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A Heat Kernel Lower Bound for Integral Ricci Curvature

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1. Introduction

The heat kernel is one of the most fundamental quantities in geometry. It can be estimated both from above and below in terms of Ricci curvature (see [1; 2; 7]). The heat kernel upper bound has been extended to integral Ricci curvature by Gallot in [4]. Here we extend Cheeger and Yau's [2] lower bound to integral Ricci curvature.

Our notation for the integral curvature bounds on a Riemannian manifold (M, g) is as follows. For each $x \in M$ let $r(x)$ denote the smallest eigenvalue for the Ricci tensor $\mathrm{Ric}\colon T_xM \to T_xM$, and for any fixed number λ define

$$\rho(x) = |\min\{0, r(x) - (n-1)\lambda\}|.$$

Then set

$$k(p,\lambda,R) = \sup_{x\in M}\left(\int_{B(x,R)} \rho^p\right)^{1/p},$$

$$\bar{k}(p,\lambda,R) = \sup_{x\in M}\left(\frac{1}{\operatorname{vol} B(x,R)} \cdot \int_{B(x,R)} \rho^p\right)^{1/p}.$$

These curvature quantities evidently measure how much Ricci curvature lies below $(n-1)\lambda$ in the (normalized) integral sense. Observe that $\bar{k}(p,\lambda,R) = 0$ if and only if $\mathrm{Ric} \ge (n-1)\lambda$.

Let $E(x,y,t)$ denote the heat kernel of the Laplace–Beltrami operator on a closed manifold (M, g). For any real number λ, we use $E_\lambda(\overline{x,y},t)$ to denote the heat kernel on the model space of constant curvature λ. Our main result is as follows.

Theorem 1.1. *Let $n > 0$ be an integer, let $p > n/2$ and $\lambda \le 0$ be real numbers, and let $D > 0$. Then there exists an explicitly computable $\varepsilon_0 = \varepsilon(n,p,\lambda,D)$ such that, for any (M,g) with* $\operatorname{diam} M \le D$ *and for $\bar{k}(p,\lambda,D) \le \varepsilon_0$ and $k(p,\lambda,R) \le 1$,*

$$E(x,y,t) \ge E_\lambda(\overline{x,y},t) - (k(p,\lambda,D))^{1/2} C(n,p,\lambda,D)(t^{-(n+1)/2} + 1)$$

for any $x, y \in M$ and $t > 0$.

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REMARK. When $k(p, \lambda, R) = 0$, this is the estimate of Cheeger and Yau. When $k(p, \lambda, R) > 0$, however, the estimate becomes trivial for small and large t. One can pick any $0 < T_1 < T_2 < \infty$ in advance so that the estimate gives a positive lower bound for $T_1 \le t \le T_2$. The constant ε_0 will now depend on T_1 and T_2 as well.

The basic strategy is the same as in Cheeger–Yau; namely, one transplants the heat kernel on the model space to M and then compares using Duhamel's principle. The new difficulty lies in controlling an error term that would be zero in the presence of the pointwise Ricci curvature bound. Considerable care is needed for controlling this error term. We prove a new comparison estimate of volume elements integrated over the directional spheres for integral Ricci curvature, which should be of independent interest. We also need a Gaussian upper bound for the heat kernel derived by combining Gallot's upper bound estimate [4] of the heat kernel with a remarkable result of Grigor'yan [6].

We refer to [3] for some general results concerning the on-diagonal lower bound of the heat kernel.

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2. Basic Facts on the Heat Kernel

Here we fix our notation and collect basic facts on the heat kernel that will be used in our proof.

As in [2], we can define the Laplace–Beltrami operator for generalized Dirichlet and Neumann boundary conditions on a general Riemannian manifold (possibly incomplete) by choosing appropriate domains. The two coincide for a complete manifold. The corresponding heat kernel can simply be defined by using a spectral theorem. The heat kernel thus defined is always positive [2, Lemma 1.1], which will be essential for our discussion.

The models as used in [2] need only to have the correct mean curvature on the distance spheres. Here we restrict our models to the standard ones: simply connected spaces of constant sectional curvature. The following result [2, Lemma 2.3] is critical for Cheeger–Yau's theorem and also for our work here.

LEMMA 2.1. *Let $E_\lambda(r, t)$ denote the heat kernel on the model space of constant curvature λ, where $r = d(x, y)$. Then, for all $r, t > 0$,*

$$\frac{\partial}{\partial r} E_\lambda(r, t) < 0.$$

As mentioned previously, we also need uniform upper bounds on the heat kernel. This is established in [4] for integral Ricci curvature.

THEOREM 2.2. *Given any real number $\lambda \le 0$ and given $p > n/2$ and $D > 0$, there exists an explicitly computable $\varepsilon_0 = \varepsilon(n, p, \lambda, D)$ such that, for any (M, g) with* diam $M \le D$ *and $\bar{k}(p, \lambda, D) \le \varepsilon_0$,*

$$E(x, y, t) \leq C(n, p, \lambda, D)(t^{-p} + 1)$$

for any $x, y \in M$ *and* $t > 0$.

However, this estimate is not sufficient for our purpose. Fortunately one has the following amazing result of [6, Thm. 1.1], which translates Gallot's estimate into a Gaussian upper bound. We let $f(t)$ and $g(t)$ denote regular functions in the sense of [6] (which includes all piecewise power functions with nonnegative exponents).

THEOREM 2.3. *Let x and y be two points on an arbitrary smooth connected Riemannian manifold M, for which one has*

$$E(x, x, t) \leq \frac{1}{f(t)} \quad \text{and} \quad E(y, y, t) \leq \frac{1}{g(t)}$$

for all $0 < t < T \leq \infty$. *Then, for any* $C > 4$ *and some* $\delta = \delta(C) > 0$,

$$E(x, y, t) \leq \frac{4A}{\sqrt{f(\delta t)}\sqrt{g(\delta t)}} e^{-d(x,y)^2/Ct},$$

where A is a constant coming from f and g.

COROLLARY 2.4. *With the assumptions of Theorem 2.2, we have*

$$E(x, y, t) \leq C(n, p, \lambda, D)(t^{-p} + 1)e^{-d(x,y)^2/5t}.$$

The final piece of information we need is a similar Gaussian-type estimate on the derivative of the heat kernel on the model space.

PROPOSITION 2.5. *For the model space we have*

$$\left|\frac{\partial}{\partial r} E_\lambda(r, t)\right| < C(n, \lambda)(t^{-(n+1)/2} + 1)e^{-d(x,y)^2/5t}.$$

Proof. The key point here is that, at the expense of some Gaussian bound, the space derivative deteriorates the bound by a factor of only $t^{1/2}$ whereas the time derivative deteriorates the bound by a factor of t. This can be seen from, say, the gradient estimate (Harnack inequality) of Li and Yau [7], which asserts that a positive solution of the heat equation satisfies

$$\frac{|\nabla u|^2}{u^2} - \alpha \frac{u_t}{u} \leq \frac{n}{\sqrt{2}} \frac{\alpha^2}{\alpha - 1} H + \frac{n}{2} \frac{\alpha^2}{t} \quad \text{for all } \alpha > 1,$$

where H denotes the lower bound on the Ricci curvature. The required time derivative estimate follows from a result of [5, Cor. 3.2]. □

3. Comparison of the Volume Element

In [8], Petersen and Wei give a mean curvature comparison estimate in terms of $k(p, \lambda, R)$, allowing us to obtain the relative volume comparison for integral Ricci curvature. Here we need a comparison of integral of the volume element just over the directional spheres (instead of the balls).

Let M^n be a complete Riemannian manifold and let $x \in M$. Around x we use exponential polar coordinates and write the volume element as $d\,\mathrm{vol} = \omega d\theta_{n-1} \wedge dt$, where $d\theta_{n-1}$ is the standard volume element on the unit sphere $S^{n-1}(1)$. As t increases ω becomes undefined, but we can simply define it to be zero at those t. We have the important equation $\omega' = m\omega$, where the prime indicates differentiation along the radial direction and m is the mean curvature of the distance spheres around x.

In the space form M^n_λ of constant sectional curvature λ, we can similarly write the volume element as $d\,\mathrm{vol} = \omega_\lambda d\theta_{n-1} \wedge dt$ and $\omega'_\lambda = m_\lambda \omega_\lambda$.

Define $\psi = \psi(t, \theta) = \max\{0, m(t, \theta) - m_\lambda(t, \theta)\}$ and let $\psi = 0$ whenever it becomes undefined. The following mean curvature comparison estimate is established in [8].

THEOREM 3.1. *For $p > n/2$ and $\lambda \leq 0$,*

$$\left(\int_{B(x,r)} \psi^{2p}\, d\,\mathrm{vol}\right)^{1/2p} \leq C(n, p)(k(p, \lambda, r))^{1/2}. \tag{3.1}$$

With the help of the mean curvature comparison estimate, we now deduce a comparison estimate for the volume element.

LEMMA 3.2. *There is a constant $C(n, p, \lambda, R)$ such that, for any $p > n/2$, $\lambda \leq 0$, and $r \leq R$, if $k(p, \lambda, R) \leq 1$ then*

$$\frac{\int_{S^{n-1}} \omega(r, \theta)\, d\theta_{n-1}}{\int_{S^{n-1}} \omega_\lambda(r, \theta)\, d\theta_{n-1}} \leq 1 + C(n, p, \lambda, R)(k(p, \lambda, R))^{1/2}. \tag{3.2}$$

REMARK. The assumption $k(p, \lambda, R) \leq 1$ is only for the simplicity of the statement.

Proof of Lemma 3.2. We will prove a more general relative version. Define

$$u(r) = \frac{\int_{S^{n-1}} \omega(r, \theta)\, d\theta_{n-1}}{\int_{S^{n-1}} \omega_\lambda(r, \theta)\, d\theta_{n-1}}.$$

From the beginning of the proof of Lemma 2.1 in [8], for $0 \leq r_1 < r_2 \leq R$ we have

$$u(r_2) - u(r_1) \leq \frac{1}{\mathrm{vol}\, S^{n-1}} \int_{r_1}^{r_2} \int_{S^{n-1}} \psi \frac{\omega}{\omega_\lambda}\, d\theta_{n-1} \wedge dt.$$

Using Hölder's inequality yields

$$\begin{aligned}
&\int_{r_1}^{r_2} \int_{S^{n-1}} \psi \frac{\omega}{\omega_\lambda}\, d\theta_{n-1} \wedge dt \\
&\quad \leq \left(\int_0^R \int_{S^{n-1}} \psi^{2p} \omega\, d\theta_{n-1} \wedge dt\right)^{\frac{1}{2p}} \cdot \left(\int_{r_1}^{r_2} \left(\omega_\lambda^{-\frac{1}{2p-1}} \int_{S^{n-1}} \frac{\omega}{\omega_\lambda}\, d\theta_{n-1}\right) dt\right)^{1-\frac{1}{2p}} \\
&\quad \leq C(n, p)(k(p, \lambda, R))^{\frac{1}{2}} \left(\int_0^R \omega_\lambda^{-\frac{1+\alpha}{2p-1}}\, dt\right)^{\frac{1}{1+\alpha}\cdot 1-\frac{1}{2p}} \\
&\qquad \cdot \left(\int_{r_1}^{r_2} \left(\int_{S^{n-1}} \frac{\omega}{\omega_\lambda}\, d\theta_{n-1}\right)^{1+\frac{1}{\alpha}} dt\right)^{\frac{\alpha}{\alpha+1}\cdot\left(1-\frac{1}{2p}\right)},
\end{aligned}$$

where $\alpha > 0$ is chosen so that $p > \frac{(1+\alpha)(n-1)+1}{2}$; therefore, $\int_0^R \omega_\lambda^{-\frac{1+\alpha}{2p-1}}\, dt$ is integrable. Thus $u(r_2)$ satisfies the integral inequality

$$u(r_2) - u(r_1) \le C(n,p,\lambda,R)(k(p,\lambda,R))^{\frac{1}{2}} \left(\int_{r_1}^{r_2} (u(t))^{1+\frac{1}{\alpha}}\, dt \right)^{\frac{\alpha}{\alpha+1}\cdot\left(1-\frac{1}{2p}\right)}.$$

This implies

$$(u(r_2) - u(r_1))_+ \le Ck^{\frac{1}{2}} \left(\int_{r_1}^{r_2} (u(t))^{1+\frac{1}{\alpha}}\, dt \right)^{\frac{\alpha}{\alpha+1}\cdot\left(1-\frac{1}{2p}\right)}.$$

Let $v = \max\{u - u(r_1), 0\} = (u - u(r_1))_+$. Then $u \le v + u(r_1)$ and we have

$$v \le Ck^{\frac{1}{2}} \left(\int_{r_1}^{r_2} (v(t) + u(r_1))^{1+\frac{1}{\alpha}}\, dt \right)^{\frac{\alpha}{\alpha+1}\cdot\left(1-\frac{1}{2p}\right)}$$

or

$$v^{\frac{\alpha+1}{\alpha}\cdot\frac{2p}{2p-1}} \le \left(Ck^{\frac{1}{2}}\right)^{\frac{\alpha+1}{\alpha}\cdot\frac{2p}{2p-1}} \int_{r_1}^{r_2} (v(t) + u(r_1))^{\frac{\alpha+1}{\alpha}}\, dt. \tag{3.3}$$

Write

$$(v(t) + u(r_1))^{\frac{\alpha+1}{\alpha}} = \left[(v(t) + u(r_1))^{\frac{\alpha+1}{\alpha}\cdot\frac{2p}{2p-1}}\right]^{1-\frac{1}{2p}}.$$

Now we use the inequality

$$(a+b)^q \le 2^{q-1}(a^q + b^q), \quad a, b \ge 0,\ q \ge 1,$$

to obtain

$$(v(t) + u(r_1))^{\frac{\alpha+1}{\alpha}\frac{2p}{2p-1}} \le \left[2^{\frac{\alpha+1}{\alpha}\cdot\frac{2p}{2p-1}-1}\left(v(t)^{\frac{\alpha+1}{\alpha}\cdot\frac{2p}{2p-1}} + u(r_1)^{\frac{\alpha+1}{\alpha}\cdot\frac{2p}{2p-1}}\right)\right]^{1-\frac{1}{2p}}.$$

Now letting $w = v^{\frac{\alpha+1}{\alpha}\cdot\frac{2p}{2p-1}}$, (3.3) becomes

$$w \le \left(Ck^{\frac{1}{2}}\right)^{\frac{\alpha+1}{\alpha}\cdot\frac{2p}{2p-1}} \int_{r_1}^{r_2} 2^{\frac{1}{\alpha}+\frac{1}{2p}} \left(w(t) + u(r_1)^{\frac{\alpha+1}{\alpha}\cdot\frac{2p}{2p-1}}\right)^{1-\frac{1}{2p}}\, dt.$$

Let $\bar{w}$ be the solution of

$$\begin{cases} \bar{w}' = 2^{\frac{1}{\alpha}+\frac{1}{2p}}\left(Ck^{\frac{1}{2}}\right)^{\frac{\alpha+1}{\alpha}\cdot\frac{2p}{2p-1}}\left(\bar{w}(t) + u(r_1)^{\frac{\alpha+1}{\alpha}\cdot\frac{2p}{2p-1}}\right)^{1-\frac{1}{2p}}, \\ \bar{w}(r_1) = 0. \end{cases}$$

Then

$$\bar{w}(r_2) = \left[\left(u(r_1)^{\frac{\alpha+1}{\alpha}\cdot\frac{2p}{2p-1}}\right)^{\frac{1}{2p}} + \frac{1}{2p}2^{\frac{1}{\alpha}+\frac{1}{2p}}\left(Ck^{\frac{1}{2}}\right)^{\frac{\alpha+1}{\alpha}\cdot\frac{2p}{2p-1}}(r_2 - r_1)\right]^{2p} - u(r_1)^{\frac{\alpha+1}{\alpha}\cdot\frac{2p}{2p-1}}.$$

By Gronwall's inequality we have

$$w \le \bar{w},$$

which means that

$$(u(r_2) - u(r_1))_+ \le \left[\left(u(r_1)^{\frac{\alpha+1}{\alpha}\cdot\frac{1}{2p-1}} + \frac{1}{2p}2^{\frac{1}{\alpha}+\frac{1}{2p}}\left(Ck^{\frac{1}{2}}\right)^{\frac{\alpha+1}{\alpha}\cdot\frac{2p}{2p-1}}(r_2 - r_1)\right)^{2p} - u(r_1)^{\frac{\alpha+1}{\alpha}\cdot\frac{2p}{2p-1}}\right]^{\frac{\alpha}{\alpha+1}\cdot\left(1-\frac{1}{2p}\right)}.$$

Using the inequality

$$(x+a)^q - a^q \le qx(x+a)^{q-1}, \quad q \ge 1,$$

we obtain

$$\begin{aligned}
&(u(r_2)-u(r_1))_+ \\
&\quad \le \left(2^{\frac{1}{\alpha}+\frac{1}{2p}}(r_2-r_1)\right)^{\frac{\alpha}{\alpha+1}\cdot\left(1-\frac{1}{2p}\right)} Ck^{\frac{1}{2}} \Bigg(u(r_1)^{\frac{\alpha+1}{\alpha}\cdot\frac{1}{2p-1}} \\
&\qquad\qquad + \frac{1}{2p} 2^{\frac{1}{\alpha}+\frac{1}{2p}} \left(Ck^{\frac{1}{2}}\right)^{\frac{\alpha+1}{\alpha}\cdot\frac{2p}{2p-1}} (r_2-r_1) \Bigg)^q,
\end{aligned}$$

where $q = (2p-1)\frac{\alpha}{\alpha+1}\left(1-\frac{1}{2p}\right)$.

In particular, when $r_1 = 0$ and $k(p,\lambda,R) \le 1$ it follows that

$$u(r_2) - 1 \le C(n,p,\lambda,R)(k(p,\lambda,R))^{\frac{1}{2}}. \qquad \square$$

4. Proof of Theorem 1.1

We follow the same basic strategy as in Cheeger and Yau, starting with Duhamel's principle (which needs to be justified because of the singularity of the distance function at the cut locus).

Using integration by parts and the heat equation, we have

$$\begin{aligned}
&E(x,y,t) - E_\lambda(\overline{x,y},t) \\
&\quad = -\int_0^t \int_M \frac{d}{ds}(E_\lambda(\overline{x,w},t-s)) \cdot E(w,y,s)\, d\,\mathrm{vol}\, ds \\
&\qquad + \int_0^t \int_M E_\lambda(\overline{x,w},t-s) \cdot \frac{d}{ds}(E(w,y,s))\, d\,\mathrm{vol}\, ds \\
&\quad = -\int_0^t \int_M \frac{d}{ds}(E_\lambda(\overline{x,w},t-s)) \cdot E(w,y,s)\, d\,\mathrm{vol}\, ds \\
&\qquad - \int_0^t \int_M E_\lambda(\overline{x,w},t-s) \cdot \Delta E(w,y,s)\, d\,\mathrm{vol}\, ds. \qquad (4.1)
\end{aligned}$$

Now

$$\begin{aligned}
\frac{d}{ds} E_\lambda &= -\Delta_\lambda E_\lambda \\
&= \frac{\partial^2}{\partial r^2} E_\lambda + m_\lambda(r) \frac{\partial}{\partial r} E_\lambda \\
&= -\Delta E_\lambda - (m(r,\theta) - m_\lambda(r)) \frac{\partial}{\partial r} E_\lambda \\
&\le -\Delta E_\lambda - (m(r,\theta) - m_\lambda(r))_+ \frac{\partial}{\partial r} E_\lambda,
\end{aligned}$$

since $\frac{\partial}{\partial r} E_\lambda \le 0$ by Lemma 2.1. Hence the right-hand side of (4.1) is

$$
\begin{aligned}
&\geq \int_0^t \int_M \Delta E_\lambda(\overline{x,w}, t-s)\cdot E(w,y,s)\, d\,\mathrm{vol}\, ds \\
&\quad - \int_0^t \int_M E_\lambda(\overline{x,w}, t-s)\cdot \Delta E(w,y,s)\, d\,\mathrm{vol}\, ds \\
&\quad - \int_0^t \int_M (m(r,\theta)-m_\lambda(r))_+ \left|\frac{\partial E_\lambda}{\partial r}(\overline{x,w}, t-s)\right| \cdot E(w,y,s)\, d\,\mathrm{vol}\, ds.
\end{aligned}
$$

The first two terms combined can be shown to be nonnegative by the same argument as in [2] (using the self-adjointness of the Laplacian and certain convexity properties of the distance function at the cut locus). The last term is the extra error term, which is

$$
\begin{aligned}
\geq -\int_0^t &\left(\int_M (m(r,\theta)-m_\lambda(r))_+^q \, d\,\mathrm{vol}\right)^{1/q} \\
&\cdot \left(\int_M \left|\frac{\partial E_\lambda}{\partial r}(\overline{x,w}, t-s) E(w,y,s)\right|^{q'} d\,\mathrm{vol}\right)^{1/q'} ds
\end{aligned}
$$

for some $q \leq 2p$ to be chosen later. Here $q' = q/(q-1)$.

Now the first factor is controlled by Theorem 3.1 and the volume comparison estimate from [8, Thm. 1.1]. For the second factor, by Corollary 2.4 and Proposition 2.5 we have

$$
\begin{aligned}
&\left|\frac{\partial E_\lambda}{\partial r}(\overline{x,w}, t-s) E(w,y,s)\right| \\
&\qquad\qquad \leq C[(t-s)^{-(n+1)/2}+1][s^{-p_1}+1]e^{-d^2(x,w)/5(t-s)}e^{-d^2(w,y)/5s}.
\end{aligned}
$$

Here $p_1 = n/2+\alpha$ will be chosen so that $\alpha > 0$ is suitably small. In order to apply Corollary 2.4 we now need $\bar{k}(\lambda, p_1, D)$ to be smaller than an explicit constant ε_0 (as determined by Gallot [4]).

We have to deal with the singularity caused by the heat kernel at $t = 0$. Divide $\int_0^t$ into $\int_0^{t/2} + \int_{t/2}^t$. If $t > 1$ then we divide further so that $\int_0^t = \int_0^{1/2} + \int_{1/2}^{(t-1)/2} + \int_{(t-1)/2}^t$. In the latter case the estimate for the middle term is straightforward. By our previous remarks we may assume that $s \leq 1/2$, so for $0 \leq s \leq t/2$ we have

$$
(t-s)^{-(n+1)/2} \leq (t/2)^{-(n+1)/2}, \qquad e^{-d^2(x,w)/5(t-s)} \leq 1,
$$

which implies

$$
\begin{aligned}
&\int_0^{t/2}\left(\int_M \left|\frac{\partial E_\lambda}{\partial r}(\overline{x,w}, t-s)E(w,y,s)\right|^{q'} d\,\mathrm{vol}\right)^{1/q'} ds \\
&\qquad \leq C(t^{-(n+1)/2}+1)\int_0^{t/2}(s^{-p_1}+1)\left(\int_M e^{-q'd^2(w,y)/5s}\, d\,\mathrm{vol}\right)^{1/q'} ds.
\end{aligned}
$$

Now, writing out the integral over the space using the exponential polar coordinate around y yields

$$\int_M e^{-q'd^2(w,y)/5s}\,d\,\mathrm{vol} = \int_0^D e^{-q'r^2/5s}\left(\int_{S^{n-1}} \omega(r,\theta)\,d\theta\right) dr.$$

Here we have used the fact that the integral is over the whole manifold. With the curvature assumption on $k(p, \lambda, D)$, we can apply the comparison estimate for the volume element (Lemma 3.2) and obtain

$$\int_M e^{-q'd^2(w,y)/5s}\,d\,\mathrm{vol} \le C \int_0^D e^{-q'r^2/5s}\omega_\lambda(r)\,dr.$$

We then make a change of coordinate $r_1 = r/\sqrt{s}$ and deduce that

$$\int_M e^{-q'd^2(w,y)/5s}\,d\,\mathrm{vol} \le Cs^{n/2} \int_0^\infty e^{-q'r_1^2}\,\frac{\omega_\lambda(r_1 s^{1/2})}{s^{(n-1)/2}}\,dr_1.$$

Making use of the inequality

$$\frac{\omega_\lambda(r_1 s^{1/2})}{s^{(n-1)/2}} \le r_1^{n-1}\exp\left\{(n-1)\sqrt{|\lambda| s}\,r_1\right\}$$

(which can be easily verified) and noticing that since $s \le 1$ this term is dominated by $e^{-q'r_1^2}$, we finally arrive at the following estimate for the $\int_0^{t/2}$ part of the error term

$$C(t^{-(n+1)/2-p_1+n/2q'+1}+1),$$

where p_1 and q must be chosen to satisfy the inequality

$$-p_1 + \frac{n}{2q'} + 1 > 0.$$

Similarly (this time using the exponential polar coordinate around x), one has

$$\int_{t/2}^t \left(\int_M \left|\frac{\partial E_\lambda}{\partial r}(\overline{x,w}, t-s)E(w,y,s)\right|^{q'} d\,\mathrm{vol}\right)^{1/q'} ds$$
$$\le C(t^{-(n+1)/2-p_1+n/2q'+1}+1).$$

Finally, we note that suitable choices for p_1 and q can be easily made. For example, $q = n+1$ and $p_1 = (n+1)/2$ will do. □

References

[1] J. Cheeger, M. Gromov, and M. Taylor, *Finite propgation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds,* J. Differential Geom. 17 (1982), 15–53.

[2] J. Cheeger and S. T. Yau, *A lower bound for the heat kernel,* Comm. Pure Appl. Math. 34 (1981), 465–480.

[3] T. Coulhon and A. Grogor'yan, *On-diagonal lower bounds for heat kernels and Markov chains,* Duke Math. J. 89 (1997), 133–199.

[4] S. Gallot, *Isoperimetric inequalities based on integral norms of Ricci curvature,* Astérisque 157/158 (1988), 191–216.

[5] A. Grogor'yan, *Upper bounds of derivatives of the heat kernel on an arbitrary complete manifold,* J. Funct. Anal. 127 (1995), 363–389.

[6] ———, *Gaussian upper bounds for the heat kernel on arbitrary manifolds,* J. Differential Geom. 45 (1997), 33–52.
[7] P. Li and S. T. Yau, *On the parabolic kernel of the Schrödinger operator,* Acta Math. 156 (1986), 153–201.
[8] P. Petersen and G. Wei, *Relative volume comparison with integral curvature bounds,* Geom. Funct. Anal. 7 (1997), 1031–1045.

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