), \tag{8}$$

where n, f, and r are as before.

An explicit computation of the center and radius of the disc (7) reveals that the domain of variability of $a_n(f)$, $f \in U_r$, is an annulus at least for $r < 1/\sqrt{n}$.

Concerning the Taylor coefficients $A_n(f^{-1})$, from formula (5) (where the quantities L_n are defined) and formulas (6)–(8) we obtain that the inequalities

$$\left|\frac{(-1)^{n-1} - p^{n-1}A_n(f^{-1})}{(-1)^{n-1} - L_n^{-2}p^{n-1}A_n(f^{-1})}\right| \leq L_n r^2 \tag{9}$$

and the asymptotically sharp equalities

$$p^{n-1}A_n(f^{-1}) = (-1)^{n-1} + O(r^2) \tag{10}$$

are valid in the range of parameters under discussion.

Analogous computations yield that the domain of variability of $A_n(f^{-1})$, $f \in U_r$, is an annulus for $r < 1/\sqrt{L_n}$.

In Section 4 we derive positive lower bounds that are not dependent on n for $\mathrm{Re}(p^{n-1}a_n(f))$, $f \in U_r$, if $r < (e^{\pi/2} - 1)/(e^{\pi/2} + 1) = 0.656\ldots$. This improves the results of Section 3 concerning the existence of an annulus as domain of variability of $a_n(f)$.

Using these results, we conclude by showing that there is a positive answer to a question posed by Livingston [17].

2. The Case $n = 2$ and Examples for $n \geq 2$

Lewandowski and Zlotkiewicz [15] observed that the set $\Omega_2(r) := \{a_2(f) \mid f \in U_r\}$ is given by Goluzin's inequality,

$$\left| pa_2(f) + 1 - r^2 - 2\frac{E(r)}{K(r)} \right| \leq 2\left(1 - \frac{E(r)}{K(r)}\right)$$

(see [8]), where E and K denote the complete elliptic integrals defined by

$$E(r) := \int_0^{\pi/2} \sqrt{1 - r^2 \sin^2 t}\, dt \quad \text{and} \quad K(r) := \int_0^{\pi/2} \frac{dt}{\sqrt{1 - r^2 \sin^2 t}}.$$

In particular, for $r \in (0, 1)$ we have

$$\max\{|a_2(f)| \mid f \in U_r\} = r + \frac{1}{r}$$

(see also [7] and [14]) and

$$\min\{|a_2(f)| \mid f \in U_r\} = \max\left\{0, \frac{1}{r}\left(r^2 - 3 + 4\frac{E(r)}{K(r)}\right)\right\}.$$

Now let $r_0 = 0.865\ldots$ be defined by

$$r_0^2 - 3 + 4\frac{E(r_0)}{K(r_0)} = 0.$$

From the foregoing we immediately conclude that for $r \in [r_0, 1)$ the set $\Omega_2(r)$ is a disc of radius $r + r^{-1}$, whereas for $r \in (0, r_0)$ the domain of variability $\Omega_2(r)$ is an annulus.

The following examples show that, for any $n \in \mathbf{N}$, there exist a parameter $r \in (0, 1)$ and a function $f_n \in U_r$ such that $a_{n+1}(f_n) = 0$. This implies that the domain of variability of $a_{n+1}(f)$, $f \in U_r$, is a disc in this case. A simple example is given by

$$f_n(z) = \frac{g(r)g(z)}{g(r) - g(z)},$$

where

$$g(z) = \frac{z}{(1 + z^n)^{1/n}(1 - z^{2n})^{1/(2n)}}.$$

The function g is univalent and starlike in D (see e.g. [8]). Computations show that

$$a_{n+1}(f_n) = -\frac{1}{n} + \frac{(1 + r^n)\sqrt{1 - r^{2n}}}{r^n}.$$

There exists a parameter $r = r(n) \in (0, 1)$ for which the expression on the right side of this equation vanishes. In particular, $a_2(f_1) = 0$ for $r = 0.883\ldots$.

The contents of this section show that there is a negative answer to the question of Fang [6] concerning whether

$$|a_{n+1}(f)| \geq (1 - r^2) r^{-n}$$

for all $n \in \mathbf{N}$, all $f \in U_r$, and all $r \in (0, 1)$.

3. Asymptotic Behavior of the Coefficients

THEOREM 1. *For $n \geq 2$, $r \in (0, 1)$, and $f \in U_r$, the estimate*

$$\left| \frac{1 - p^{n-1} a_n(f)}{1 - n^{-2} p^{n-1} a_n(f)} \right| \leq n r^2$$

from (7) *is valid.*

For the proof of Theorem 1 we need the following two lemmas.

LEMMA 1. *For $r \in (0, 1)$, the inequalities*

$$\frac{r^2}{2} \leq 1 - \frac{E(r)}{K(r)} \leq r^2 \tag{11}$$

are valid. These inequalities are asymptotically sharp because

$$\lim_{r \to 0} \frac{1}{r^2} \left(1 - \frac{E(r)}{K(r)} \right) = \frac{1}{2}$$

and

$$\lim_{r \to 1} \frac{1}{r^2} \left(1 - \frac{E(r)}{K(r)} \right) = 1.$$

Proof. A straightforward computation yields

$$X(r) := 1 - \frac{E(r)}{K(r)} = r^2 \frac{\int_0^{\pi/2} \frac{\sin^2 t \, dt}{\sqrt{1 - r^2 \sin^2 t}}}{\int_0^{\pi/2} \frac{dt}{\sqrt{1 - r^2 \sin^2 t}}}.$$

This immediately implies the second inequality in (11).

The first inequality in (11) is equivalent to the assertion

$$\int_0^{\pi/4} \frac{1 - 2\sin^2 t}{\sqrt{1 - r^2 \sin^2 t}} \, dt \leq \int_{\pi/4}^{\pi/2} \frac{2 \sin^2 t - 1}{\sqrt{1 - r^2 \sin^2 t}} \, dt.$$

By the change of variable $t = \pi/2 - \tau$ in the second integral we get the equivalent inequality

$$\int_0^{\pi/4} \frac{1 - 2\sin^2 t}{\sqrt{1 - r^2 \sin^2 t}} \, dt \leq \int_0^{\pi/4} \frac{1 - 2\sin^2 \tau}{\sqrt{1 - r^2 \cos^2 \tau}} \, d\tau,$$

which is valid since $\sin^2 t \leq \cos^2 t$ for $t \in [0, \pi/4]$.

Therefore, it is now clear that

$$\lim_{r \to 0} \frac{X(r)}{r^2} = \frac{1}{2} \quad \text{and} \quad \lim_{r \to 1} \frac{X(r)}{r^2} = 1. \qquad \square$$

LEMMA 2. *For any $n \geq 2$, there exists a positive constant $C(n)$ such that*

$$|p^{n-1}a_n(f) - 1| \leq r^2 C(n), \quad f \in U_r,\ r \in (0,1). \tag{12}$$

Proof. Let $f \in U_r$ and choose $c \in \mathbf{C}$ such that $c \notin f(D)$. Then the function

$$g(z) := \frac{cf(z)}{c - f(z)} = z + \sum_{n=2}^{\infty} a_n(g)z^n, \quad z \in D,$$

is a member of the class S. We have

$$c = -g(p) \quad \text{and} \quad f(z) = \frac{g(z)}{1 + g(z)/c} = \sum_{n=1}^{\infty} \frac{g(z)^n}{g(p)^{n-1}},$$

where the expansion is valid in some neighborhood of the origin. Straightforward computations using the last formula yield

$$a_2(f) = a_2(g) + \frac{1}{g(p)}, \qquad a_3(f) = a_3(g) + \frac{2a_2(g)}{g(p)} + \frac{1}{g(p)^2},$$

and it is easy to see that

$$a_n(f) = \sum_{k=1}^{n} \frac{1}{g(p)^{k-1}} \sum^{*} a_{j_1}(g)a_{j_2}(g)\cdots a_{j_k}(g),$$

where the sum $\sum^*$ ranges over all $(j_1, j_2, \ldots, j_k) \in \mathbf{N}^k$ such that $j_1 + j_2 + \cdots + j_k = n$.

It is evident that

$$a_n(f) = a_n(g) + \frac{P_{n,1}}{g(p)} + \cdots + \frac{P_{n,n-2}}{g(p)^{n-2}} + \frac{1}{g(p)^{n-1}}, \tag{13}$$

where $P_{n,k}$ $(k = 1, \ldots, n-2)$ denotes a polynomial in $a_2(g), \ldots, a_{n-k}(g)$ with coefficients in $\mathbf{N}$. In particular,

$$P_{n,n-2} = (n-1)a_2(g). \tag{14}$$

Using (13) and (14), the inequalities $|a_n(g)| \leq n$, and Koebe's $\frac{1}{4}$-theorem

$$\frac{1}{4} \leq \left|\frac{g(p)}{p}\right| \quad \text{for } g \in S \text{ and } p \in D \setminus \{0\},$$

we obtain the existence of a constant $C_1(n)$ such that

$$\left| p^{n-1}a_n(f) - \frac{p^{n-1}}{g(p)^{n-1}} - \frac{(n-1)a_2(g)p^{n-1}}{g(p)^{n-2}} \right| \leq r^2 C_1(n). \tag{15}$$

Now, we use formula (31) from [8, Chap. IV.3], which may be expressed in our notation as

$$\left| \frac{p}{g(p)} + a_2(g)p + 1 - r^2 - 2\frac{E(r)}{K(r)} \right| \leq 2\left(1 - \frac{E(r)}{K(r)}\right).$$

Together with the inequalities (11) of Lemma 1, this yields

$$\frac{p}{g(p)} + a_2(g)p = 1 + 3r^2\omega, \quad \text{where } |\omega| \le 1. \tag{16}$$

The previous formula for $a_2(f)$ and (16) imply that

$$|pa_2(f) - 1| \le 3r^2.$$

In order to prove Lemma 2 for $n \ge 3$, we evaluate

$$\left(\frac{p}{g(p)} + a_2(g)p\right)^{n-1} = (1 + 3r^2\omega)^{n-1}$$

with the help of the binomial theorem, using $|a_2(g)| \le 2$, $|\omega| \le 1$, and (once again) Koebe's $\frac{1}{4}$-theorem. These computations give the existence of a constant $C_2(n)$ such that

$$\left|\frac{p^{n-1}}{g(p)^{n-1}} + \frac{(n-1)a_2(g)p^{n-1}}{g(p)^{n-2}} - 1\right| \le r^2 C_2(n).$$

This estimate, together with (15), yields (12) for $n \ge 3$ with $C(n) = C_1(n) + C_2(n)$. □

Proof of Theorem 1. For $f \in U_r$ we choose the function $g \in S$ as in the proof of Lemma 2. We recall that

$$f(z) = \frac{g(p)g(z)}{g(p) - g(z)}, \quad z \in D.$$

For this fixed $g \in D$ we consider the functions $f_\zeta \in U_{|\zeta|}$, $\zeta \in D \setminus \{0\}$, defined as

$$f_\zeta(z) = \frac{g(\zeta)g(z)}{g(\zeta) - g(z)}, \quad z \in D.$$

The formula (13) yields that the function

$$\varphi(\zeta) := \zeta^{n-1} a_n(f_\zeta), \quad \zeta \in D \setminus \{0\},$$

is holomorphic in $D \setminus \{0\}$. By Lemma 2, the function $\varphi(\zeta)$ has a holomorphic completion at the origin that we also call φ. Lemma 2 likewise implies that this completion fulfills

$$\varphi(0) = 1 \quad \text{and} \quad \varphi'(0) = 0.$$

From (3) we obtain

$$|\varphi(\zeta)| \le \sum_{k=0}^{n-1} |\zeta|^{2k} \le n, \quad \zeta \in D.$$

Hence, the Schwarz lemma yields

$$\left|\frac{\varphi(\zeta) - 1}{1 - \varphi(\zeta)/n^2}\right| \le n|\zeta|^2, \quad \zeta \in D,$$

which implies the assertion of Theorem 1. □

Using Theorem 1 allows us to derive asymptotic formulas for functionals over U_r, $r \to 0$. In particular, the formulas (8),

$$p^{n-1}a_n(f) = 1 + O(r^2), \quad n \geq 2,$$

are immediate consequences of (12). Another example for this possibility is given by the following.

COROLLARY 1. *For any $n \geq 2$ there exists a positive constant $D(n)$ such that, for all $f \in U_r$, $r \in (0, 1)$, the inequalities* (*cf.* (10))

$$|p^{n-1}A_n(f^{-1}) - (-1)^{n-1}| \leq r^2 D(n)$$

are valid.

Proof. It is known (see e.g. [10, v. 1, p. 56]) that

$$A_2(f^{-1}) = -a_2(f), \qquad A_3(f^{-1}) = -a_3(f) + 2a_2(f)^2,$$

and, in general,

$$A_n(f^{-1}) = \sum_{k=1}^{n-1}(-1)^k \frac{(n+k-1)!}{n!} \sum^{**} \frac{a_2(f)^{j_1} a_3(f)^{j_2} \cdots a_n(f)^{j_{n-1}}}{j_1!\, j_2! \cdots j_{n-1}!}, \tag{17}$$

where the sum $\sum^{**}$ ranges over all $(j_1, j_2, \ldots, j_{n-1}) \in (\mathbf{N} \cup \{0\})^{n-1}$ such that

$$j_1 + j_2 + \cdots + j_{n-1} = k \quad \text{and} \quad j_1 + 2j_2 + \cdots + (n-1)j_{n-1} = n-1.$$

Now, according to (12), we may insert into (17)

$$a_k(f) = (1 + r^2\delta_k(f))p^{1-k},$$

where $|\delta_k(f)| \leq C(k)$. This yields a representation

$$A_n(f^{-1}) = (C_n + r^2 D(n; r, f))p^{1-n},$$

where

$$C_n = \sum_{k=1}^{n-1}(-1)^k \frac{(n+k-1)!}{n!} \sum^{**} \frac{1}{j_1!\, j_2! \cdots j_{n-1}!}.$$

Because of the inequalities $|\delta_k(f)| \leq C(k)$, there exists an uniform upper bound $D(n)$ for $|D(n; r, f)|$ with $r \in (0, 1)$ and $f \in U_r$. On the other hand, for

$$f(z) = \frac{z}{1 - z/p}$$

we have that

$$f^{-1}(w) = \frac{w}{1 + w/p}.$$

In this case, formula (17) becomes

$$(-1)^{n-1} = \sum_{k=1}^{n-1}(-1)^k \frac{(n+k-1)!}{n!} \sum^{**} \frac{1}{j_1!\, j_2! \cdots j_{n-1}!}.$$

Hence, $C_n = (-1)^{n-1}$. This, when combined with our previous results, proves Corollary 1 and so (10) is true. □

Now it is easy to prove (9) using the same technique as in the proof of Theorem 1. In this proof, one need only replace Lemma 2 by Corollary 1 and replace (1) and (3) by (5) and (6), respectively.

REMARK 1. It is easy to prove that the domain of variability of $a_n(f)$, $f \in U_r$, is a disc or an annulus that contains the circle $\{w \mid |w| = r^{1-n}\}$ by using the transform

$$f(z,t) = \begin{cases} \frac{g(tp)g(tz)}{t(g(tp)-g(tz))}, & 0 < |t| \le 1, \\ \frac{z}{(1-z/p)}, & t = 0, \end{cases}$$

where g is fixed by f as before.

4. Lower Bounds for the Coefficients for Little *r*

In Section 2 we derived the existence of a positive lower bound for $|a_2(f)|$ $(f \in U_r)$ if and only if $r < r_0 = 0.865\ldots$, and in Section 3 it was shown that such a positive lower bound for $|a_n(f)|$ $(f \in U_r)$ exists if $r < 1/\sqrt{n}$. We did not mention these bounds explicitly because in the present section we shall prove an improvement of this assertion.

THEOREM 2. *Let $f \in U_r$. Then, for all $n \ge 2$, the inequalities*

$$\operatorname{Re}(p^{n-1}a_n(f)) \ge \begin{cases} \frac{1}{3}, & r \in (0, r_1], \\ \frac{1}{3}\cos\left(\ln\frac{1+r}{1-r}\right), & r \in (r_1, r_2), \end{cases}$$

are valid, where

$$r_1 = \frac{e-1}{e+1} = 0.461\ldots \quad \text{and} \quad r_2 = \frac{e^{\pi/2}-1}{e^{\pi/2}+1} = 0.656\ldots.$$

In the proof of this theorem we only consider functions $f \in U_r$ having their pole in the real point $z = r$. One easily obtains the general case by a rotation.

The proof is based on two lemmas. The first one is a modified version of a classical formula used in the proof of Carathéodory's theorem on functions with positive real part (cf. [2; 5; 21, Chap. V.3]).

LEMMA 3. *Let $f \in U_r$ with its pole in $z = r$, $r \in (0,1)$. Then, for almost all $\theta \in [0, 2\pi)$, the limit*

$$f(e^{i\theta}) = \lim_{R \to 1-0} f(Re^{i\theta})$$

exists and

$$a_n(f) = -\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\left(r^n + \frac{1}{r^n} - 2\cos(n\theta)\right)d\theta. \tag{18}$$

Proof. The existence of the angular limit $f(e^{i\theta})$ for θ-a.e. in $[0, 2\pi)$ is implied by Fatou's theorem (see [8, Chap. IX]), since the function

$$h(z) := \begin{cases} (1 - z^n(r^n + 1/r^n) + z^{2n})f(z), & z \in D \setminus \{r\}, \\ \lim_{0<|w-r|\to 0} h(w), & z = r, \end{cases}$$

is bounded and holomorphic for $n \geq 1$. It is clear that

$$a_n(h) = a_n(f), \quad n \geq 1.$$

As a consequence of a theorem of F. Riesz (see e.g. [8, p. 404]) we get

$$\lim_{R \to 1-0} \int_0^{2\pi} |h(Re^{i\theta}) - h(e^{i\theta})| \, d\theta = 0.$$

This together with the residue theorem yields

$$a_n(h) = \frac{1}{2\pi} \int_0^{2\pi} h(e^{i\theta}) e^{-in\theta} \, d\theta,$$

which is equivalent to the assertion of Lemma 3. □

LEMMA 4. *Let $g \in S$ and $r \in (0,1)$. Then*

$$\operatorname{Re}(g(r)) \geq \begin{cases} \frac{r}{(1+r)^2}, & r \in (0, r_1], \\ \frac{r}{(1+r)^2} \cos\left(\ln \frac{1+r}{1-r}\right), & r \in (r_1, r_2), \end{cases}$$

where

$$r_1 = \frac{e-1}{e+1} = 0.461\ldots \quad \textit{and} \quad r_2 = \frac{e^{\pi/2}-1}{e^{\pi/2}+1} = 0.656\ldots.$$

Proof. Grunsky's inequality (see [8; 11; 13]) implies that, for $g \in S$ and $r \in (0,1)$,

$$\left| \ln \frac{g(r)}{r} + \ln(1-r^2) \right| \leq \ln \frac{1+r}{1-r} =: \alpha.$$

Thus,

$$\frac{g(r)(1-r^2)}{r} \in \Psi(\bar{D}; \alpha), \quad \Psi(z; \alpha) := e^{\alpha z}.$$

If $r \in (0, r_1]$ then $\alpha \in (0, 1]$, hence

$$\operatorname{Re}\left(1 + z \frac{\Psi''(z; \alpha)}{\Psi'(z; \alpha)}\right) = \operatorname{Re}(1 + \alpha z) \geq 0, \quad z \in \bar{D}.$$

Therefore, the function $\Psi(\cdot; \alpha)$ is univalent in D and $\Psi(\bar{D}; \alpha)$ is a convex set. On the other hand, $\Psi(\bar{D}; \alpha)$ is symmetric with respect to the real axis. Hence

$$\min_{|z| \leq 1} \operatorname{Re}(e^{\alpha z}) = \min_{x \in [-1,1]} e^{\alpha x} = e^{-\alpha}$$

for $\alpha \in (0, 1]$, and

$$\operatorname{Re} \frac{g(r)(1-r^2)}{r} \geq e^{-\alpha} = \frac{1-r}{1+r}, \quad r \in (0, r_1],$$

which is equivalent to the first part of the assertion.

If $r \in (r_1, r_2)$ then $\alpha \in (1, \pi/2)$. Hence, $\cos\alpha > 0$ and

$$\min_{|z| \leq 1} \operatorname{Re}(e^{\alpha z}) = \min_{\theta \in [0, 2\pi)} e^{\alpha \cos\theta} \cos(\alpha \sin\theta) > e^{-\alpha} \cos\alpha,$$

which implies the second inequality of Lemma 4. □

REMARK 2. Since Grunsky's inequality describes the exact domain of variability of $g(r)$, $g \in S$, the values r_1 and r_2 in Lemma 4 are best possible in the following sense. For $r \in (0, r_1]$, the lower bound given in Lemma 4 is best possible. In the case $r \in (r_1, r_2)$, we preferred an explicit estimate that is not sharp. It is also clear that, for any $r \in (r_2, 1)$, there exist functions $g \in S$ such that $\operatorname{Re}(g(r)) < 0$.

Proof of Theorem 2. Let $f(e^{i\theta})$ be an angular limit of a function $f \in U_r$ that has its pole in $z = r$. We fix θ and consider the function $g(\cdot;\theta)$ defined by

$$g(z;\theta) = \frac{f(e^{i\theta})f(z)}{f(e^{i\theta}) - f(z)}, \quad z \in D.$$

One has

$$g(r;\theta) = -f(e^{i\theta}),$$

and $g(\cdot;\theta) \in S$ for almost all $\theta \in [0, 2\pi)$ (cf. [13]). Using Lemma 4 yields

$$\operatorname{Re}(f(e^{i\theta})) \le \begin{cases} -\frac{r}{(1+r)^2}, & r \in (0, r_1], \\ -\frac{r}{(1+r)^2}\cos\left(\ln\frac{1+r}{1-r}\right), & r \in (r_1, r_2). \end{cases}$$

These inequalities together with (18) imply

$$\operatorname{Re}(r^{n-1}a_n(f)) \ge \begin{cases} \frac{1+r^{2n}}{(1+r)^2}, & r \in (0, r_1], \\ \frac{1+r^{2n}}{(1+r)^2}\cos\left(\ln\frac{1+r}{1-r}\right), & r \in (r_1, r_2). \end{cases}$$

This completes the proof of Theorem 2, since

$$\frac{1+r^{2n}}{(1+r)^2} > \frac{1}{(1+r)^2} > \frac{1}{3} \quad \text{for } r \in (0, r_2). \qquad \square$$

REMARK 3. Let $f \in U_r$ with its pole in the real point $z = r$, $r \in (0,1)$. Using the sharp estimate of Komatu (see [14]),

$$|\operatorname{res}(f, z = r)| \ge r^2(1 - r^2),$$

and the relation

$$\lim_{n\to\infty}(r^{n+1}a_n(f)) = -\operatorname{res}(f, z = r),$$

which follows from Lemma 3, we get a theorem proved by Fang [6]—namely,

$$\lim_{n\to\infty}(r^{n-1}|a_n(f)|) \ge 1 - r^2.$$

Here, equality is attained if and only if

$$f(z) = \frac{z(1 - rz)}{1 - z/r}.$$

Hence, for any $f \in U_r$, $r \in (0,1)$, there exists a natural number $n(f)$ such that $a_n(f) \ne 0$ for all $n > n(f)$. If $r < r_2$, then $n(f) = 1$ by Theorem 2. It is not clear whether the quantity

$$n_r := \sup\{n(f) \mid f \in U_r\}$$

is finite for $r \in [r_2, 1)$.

REMARK 4. Livingston [17] investigated the set Λ_r^* of weakly starlike meromorphic functions. The set Λ_r^* is related to the class U_r as follows. Any function $F \in \Lambda_r^*$ has an expansion

$$F(z) = 1 + \sum_{n=1}^{\infty} a_n(F) z^n, \quad |z| < r,$$

and the function

$$f(z) := \frac{F(z) - 1}{a_1(F)}$$

is a member of the class U_r. In [17] Livingston posed the question of whether there is a positive lower bound for $|a_n(F)|,\ F \in \Lambda_r^*$.

The estimate

$$|a_1(F)| \geq \frac{(1-r)^2}{r}, \quad F \in \Lambda_r^*,$$

proved in [16] implies that

$$|a_n(F)| \geq \frac{(1-r)^2}{r} |a_n(f)|, \quad F \in \Lambda_r^*,\ \ f \in U_r,$$

for $n \geq 2$. Hence, Theorem 2 implies that Livingston's question has a positive answer for $r < r_2$ and $n \geq 2$.

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