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Rational Approximation on the Unit Sphere in $\mathbf{C}^2$

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1. Introduction

For a compact set $X \subset \mathbf{C}^n$, we denote by $R(X)$ the closure in $C(X)$ of the set of rational functions holomorphic in a neighborhood of X. We are interested in finding conditions on X which imply that $R(X) = C(X)$, that is, conditions implying that each continuous function on X is the uniform limit of a sequence of rational functions holomorphic in a neighborhood of X.

When $n = 1$, the theory of rational approximation is well developed. Examples of sets without interior for which $R(X) \neq C(X)$ are well known, the "Swiss cheese" being a prime example. On the other hand, the Hartogs–Rosenthal theorem states that if the two-dimensional Lebesgue measure of X is zero, then $R(X) = C(X)$.

In higher dimensions, there is an obstruction to rational approximation that does not appear in the plane. For $X \subset \mathbf{C}^n$, we denote by $\hat{X}_r$ the rationally convex hull of X, which can be defined as the set of points $z \in \mathbf{C}^n$ such that every polynomial Q with $Q(z) = 0$ vanishes at some point of X. The condition $X = \hat{X}_r$ (X is rationally convex) is both necessary for rational approximation and difficult to establish, in practice, when $n > 1$; in the plane, every compact set is rationally convex.

We will consider primarily subsets of the unit sphere ∂B in $\mathbf{C}^2$. We have been motivated by a desire to obtain an analogue of the Hartogs–Rosenthal theorem in this setting. Basener [5] has given examples of rationally convex sets $X \subset \partial B$ for which $R(X) \neq C(X)$; his examples have the form $\{(z, w) \in \partial B : z \in E\}$, where $E \subset \mathbf{C}$ is a suitable Swiss cheese. These sets have the property that $\sigma(X) > 0$, where σ is three-dimensional Hausdorff measure on ∂B. It is reasonable to conjecture that if X is rationally convex and $\sigma(X) = 0$, then $R(X) = C(X)$. This paper contains several contributions to the study of this question.

In Section 2 we employ a construction of Henkin [10]. For a measure μ supported on ∂B orthogonal to polynomials, Henkin produced a function $K_\mu \in L^1(d\sigma)$ satisfying $\bar\partial_b K_\mu = -4\pi^2\mu$. Lee and Wermer established that, if $X \subset \partial B$ is rationally convex and if $\mu \in R(X)^\perp$ (i.e., $\int g\, d\mu = 0$ for all $g \in R(X)$), then K_μ extends holomorphically to the unit ball. We show that, if the extension belongs

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to the Hardy space $H^1(B)$, then μ must be the zero measure. Under an assumption on the size of the rational hull of small tubular neighborhoods of X, which we call the *hull-neighborhood property,* we are able to show that K_μ satisfies a certain boundedness condition (see Lemma 2.4). From this we deduce (in the proof of Theorem 2.5) that $K_\mu \in H^1(B)$ if X is a subset of a Lipschitz graph lying in ∂B. Hence, in this case the only measure $\mu \in R(X)^\perp$ is the zero measure, so $R(X) = C(X)$. (In Section 4 we show how the same result can be established for graphs of Hölder functions.) Section 2 also includes an example of a class of sets satisfying the hull-neighborhood property.

In Section 3 we study the algebra generated by $R(E)$ and a smooth function f on a plane set E; we show that, if this algebra has maximal ideal space E but does not contain all continuous functions on E, then there is a subset E_0 of E on which $f \in R(E_0)$ and $R(E_0) \neq C(E_0)$. We then use this result to establish rational approximation on certain graphs lying in ∂B.

We use the following notation in addition to that already introduced: B will denote the unit ball in $\mathbf{C}^2$; coordinates of points in $\mathbf{C}^2$ will be denoted either by using subscripts such as $z = (z_1, z_2)$ or by $p = (z, w)$, according to the context. We use π to denote projection to the first coordinate; that is, $\pi(z, w) = z$. If z, ζ are points in $\mathbf{C}^2$, then $\langle z, \zeta\rangle$ will denote the usual Hermitian inner product of z and ζ.

2. Rational Approximation and the Henkin Transform

A basic tool of approximation theory in the plane is the Cauchy transform $\hat{\mu}$ of a measure μ. If μ is a finite complex measure with compact support, then

$$\hat{\mu}(z) = \int \frac{d\mu(\zeta)}{\zeta - z}.$$

The Cauchy transform $\hat{\mu}(z)$ is integrable with respect to Lebesgue measure m on the plane, is analytic in z off the support of μ, and satisfies the fundamental relation

$$\frac{\partial \hat{\mu}}{\partial \bar{z}} = -\pi\mu$$

in the sense of distributions; that is,

$$\int \phi \, d\mu = \frac{1}{\pi} \int_{\mathbf{C}} \frac{\partial \phi}{\partial \bar{z}} \hat{\mu} \, dm. \tag{1}$$

In [10], Henkin studied global solutions to the inhomogeneous tangential Cauchy–Riemann equations on the boundary of strictly convex domains in $\mathbf{C}^n$. His work produced transforms analogous in certain respects to the Cauchy transform. In the particular case that concerns us, the boundary of the unit ball in $\mathbf{C}^2$, Henkin introduced the kernel

$$H(z, \zeta) = \frac{\langle Tz, \zeta\rangle}{|1 - \langle z, \zeta\rangle|^2},$$

where $Tz = (\overline{z_2}, -\overline{z_1})$. Given a measure μ supported on a set $X \subset \partial B$, the Henkin transform of μ is defined by

$$K_\mu(z) = \int_X H(z, \zeta)\, d\mu(\zeta).$$

Henkin showed that the integral defining K_μ converges σ-a.e. on ∂B and that K_μ is integrable with respect to $d\sigma$ on ∂B and is smooth on $\partial B \setminus X$. Further, if μ satisfies the condition

$$\int_X P\, d\mu = 0 \quad \text{for all polynomials } P, \tag{2}$$

then K_μ satisfies

$$\bar\partial_b K_\mu = -4\pi^2 \mu. \tag{3}$$

Here $\bar\partial_b$ is the tangential Cauchy–Riemann operator on ∂B. Equation (3) means that

$$\int \phi\, d\mu = \frac{1}{4\pi^2} \int_{\partial B} K_\mu\, \bar\partial\phi \wedge \omega \tag{4}$$

for all functions ϕ smooth in a neighborhood of ∂B, where $\omega(z) = dz_1 \wedge dz_2$. An elementary proof of (4) is presented in Lee's thesis [14]; Varopoulos [19, Sec. 3.2] has also given an exposition of Henkin's results for the case of the ball.

Note that the condition (2) that μ be orthogonal to polynomials (satified by all $\mu \in R(X)^\perp$) is necessary for the solution of (3), and observe that (3) implies K_μ is a CR function on $\partial B \setminus X$. Lee and Wermer [15] proved that, if X is rationally convex, then K_μ extends holomorphically from $\partial B \setminus X$ to B for any $\mu \in R(X)^\perp$. We may state this as follows.

THEOREM 2.1. *Suppose X is a rationally convex subset of ∂B. Let μ be a measure on X such that $\mu \in R(X)^\perp$, and let K_μ be its Henkin transform. Then there exists a function k_μ, holomorphic in a neighborhood of $\bar B \setminus X$, with $k_\mu = K_\mu$ on $\partial B \setminus X$.*

We let $H^1(B)$ denote the Hardy space of functions g holomorphic on B satisfying

$$\sup\left\{ \int_{\partial B} |g^{(r)}|\, d\sigma : r < 1 \right\} < \infty,$$

where $g^{(r)}(z) \equiv g(rz)$ for $z \in \partial B$. For $g \in H^1(B)$, $\lim_{r\to 1} g^{(r)} \equiv g^*$ exists σ-a.e. on ∂B, and $g^{(r)} \to g^*$ as $r \to 1$ in $L^1(d\sigma)$.

LEMMA 2.2. *Let X be a rationally convex subset of ∂B with $\sigma(X) = 0$. Let μ be a measure on X with $\mu \perp R(X)$, and let k_μ be the holomorphic extension of K_μ to B (as in Theorem 2.1). If $k_\mu \in H^1(B)$, then μ is the zero measure.*

Proof. It suffices to show that $\int \phi\, d\mu = 0$ for every function $\phi \in C^1(\mathbf{C}^2)$. Note that $\sigma(X) = 0$ implies that $k_\mu^* = K_\mu$ at σ-almost all points of ∂B, and so by (4) we have

$$\int_X \phi\, d\mu = \frac{1}{4\pi^2} \int_{\partial B} k_\mu^*\, \bar\partial\phi \wedge \omega = \lim_{r\to 1} \frac{1}{4\pi^2} \int_{\partial B} k_\mu^{(r)}\, \bar\partial\phi \wedge \omega.$$

By Stokes's theorem,

$$\int_{\partial B} k_\mu^{(r)}\,\bar\partial\phi\wedge\omega = \int_B \bar\partial(k_\mu^{(r)}\,\bar\partial\phi\wedge\omega) = \int_B \bar\partial(k_\mu^{(r)})\wedge\bar\partial\phi\wedge\omega = 0$$

for fixed r, since $k_\mu^{(r)}$ is holomorphic in B. This shows that $\int\phi\,d\mu = 0$ for all $\phi\in C^1(\mathbf{C}^2)$ and completes the proof. $\square$

Thus, to prove that $R(X) = C(X)$ for a rationally convex subset of ∂B with $\sigma(X) = 0$, it suffices to show that $k_\mu \in H^1(B)$ for every $\mu \perp R(X)$. We will use this approach to establish rational approximation on certain subsets of ∂B. It should be noted that the condition $\sigma(X) = 0$ is necessary in the preceding lemma. If X is the rationally convex set constructed by Basener, then $R(X) \neq C(X)$ and there exist nonzero measures $\mu \in R(X)^\perp$ for which $k_\mu \in H^1(B)$ [3].

We begin with a general estimate on the Henkin transform.

LEMMA 2.3. *If $X\subset\partial B$, μ is a measure supported on X, and $z\in\partial B$, then*

$$|K_\mu(z)| \le \frac{4\|\mu\|}{\operatorname{dist}^4(z,X)}. \tag{5}$$

Proof. For any $\zeta, z\in\partial B$,

$$|z-\zeta|^2 = |z|^2+|\zeta|^2-2\operatorname{Re}(\langle z,\zeta\rangle) = 2\operatorname{Re}(1-\langle z,\zeta\rangle) \le 2|1-\langle z,\zeta\rangle|;$$

thus, for $\zeta\in X$ and $z\in\partial B$ we have

$$\operatorname{dist}^2(z,X) \le 2|1-\langle z,\zeta\rangle|. \tag{6}$$

From this we obtain an estimate on Henkin's kernel H: for $z\in\partial B$ and $\zeta\in X$,

$$|H(z,\zeta)| = \frac{|\langle Tz,\zeta\rangle|}{|1-\langle z,\zeta\rangle|^2} \le \frac{4|Tz||\zeta|}{\operatorname{dist}^4(z,X)} = \frac{4}{\operatorname{dist}^4(z,X)},$$

from which (5) follows immediately by the definition of K_μ. $\square$

We would like to establish an estimate similar to (5) for the holomorphic extension k_μ of K_μ to B given by Theorem 2.1 for rationally convex X. We shall do this for the class of sets satisfying the following strong notion of convexity with respect to rational functions.

DEFINITION. Given $X\subset\mathbf{C}^2$, let $X_\varepsilon = \{z\in\mathbf{C}^n : \operatorname{dist}(z,X) < \varepsilon\}$. We say that X has the *hull-neighborhood* property (abbreviated (H-N)) if there exists $k>0$ such that, if we put $E=\pi(X)$, then for all $\varepsilon>0$ we have

$$[X_\varepsilon]\hat{}_r \cap \pi^{-1}(E) \subset X_{k\varepsilon}. \tag{7}$$

In other words, given $z\in\mathbf{C}^2$ with $\pi(z)\in\pi(X)$ and $\varepsilon>0$ so that $\operatorname{dist}(z,X) > k\varepsilon$, there exists a polynomial Q with $Q(z)=0$ whose zero set does not meet X_ε. Since $\pi(\hat X_r) = \pi(X)$, it is clear that if X has property (H-N) then X is rationally convex. Also, for $X\subset\partial B$, $[X_\varepsilon]\hat{}_r$ is contained in the ball of radius $1+\varepsilon$ centered at the origin, so $[X_\varepsilon]\hat{}_r \subset X_{2+\varepsilon}$. Hence, for $X\subset\partial B$, there exists $k>0$ such that (7) holds for all $\varepsilon>0$ if and only if there exists $k>0$ such that (7) holds for all sufficiently small ε.

LEMMA 2.4. *Assume $X \subset \partial B$ has property* (H-N). *Then there exists a constant c such that, for all $p \in B$ with $\pi(p) \in \pi(X)$ and all $\mu \in R(X)^\perp$, we have*

$$|k_\mu(p)| \le \frac{c\|\mu\|}{\operatorname{dist}^4(p, X)}. \tag{8}$$

Proof. Fix $p \in B$ and set $\delta = \operatorname{dist}(p, X)$. If $\varepsilon > 0$ satisfies $k\varepsilon < \delta$ then, by hypothesis $p \notin [X_\varepsilon]\hat{}_r$. Hence there exists a polynomial Q with $Q(p) = 0$ such that the zero set V of Q does not meet X_ε. Note that k_μ is continuous on $V \cap \bar{B}$ with boundary values K_μ on $V \cap \partial B$. By the maximum principle, $|k_\mu|$ attains its maximum on $V \cap \bar{B}$ at a point $p_0 \in \partial B \cap V$ and so, by Lemma 2.3,

$$|k_\mu(p)| \le |K_\mu(p_0)| \le \frac{4\|\mu\|}{\operatorname{dist}^4(p_0, X)} \le \frac{4\|\mu\|}{\varepsilon^4}.$$

Since the preceding inequality holds whenever $k\varepsilon < \delta$, we obtain (8). □

Let $\triangle$ denote the closed unit disk in the complex plane. For a function defined on $\triangle$, we let $\Gamma(f) \subset \mathbf{C}^2$ denote the graph of f over $\triangle$. $\operatorname{Lip}(\triangle)$ will denote the set of Lipschitz functions on $\triangle$—that is, those functions f for which there exists a constant $M > 0$ such that $|f(z) - f(z')| \le M|z - z'|$ for all $z, z' \in \triangle$; the least such M we call the *Lipschitz constant* for f. The main result of this section is the following approximation theorem for subsets of Lipschitz graphs with the hull-neighborhood property.

THEOREM 2.5. *Let $f \in \operatorname{Lip}(\triangle)$, and assume $\Gamma(f) \subset \partial B$. If $X \subset \Gamma(f)$ has property* (H-N), *then $R(X) = C(X)$.*

Proof. We will show that, under the hypotheses of Theorem 2.5, $k_\mu \in H^1(B)$ for each $\mu \in R(X)^\perp$. By Lemma 2.2, since $\sigma(\Gamma(f)) = 0$ this will imply that every measure in $R(X)^\perp$ is identically zero and hence $R(X) = C(X)$. Fix $\mu \in R(X)^\perp$ and write $k = k_\mu$. Let (z, w) denote the coordinates in $\mathbf{C}^2$. We show that $k \in H^1(B)$ by estimating k on the slices $z =$ constant. To do this, we first introduce some notation and prove a lemma.

For $z \in \triangle$, let $D_z = \left\{w : |w| < \sqrt{1 - |z|^2}\right\}$ and let γ_z be the boundary of D_z. If g is a function holomorphic in B and $z \in \triangle$, we let g_z denote the slice function $g_z(w) = g(z, w)$ with $w \in D_z$. If for some $s > 0$ we have $g_z \in H^s(D_z)$, that is,

$$\sup\left\{\int_0^{2\pi} \left|g_z\big(r\sqrt{1 - |z|^2}e^{i\theta}\big)\right|^s d\theta : 0 < r < 1\right\} < \infty, \tag{9}$$

then $g_z^*(w) = \lim_{r\to 1} g_z(rw)$ exists for almost all $w \in \gamma_z$. If also $g_z^*(w) \in L^1$ with respect to linear measure on γ_z, then in fact $g_z \in H^1(D_z)$ (see [8, Thm. 2.11]) and $\int_0^{2\pi} \left|g\big(z, r\sqrt{1 - |z|^2}e^{i\theta}\big)\right| d\theta$ is increasing in r.

LEMMA 2.6. *Let X be a subset of ∂B with $\sigma(X) = 0$. Suppose g is holomorphic in a neighborhood of $\bar{B} \setminus X$, $g|_{\partial B} \in L^1(d\sigma)$, and, for some $s > 0$, $g_z \in H^s(D_z)$ for almost all $z \in \triangle$. Then $g \in H^1(B)$.*

Proof. First note that if f is any positive function defined (σ-a.e.) on ∂B, then (see [17, Prop. 1.47])

$$\int_{\partial B} f\,d\sigma = \int_{\triangle} dm(z) \int_0^{2\pi} f_z\bigl(\sqrt{1-|z|^2}e^{i\phi}\bigr)\,d\phi. \tag{10}$$

Set $G = g|_{\partial B}$. The hypotheses imply, for m-almost all $z \in \triangle$, that $G|_{\gamma_z} = g_z^*$ is defined almost everywhere and is integrable with respect to linear measure on γ_z and that $g_z \in H^1(D_z)$. Thus, if $r < 1$ then by (10) we have

$$\begin{aligned}\int_{\partial B} |g^{(r)}|\,d\sigma &= \int_{\triangle} dm(z) \int_0^{2\pi} \bigl|g_{rz}\bigl(r\sqrt{1-|z|^2}e^{i\phi}\bigr)\bigr|\,d\phi \\ &\le \int_{\triangle} dm(z) \int_0^{2\pi} \bigl|g_{rz}^*\bigl(\sqrt{1-|rz|^2}e^{i\phi}\bigr)\bigr|\,d\phi.\end{aligned}$$

The change of variables $z' = rz$ converts the last integral above to

$$\frac{1}{r^2}\int_{|z'|\le r} dm(z') \int_0^{2\pi} \bigl|G\bigl(z', \sqrt{1-|z'|^2}e^{i\phi}\bigr)\bigr|\,d\phi \le \frac{1}{r^2}\int_{\partial B} |G|\,d\sigma,$$

again by (10). Since $G \in L^1(d\sigma)$, we find that $\int_{\partial B} |g^{(r)}|\,d\sigma$ is bounded independently of r, so $g \in H^1(B)$. $\square$

By Lemma 2.6, the proof of Theorem 2.5 will be complete if we can show that, for some $s > 0$, $k_z \in H^s(D_z)$ for almost all $z \in \triangle$. Fix $z \in \triangle$. We may assume $z \in \pi(X)$, for if $z \notin \pi(X)$ then k_z is holomorphic in a neighborhood of the closure of D_z and there is nothing to prove. If $p = (z, w)$ with $w \in D_z$, then for any $p' = (z', f(z'))$ we have

$$\begin{aligned}|w - f(z)| &\le |w - f(z')| + |f(z') - f(z)| \\ &\le |w - f(z')| + M|z - z'| \\ &\le \sqrt{M^2+1}\,|p - p'|\end{aligned}$$

by the Cauchy–Schwarz inequality, so

$$|w - f(z)| \le \sqrt{M^2+1}\,\operatorname{dist}(p, X). \tag{11}$$

By Lemma 2.4, it follows that

$$|k(p)| \le \frac{C}{\operatorname{dist}^4(p, X)} \le \frac{C'}{|w - f(z)|^4} \tag{12}$$

for some constant C'. Write $f(z) = \sqrt{1-|z|^2}e^{i\phi}$. Then, using (12), for $r < 1$ we obtain

$$\begin{aligned}\int_0^{2\pi} \bigl|k_z\bigl(r\sqrt{1-|z|^2}e^{i\theta}\bigr)\bigr|^{1/8}\,d\theta &\le \frac{(C')^{1/8}}{(1-|z|^2)^{1/4}} \int_0^{2\pi} \frac{1}{|re^{i\theta} - e^{i\phi}|^{1/2}}\,d\theta \\ &= C'' \int_0^{2\pi} \frac{1}{|re^{i\theta} - 1|^{1/2}}\,d\theta.\end{aligned}$$

For $|\theta| \le \pi/3$ we have $\cos(\theta) \le 1 - \theta^2/4$, which implies

$$|1 - re^{i\theta}|^{1/2} = [1 + r^2 - 2r\cos(\theta)]^{1/4} \ge [(1-r)^2 + r\theta^2/2]^{1/4} \ge (r/2)^{1/4}\sqrt{\theta}.$$

It follows from this that the last integral is bounded independently of r, for r near 1, and so $k \in H^{1/8}(D_z)$ for all $z \in \triangle$. This completes the proof. $\square$

REMARK. The special case of Theorem 2.5 when f is continuously differentiable on $\triangle$ can also be obtained as a direct consequence of Theorem 4.3 of [4].

We close this section by exhibiting a class of sets with the hull-neighborhood property. Recall that a real submanifold of $\mathbf{C}^n$ is said to be *totally real* if, at each point, its tangent space contains no complex line.

THEOREM 2.7. *Let $f \in C^\infty(\triangle)$, and assume $\Gamma(f)$ is a totally real submanifold of $\mathbf{C}^2$. If X is a compact polynomially convex subset of $\Gamma(f)$, then X has property* (H-N).

Proof. For $p \in \mathbf{C}^2$, let $\delta(p) = \operatorname{dist}(p, \Gamma(f))$. Since $\Gamma(f)$ is totally real, a result of Hörmander and Wermer ([12], or see [1, Lemma 17.2]) implies that there is a neighborhood U of X in $\mathbf{C}^2$ such that δ^2 is strictly plurisubharmonic on U.

Since X is polynomially convex, there exists a compact polynomial polyhedron Π, $X \subset \Pi \subset U$, where $\Pi = \{|P_j| \le 1,\ j = 1, \dots, k\}$ with each P_j a polynomial. We may assume that $|P_j| \le 1/2$ on X for each j, and that the coordinate functions are contained among the P_j. Define a function Ψ on $\mathbf{C}^2$ by

$$\Psi = \max\{|P_1|, \dots, |P_k|\} - \tfrac{3}{4}.$$

Then $\Psi = 1/4$ on $\partial\Pi$ and $\Psi < 0$ on X.

Choose $\varepsilon_0 > 0$ so small that $\Psi < 0$ on X_{ε_0}. We will show that whenever $p \in \mathbf{C}^2$ satisfies $\pi(p) \in \pi(X)$ and $\operatorname{dist}(p, X) > \sqrt{M^2+1}\,\varepsilon$ for some $\varepsilon < \varepsilon_0$, where M is the Lipschitz constant for f, then there is a polynomial Q with $Q(p) = 0$ whose zero set does not meet X_ε. By the remarks following the definition of (H-N), this will complete the proof.

Choose a constant $\kappa > 0$ so that $\kappa\delta^2(p) < 1/4$ for all $p \in \partial\Pi$. Then, on a neighborhood N of $\partial\Pi$, we have $\kappa\delta^2 < \Psi$. Define

$$F = \begin{cases} \max(\Psi, \kappa\delta^2) & \text{on } \Pi \cup N, \\ \Psi & \text{on } \mathbf{C}^2 \setminus \Pi. \end{cases}$$

Then F is well-defined and plurisubharmonic on $\mathbf{C}^2$. For $\varepsilon < \varepsilon_0$, set

$$\Lambda_\varepsilon = \{q \in \mathbf{C}^2 : F(q) \le \kappa\varepsilon^2\}.$$

Then Λ_ε is compact and $X_\varepsilon \subset \Lambda_\varepsilon$, for if $\operatorname{dist}(q, X) < \varepsilon$ then $\Psi(q) < 0$; hence

$$F(q) = \kappa\delta^2(q) \le \kappa \operatorname{dist}^2(q, X) < \kappa\varepsilon^2,$$

implying $q \in \Lambda_\varepsilon$. Also, since F is plurisubharmonic, Λ_ε is polynomially convex (this follows from [11, Thm. 4.3.4]). Suppose p satisfies $\operatorname{dist}(p, X) > \sqrt{M^2+1}\,\varepsilon$. We distinguish two cases: either (i) $F(p) = \kappa\delta^2(p)$ or (ii) $F(p) = \Psi(p)$. In the first case, we find (as in the proof of Theorem 2.5) that if we write p in coordinates as $p = (z, w)$ then $|w - f(z)| \le \sqrt{M^2 + 1}|p - p'|$ whenever $p' \in \Gamma(f)$, implying $\operatorname{dist}(p, X) \le \sqrt{M^2 + 1}\delta(p)$, and so

$$F(p) \ge \frac{\kappa \operatorname{dist}^2(p, X)}{M^2 + 1} > \kappa\varepsilon^2$$

and thus $p \notin \Lambda_\varepsilon$. By the polynomial convexity of Λ_ε, there exists a polynomial Q, nonvanishing on Λ_ε, with $Q(p) = 0$; since $X_\varepsilon \subset \Lambda_\varepsilon$, Q does not vanish on X_ε. In the second case, we must have $\Psi(p) > 0$ and so $|P_j(p)| > 3/4$ for some j. Set $Q = P_j - P_j(p)$. Then $Q(p) = 0$, but since $\Psi < 0$ on X_ε it follows that $|P_j| < 3/4$ on X_ε, so Q cannot vanish on X_ε. In both cases, we have found the required polynomial Q, and the proof is complete. □

Finally we note that the approach in this section is related to the problem of determining when X is a removable singularity for integrable CR functions. In this context, we may say that X is *removable* for L^1 CR functions if X has the property that, whenever $g \in L^1(d\sigma)$ and $\bar{\partial}_b g = 0$ off X, g extends to a function in $H^1(B)$ (see [2]). By (3), $\bar{\partial}_b K_\mu = 0$ off X whenever $\mu \in R(X)^\perp$ and so, by the remarks following Lemma 2.2, $R(X) = C(X)$ for any subset of ∂B with $\sigma(X) = 0$ that is removable for L^1 CR functions. An extensive bibliography on this question and a survey of recent results is contained in [16].

3. The Algebra Generated by $R(E)$ and a Smooth Function

In this section we study the algebra generated by $R(E)$ and a smooth function on a planar set E. We then apply our results to the question of rational approximation on certain subsets of ∂B.

If $\mathcal{A}$ is a uniform algebra on a compact space X, we write $\mathcal{M}(\mathcal{A})$ for its maximal ideal space and view elements of $\mathcal{M}(\mathcal{A})$ as homomorphisms $m: \mathcal{A} \to \mathbf{C}$. We will identify each point $x \in X$ with the point evaluation $m_x \in \mathcal{M}(\mathcal{A})$ defined by $m_x(h) = h(x)$. If $\mathcal{A} = R(X)$ for some compact subset $X \subset \mathbf{C}^n$, then $\mathcal{M}(\mathcal{A})$ can be identified with $\hat{X}_r$ via $m \in \mathcal{M}(\mathcal{A}) \to (m(z_1), \dots, m(z_n))$, where $(z_1, \dots, z_n)$ are the coordinate functions. This correspondence is a homeomorphism.

If $\mathcal{F}$ is a family of continuous functions on a compact space X, then $[\mathcal{F}]$ will denote the algebra generated by $\mathcal{F}$, that is, the smallest closed subalgebra of $C(X)$ containing $\mathcal{F}$. In [20], Wermer studied the algebra $\mathcal{A} = [z, f]$ on $\triangle$ generated by the identity function z and a smooth function f. Under the assumption that $\mathcal{M}(\mathcal{A}) = \triangle$, he showed that $\mathcal{A}$ consists of those continuous functions on $\triangle$ whose restrictions to the zero set E of $\partial f/\partial\bar{z}$ lie in $R(E)$. We will make use of the following generalization of Wermer's result due to Anderson and Izzo [4, Thm. 4.2].

LEMMA 3.1. *Let $\mathcal{G}$ be a collection of continuously differentiable functions on $\triangle$, and set $\mathcal{A} = [\mathcal{G}]$. Assume the function z lies in $\mathcal{A}$ and that $\mathcal{M}(\mathcal{A}) = \triangle$. Set $T = \left\{\zeta \in \triangle : \frac{\partial g}{\partial\bar{z}}(\zeta) = 0 \ \forall g \in \mathcal{G}\right\}$. Then $\mathcal{A} = \{g \in C(\triangle) : g|_T \in R(T)\}$.*

In order to pass from algebras on compact subsets of the disk to algebras on the disk, we will need two results on extension algebras. The first is due to Bear [6].

LEMMA 3.2. *Let $\mathcal{A}_0$ be a uniform algebra on a compact subset X_0 of a compact space X. Put $\mathcal{A} = \{h \in C(X) : h|_{X_0} \in \mathcal{A}_0\}$. If $\mathcal{M}(\mathcal{A}_0) = X_0$, then $\mathcal{M}(\mathcal{A}) = X$.*

LEMMA 3.3. *Let $\mathcal{A}$, $\mathcal{A}_0$, X, and X_0 be as in Lemma 3.2. Assume $\mathcal{G}_0$ is a subset of $C(X_0)$ with $[\mathcal{G}_0] = \mathcal{A}_0$. Let $\mathcal{G} \subset C(X)$, and assume that* (i) *$[\mathcal{G}]$ contains all continuous functions on X vanishing in a neighborhood of X_0 and* (ii) *$\mathcal{G}|_{X_0} = \mathcal{G}_0$. Then $[\mathcal{G}] = \mathcal{A}$.*

Proof. Clearly $\mathcal{G} \subset \mathcal{A}$ and so it suffices to show, given $h \in \mathcal{A}$, that $\int h\,d\mu = 0$ for all measures $\mu \in [\mathcal{G}]^{\perp}$. For any such measure the hypothesis that $[\mathcal{G}]$ contains all continuous functions vanishing near X_0 implies $\operatorname{supp}(\mu) \subset X_0$. Since $h|_{X_0} \in \mathcal{A}_0$, we may choose a sequence h_j of polynomials in elements of $\mathcal{G}_0$ converging to h on X_0. By hypothesis (ii), we may assume each h_j is the restriction to X_0 of an element of $[\mathcal{G}]$. Then

$$\int_X h\,d\mu = \int_{X_0} h\,d\mu = \lim_{j\to\infty} \int_{X_0} h_j\,d\mu = 0$$

since $\mu \in [\mathcal{G}]^{\perp}$. □

Given a compact $E \subset \mathbf{C}$, we write $f \in C^1(E)$ if f is the restriction to E of a function that is continuously differentiable in some neighborhood of E.

THEOREM 3.4. *Let E be a compact subset of $\mathbf{C}$, and take $f \in C^1(E)$. Assume that $\mathcal{M}([R(E), f]) = E$. If $[R(E), f] \neq C(E)$, then there exists a compact subset E_0 of E such that $R(E_0) \neq C(E_0)$ and $f|_{E_0} \in R(E_0)$.*

Proof. Let E and f satisfy the hypotheses of the theorem. Without loss of generality, E is a compact subset of the open unit disk. Set $\mathcal{A} = \{h \in C(\triangle) : h|_E \in [R(E), f]\}$. Since $\mathcal{M}([R(E), f]) = E$ by hypothesis, Lemma 3.2 implies that $\mathcal{M}(\mathcal{A}) = \triangle$. Fix any smooth extension of f to $\triangle$ (we denote the extension by f, also). Since $R(E)$ is generated by the set of functions holomorphic in a neighborhood of E, Lemma 3.3 implies that $\mathcal{A}$ is generated by the set $\mathcal{G}$ consisting of f together with all functions smooth on $\triangle$ and holomorphic in a neighborhood of E. Set $E_0 = \left\{\zeta \in \triangle : \frac{\partial g}{\partial \bar z}(\zeta) = 0 \ \forall g \in \mathcal{G}\right\}$. Clearly $E_0 \subset E$. By Lemma 3.1, $\mathcal{A} = \{h \in C(\triangle) : h|_{E_0} \in R(E_0)\}$. Since $f \in \mathcal{A}$, $f|_{E_0} \in R(E_0)$. If $R(E_0) = C(E_0)$, then $\mathcal{A} = C(\triangle)$ and hence $[R(E), f] = C(E)$, contrary to hypothesis. □

As mentioned in the introduction, Basener gave examples of rationally convex subsets X of ∂B with $R(X) \neq C(X)$. To explain Basener's construction, we recall the notion of a Jensen measure. Given a uniform algebra $\mathcal{A}$ on X, a probability measure σ on X is said to be a *Jensen measure for $m \in \mathcal{M}(\mathcal{A})$* if, for every $h \in \mathcal{A}$,

$$\log|m(h)| \le \int_X \log|h|\,d\sigma.$$

If m is point evaluation at some $p_0 \in X$, then the point mass δ_{p_0} at p_0 is trivially a Jensen measure for m. Every Jensen measure σ for m represents m: $m(h) = \int h\,d\sigma$ for all $h \in \mathcal{A}$. Basener's assumption for $X \subset \partial B$ was the following condition on $E = \pi(X)$:

(B) For all $z_0 \in E$, the only Jensen measure for z_0 relative to $R(E)$ is δ_{z_0}.

It can be shown (see [7, Thm. 3.4.11]) that (B) is equivalent to the condition that the set of functions harmonic in a neighborhood of E be dense in $C(E)$. Examples of sets $E \subset \mathbf{C}$ satisfying (B) for which $R(E) \neq C(E)$ can be found in [7, pp. 192–195] and [18, Sec. 27].

Basener showed that if $X \subset \partial B$ has the form $X = \{(z, w) \in \partial B : z \in E\}$, where E is a compact subset of the open unit disk satisfying (B), then X is rationally convex; in fact, his proof shows (see also [18, Sec. 19.8]) that the same is true for any $X \subset \partial B$ for which $\pi(X) = E \subset \operatorname{int}(\triangle)$ satisfies (B). Our next lemma has a similar flavor.

LEMMA 3.5. *Let E be a compact subset of* $\mathbf{C}$ *satisfying* (B), *and let $f \in C(E)$. Then $\mathcal{M}([R(E), f]) = E$.*

This can be proved by an argument essentially the same as that of Basener mentioned previously, but a simpler approach is to note that it is an immediate consequence of the following easy lemma (which strengthens Lemma 2.2 of [13]).

LEMMA 3.6. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are uniform algebras on a compact space X and that $\mathcal{A} \subset \mathcal{B}$. If $x \in X$ is such that the only Jensen measure for x relative to $\mathcal{A}$ is δ_x and if $m \in \mathcal{M}(B)$ coincides with point evaluation at x when restricted to $\mathcal{A}$, then m is point evaluation at x on all of $\mathcal{B}$.*

Proof. Let μ be a Jensen measure for m (as a functional on $\mathcal{B}$). Then obviously μ is a Jensen measure for the restriction of m to $\mathcal{A}$, that is, for point evaluation at x on $\mathcal{A}$. Hence, by hypothesis, $\mu = \delta_x$. Since μ represents m, we conclude that m is point evaluation at x on all of $\mathcal{B}$. □

If $\mathcal{A}$ is a uniform algebra on X, then a point $p \in X$ is a peak point for A if there exists a function $f \in A$ with $f(p) = 1$ while $|f| < 1$ on $X \setminus \{p\}$. When X is a compact planar set, Bishop proved that $R(X) = C(X)$ if almost every point of X is a peak point for $R(X)$.

THEOREM 3.7. *Let E be a compact subset of* $\mathbf{C}$ *satisfying* (B), *and let $f \in C^1(E)$. If almost every point of E is a peak point for* $[R(E), f]$, *then* $[R(E), f] = C(E)$.

Proof. Suppose that $[R(E), f] \neq C(E)$. By Lemma 3.5, $\mathcal{M}([R(E), f]) = E$. We may then apply Theorem 3.4 to produce a compact subset E_0 of E with $f|_{E_0} \in R(E_0)$ and $R(E_0) \neq C(E_0)$. If $z \in E_0$ is a peak point for $[R(E), f]$, choose $g \in [R(E), f]$ peaking at z. Since $g|_{E_0} \in R(E_0)$, the point z is a peak point for $R(E_0)$. By Bishop's peak-point theorem, $R(E_0) = C(E_0)$, which is a contradiction. □

Corollary 3.8. *Let E be a compact subset of the open unit disk satisfying* (B), *let $f \in C^1(E)$, and set $X = \{(z, f(z)) : z \in E\}$. If $X \subset \partial B$, then $R(X) = C(X)$.*

Proof. Let $\mathcal{A}$ be the algebra on X generated by $r(z)$ and w, where (z, w) are coordinates in $\mathbf{C}^2$ and r ranges over $R(E)$. Since $\mathcal{A} \subset R(X)$, it suffices to show that $\mathcal{A} = C(X)$. Moreover, $\mathcal{A}$ is isometrically isomorphic to the algebra on E generated by $R(E)$ and f, and it is therefore enough to show $[R(E), f] = C(E)$. Each point of ∂B is a peak point for polynomials and hence a peak point for $\mathcal{A}$, so every point of E is a peak point for $[R(E), f]$. By Theorem 3.7, $[R(E), f] = C(E)$. □

It is reasonable to conjecture that Theorems 3.4 and 3.7 remain valid if the hypothesis that $f \in C^1(E)$ is replaced by the assumption that f is merely continuous on E. We have no proof or counterexample.

Finally, we remark that Theorem 3.7 can also be obtained in a different fashion by combining our Lemma 3.5 with Theorem 4.3 of [4].

4. Approximation on Hölder Graphs

In this section we show that the hypothesis $f \in \mathrm{Lip}(\triangle)$ of Theorem 2.5 may be weakened to the assumption that f satisfies a Hölder condition with exponent α $(0 < \alpha < 1)$ on $E = \pi(X)$. That is, we assume there exists an M such that, for all $z, z' \in E$,

$$|f(z) - f(z')| \leq M|z - z'|^{\alpha}. \tag{13}$$

In order to establish Theorem 2.5 under the hypothesis that f satisfies (13), it suffices to show (cf. (11) in the proof of Theorem 2.5) that there exists a constant C such that, for $z \in E$ and $w \in D_z$,

$$|w - f(z)| \leq C \operatorname{dist}((z, w), X)^{\alpha}. \tag{14}$$

From (14) it follows, as in the proof of Theorem 2.5, that if $p = (z, w)$ then we have the estimate

$$|k(p)| \leq \frac{C'}{|w - f(z)|^{4/\alpha}},$$

from which we infer that $k \in H^{\alpha/8}(D_z)$ for all $z \in \triangle$, completing the proof.

To establish (14), we fix $p = (z, w)$ and take $p' = (z', f(z')) \in X$ so that $\operatorname{dist}(p, X) = |p - p'|$. Then

$$\begin{aligned}
|w - f(z)| &\leq |w - f(z')| + |f(z') - f(z)| \\
&\leq |w - f(z')| + M|z - z'|^{\alpha} \\
&\leq (M^2 + 1)^{1/2}(|w - f(z')|^2 + |z - z'|^{2\alpha})^{1/2}
\end{aligned}$$

and so

$$\frac{|w - f(z)|^{2/\alpha}}{\operatorname{dist}^2(p, X)} \leq \frac{(M^2 + 1)^{1/\alpha}(|w - f(z')|^2 + |z - z'|^{2\alpha})^{1/\alpha}}{|w - f(z')|^2 + |z - z'|^2}. \tag{15}$$

Set $x = |w - f(z')|$ and $y = |z - z'|$. Note that $\operatorname{dist}^2(p, X) = x^2 + y^2 \le 4$, since p and p' are points in the closed unit ball. The quantity

$$G(x, y) = \frac{(x^2 + y^{2\alpha})^{1/\alpha}}{x^2 + y^2}$$

on the right of (15) is clearly bounded on $1 \le x^2 + y^2 \le 4$ and so, to complete the proof of (14), it suffices to show that $G(x, y)$ is bounded for $x^2 + y^2 < 1$. Applying the elementary inequality $(A+B)^p \le 2^p(A^p+B^p)$ for positive A, B, p, we obtain

$$(x^2 + y^{2\alpha})^{1/\alpha} \le 2^{1/\alpha}(x^{2/\alpha} + y^2) \le 2^{1/\alpha}(x^2 + y^2),$$

where in the last inequality we have used the fact that $x < 1$. Therefore, $G(x, y) \le 2^{1/\alpha}$ for $x^2 + y^2 < 1$, and the proof is finished.

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