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Meromorphic Vector Fields and Elliptic Fibrations

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1. Introduction

The purpose of this paper is to present a classification of meromorphic semi-complete vector fields under a mild additional assumption. Motivation for this result comes in part from the fact that methods of differential equations and singularity theory can be employed to study certain problems in complex algebraic geometry. Indeed, these problems can be thought of as special cases of more general questions involving differential equations. Here we address some of these questions about differential equations. We also indicate how part of the theory of elliptic surfaces developed by Kodaira can be considered as a particular case of our methods. It should be noted that the mild assumption mentioned previously is always verified in these applications.

Another motivation for our classification is that it generalizes previous works on holomorphic semi-complete vector fields whose interest was already settled. An important reason for considering this generalization is the fact that meromorphic vector fields are more flexible than holomorphic ones in the sense that they appear in several situations where the corresponding holomorphic vector fields do not exist (a simple example being elliptic surfaces, cf. Example 2 to follow). A general classification of meromorphic semi-complete vector fields is interesting and would have additional applications (see Example 1); in fact, it would also be a rather significant generalization of certain natural questions in complex geometry. Whereas the discussion here combined with the results obtained in previous papers about *holomorphic* vector fields may lead to such classification, this attempt would take us too far from the aim of the present article.

We say that a singular holomorphic foliation $\mathcal{F}$ defined on a *neighborhood* of $(0,0) \in \mathbb{C}^2$ has infinitely many leaves accumulating on $(0,0)$ if, for a small ball $B(\varepsilon)$ centered at $(0,0)$, the singular foliation $\mathcal{F}|_B$ of $B(\varepsilon)$ obtained by restriction of $\mathcal{F}$ to $B(\varepsilon)$ possesses infinitely many leaves accumulating on $(0,0)$. The main result of this paper is the following theorem.

Main Theorem. *Let Y be a holomorphic vector field with an isolated singularity at $(0,0) \in \mathbb{C}^2$ that has only a finite number of orbits accumulating on the origin. Consider a meromorphic (nonholomorphic) function f defined on a neighborhood of $(0,0) \in \mathbb{C}^2$. Assume that $X = fY$ is a meromorphic semi-complete*

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vector field. Then, up to an inversible factor, X admits one of the following normal forms.

A. *If Y has vanishing linear part at* $(0,0)$, *then X has one of the forms*
  (i) $X_{111} = (xy(x-y))^a[x(x-2y)\partial/\partial x + y(y-2x)\partial/\partial y]$ *where* $a \in \mathbb{Z}$,
  (ii) $X_{112} = (xy(x-y)^2)^a[x(x-3y)\partial/\partial x + y(y-3x)\partial/\partial y]$ *where* $a \in \mathbb{Z}$, *or*
  (iii) $X_{123} = (xy^2(x-y)^3)^a[x(2x-5y)\partial/\partial x + y(y-4x)\partial/\partial y]$ *where* $a \in \mathbb{Z}$.

B. *If the linear part of Y at* $(0,0)$ *does not vanish but is nilpotent, then*
  (i) $Z_{112} = (y(y-x^2))^a((2y-x^2)\partial/\partial x + 2xy\partial/\partial y)$ *where* $a \in \mathbb{Z}$,
  (ii) $Z_{123} = (y(y-x^2)^2)^a((3y-x^2)\partial/\partial x + 4xy\partial/\partial y)$ *where* $a \in \mathbb{Z}$, *and*
  (iii) $Z_{32} = (x^3+y^2)^a(2y\partial/\partial x - 3x^2\partial/\partial y)$ *where* $a \in \mathbb{Z}$.

C. *If the linear part of Y at* $(0,0)$ *has an eigenvalue different from* 0, *then*
  (i) $(x^n y^m)^a f(mx\partial/\partial x - ny\partial/\partial y + \text{h.o.t.})$, *where* $a \in \mathbb{Z}$ *and* $m, n \in \mathbb{N}$;
  (ii) $(x^c y^d)^a(mx\partial/\partial x - ny\partial/\partial y)$, *where* $a \in \mathbb{Z}$, $m, n \in \mathbb{N}$, *and* $cm - nd = \pm 1$; *and*
  (iii) $(xy)^a(x-y)(x\partial/\partial x - y\partial/\partial y)$, *where* $a \in \mathbb{Z}$.

The rest of this introduction is devoted to presenting the definition of a meromorphic semi-complete vector field, which is a meromorphic variant of the definition given in [R1]. Some further examples will be provided as well.

A meromorphic vector field X on a neighborhood U of the origin $(0,\ldots,0) \in \mathbb{C}^n$ is by definition a vector field of the form

$$X = F_1 \frac{\partial}{\partial z_1} + \cdots + F_n \frac{\partial}{\partial z_n},$$

where the F_i are meromorphic functions on U (i.e., $F_i = g_i/h_i$ with g_i, h_i holomorphic on U). We denote by D_X the union of the sets $\{h_i = 0\}$. Of course, D_X is a divisor consisting of poles and indeterminacy points of X.

DEFINITION. The meromorphic vector field X is said to be *semi-complete* on U if and only if there exists a meromorphic map $\Phi_{sg} : \Omega \subseteq \mathbb{C} \times U \to U$, where Ω is an open set of $\mathbb{C} \times U$, that satisfies the following conditions:

1. $$\left.\frac{d\Phi_{sg}(T,x)}{dT}\right|_{T=0} = X(x) \quad \text{for all } x \in U \setminus D_x;$$
2. $\Phi_{sg}(T_1 + T_2, x) = \Phi_{sg}(T_1, \Phi_{sg}(T_2, x))$, provided that both sides are well-defined;
3. if (T_i, x) is a sequence of points in Ω converging to a point $(\hat{T}, x)$ in the boundary of Ω, then $\Phi_{sg}(T_i, x)$ converges to the boundary of $U \setminus D_X$ in the sense that the sequence leaves every compact subset of $U \setminus D_X$.

The map Φ_{sg} will be called the meromorphic *semi-global flow* associated to X (or induced by X). Meromorphic semi-complete vector fields arise, for instance, in the study of complete polynomial vector fields on affine algebraic surfaces.

Example 1. Consider a (nonlinear) complete polynomial vector field

$$X = P\partial/\partial x + Q\partial/\partial y$$

(P, Q polynomials) defined on $\mathbb{C}^2$. Such vector fields are studied in [CeSc], but here we propose a "dual approach" that relies on meromorphic semi-complete vector fields. Indeed, consider $\mathbb{CP}(2)$ as a compactification of $\mathbb{C}^2$. The vector fields X can naturally be identified to a meromorphic vector field, still denoted by X, defined on $\mathbb{CP}(2)$ and having "the line at infinity" as its divisor of poles. It is immediate to check that X is a meromorphic semi-complete vector field on $\mathbb{CP}(2)$. As an example, X may be chosen on $\mathbb{C}^2$ as

$$X(x, y) = x\partial/\partial x + (y + x^4)\partial/\partial y.$$

The behavior of X on a neighborhood of the "line at infinity" is strongly influenced by its "top-degree" homogeneous component. Thus, the general classification of meromorphic semi-complete vector fields will have a number of consequences on the original vector field X. This approach is expected to complement recent results of [CeSc].

Meromorphic semi-complete vector fields also appear in connection with elliptic (or ruled) surfaces, as shown by the next example.

Example 2 (Elliptic Surfaces). Let M be an elliptic surface; that is, M is a complex compact surface together with a holomorphic map $\mathcal{P}\colon M \to \mathbb{CP}(1)$ such that the generic fiber is an elliptic curve. Denote by K_M the canonical line bundle of M—namely, the holomorphic line bundle whose sections are (nondegenerated) 2-forms on M. Consider the canonical divisor of M (i.e., the divisor induced by K_M) and choose a representative D_M of this divisor such that D_M is invariant under the fibration (i.e., D_M is contained in a finite union of fibers possibly including some singular ones). Thus K_M admits a meromorphic section η whose divisor of zeros/poles coincides with D_M. We define a meromorphic vector field X on M by letting

$$\eta(p)(X(p), \cdot) = D_p\mathcal{P}$$

whenever this equation makes sense. Clearly X is a meromorphic vector field whose regular orbits are the fibers (level sets) $\mathcal{P}^{-1}(z)$ (except for finitely many $z \in \mathbb{CP}(1)$). In the next section we shall prove that such an X is semi-complete (see Corollary 2.9).

As pointed out by the referee, on a neighborhood of a singular fiber we can multiply X by an appropriate holomorphic function and so obtain a holomorphic semi-complete vector field defined on this neighborhood. Therefore, the classification of holomorphic vector fields would be sufficient for most of our applications to elliptic surfaces. Nonetheless, in other applications (such as those of Example 1) it is intrinsically important to allow strictly meromorphic vector fields.

Finally, there are many examples of meromorphic vector fields that are not proportional (by a meromorphic function) to a holomorphic semi-complete vector field. Here is one of the simplest possibilities.

EXAMPLE 3. Consider the vector field Z defined on a neighborhood of $(0, 0) \in \mathbb{C}^2$ and given by $Z = x\partial/\partial x + y^3\partial/\partial y$. First we observe that the foliation $\mathcal{F}_Z$ associated to Z cannot support a *holomorphic* semi-complete vector field, as follows from [R2]. However, by explicit integration we can see that the meromorphic vector field $Y = Z/y^2$ is semi-complete.

The method employed here will be expanded considerably in subsequent papers. In particular, we shall apply similar ideas to manifolds of dimension 3.

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2. Preliminaries

This section contains a brief summary of the main properties of semi-complete vector fields as well as a few other basic results. Most of the properties of meromorphic semi-complete vector fields discussed here are more or less immediate adaptations of the analogous properties for holomorphic ones (see [R1] and [GR]).

Let us start by recalling the notion of a *singular holomorphic foliation* (details can be found in [CeM]). A singular holomorphic foliation $\mathcal{F}$ defined on a neighborhood U of $(0, 0) \in \mathbb{C}^2$ consists of an actual (i.e. regular) holomorphic foliation defined on $U \setminus \{(0, 0)\}$. More generally, a singular holomorphic foliation with isolated singularities $p_1, \dots, p_l$ defined on a complex surface M is nothing but a regular holomorphic foliation defined on $M \setminus \{p_1, \dots, p_l\}$. It turns out that a local singular holomorphic foliation $\mathcal{F}$ is always locally given as the orbits of a (local) *holomorphic vector field having only isolated singularities.* In other words, given a point $p \in M$, there is a neighborhood U of p and a holomorphic vector field Y on U such that the orbits of Y define the foliation $\mathcal{F}$ on U. Furthermore, Y is either nonsingular or has an isolated singularity at p (in which case p is a singularity of $\mathcal{F}$). It also results from the preceding discussion that a meromorphic vector field X can always locally be given as $X = f_\alpha Y/f_\beta$, where f_α, f_β are holomorphic functions and Y is a holomorphic vector field with isolated zeros.

Consider a holomorphic vector field $Y = F\partial/\partial x + G\partial/\partial y$ (with isolated singularities) defining a singular holomorphic foliation $\mathcal{F}$ on a neighborhood of $(0, 0) \in \mathbb{C}^2$. We say that $\mathcal{F}$ has eigenvalues λ_1, λ_2 at p if the eigenvalues of X at p are λ_1, λ_2 (clearly λ_1, λ_2 are defined only up to a nonzero multiplicative constant). Notice that a singularity of a foliation $\mathcal{F}$ is said to be *simple* if it possesses at least one eigenvalue that is different from 0. More generally, the *order* of $\mathcal{F}$ at $(0, 0)$ is by definition the order of Y at $(0, 0)$; that is, it is the degree of the first nontrivial homogeneous component of the Taylor series of Y based at the origin.

The next step is to recall Seidenberg's theorem from [S]. Fix a singular foliation $\mathcal{F}$ defined on a neighborhood of $(0, 0) \in \mathbb{C}^2$. Let Y be a holomorphic vector field, $Y = F\partial/\partial x + G\partial/\partial y$, defining $\mathcal{F}$ and having an isolated singularity at the origin. Let us denote by π_1 the blow-up map from $\tilde{\mathbb{C}}^2 \to \mathbb{C}^2$ and by $\tilde{\mathcal{F}}^1 = \pi^*\mathcal{F}$ the blow-up of $\mathcal{F}$. Clearly, all the singularities of $\tilde{\mathcal{F}}^1$ are contained in the exceptional divisor $\pi^{-1}(0) = \mathcal{E}^1 \simeq \mathbb{CP}(1)$.

Now we want to define the *reduction tree* of $\mathcal{F}$. We assume that both eigenvalues of $\mathcal{F}$ at (0, 0) vanish (otherwise $\mathcal{F}$ is simple and already "reduced"). Let $\tilde{\mathcal{F}}^1$ and $\mathcal{E}^1 = \pi^{-1}(0) \simeq \mathbb{CP}(1)$ be as before. The triple $(\tilde{\mathcal{F}}^1, \mathcal{E}^1, \pi)$ is the first stage of the reduction tree of $\mathcal{F}$. The reduction tree is completed at the first stage (i.e., it consists of the triple $(\tilde{\mathcal{F}}^1, \mathcal{E}^1, \pi)$) if all the singularities of $\tilde{\mathcal{F}}^1$ are simple.

If the reduction tree is not completed at the first stage, then it continues as follows. Let $\tilde{p}_{1,1}, \ldots, \tilde{p}_{1,l}$ be those singularities of $\tilde{\mathcal{F}}^1$ that are not simple. We then blow up $\tilde{\mathcal{F}}^1$ at each of the singularities $\tilde{p}_{1,1}, \ldots, \tilde{p}_{1,l}$, producing a new foliation $\tilde{\mathcal{F}}^2$ along with a proper map Π_2 and an exceptional divisor $\mathcal{E}^2 = (\pi \circ \Pi_2)^{-1}(0)$. The triple $(\tilde{\mathcal{F}}^2, \mathcal{E}^2, \pi \circ \Pi_2)$ will be the second stage of the reduction tree. Note that Π_2 (resp., $\mathcal{E}^2$) is the composition (resp., sum or adjunction) of punctual blow-up maps $\pi_{1,1}, \ldots, \pi_{1,l}$ (resp., $\mathbb{CP}(1)$s associated to these punctual blow-up maps) based respectively at $\tilde{p}_{1,1}, \ldots, \tilde{p}_{1,l}$.

If all singularities of $\tilde{\mathcal{F}}^2$ are simple then the tree is completed. Otherwise we proceed inductively by blowing up those singularities that are not simple. With this notation we have the following.

THEOREM 2.1 [S]. *The reduction tree of $\mathcal{F}$ is finite. In other words, we have foliations $\tilde{\mathcal{F}}^i$ along with proper maps Π_i and divisors $\mathcal{E}^i$ obtained as before, with*

$$\mathcal{F} = \tilde{\mathcal{F}}^0 \xleftarrow{\Pi_1 = \pi} (\tilde{\mathcal{F}}^1, \mathcal{E}^1) \xleftarrow{\Pi_2} \cdots \xleftarrow{\Pi_r} (\tilde{\mathcal{F}}^r, \mathcal{E}^r), \tag{1}$$

such that the following statements hold.

1. *$\mathcal{E}^i$ $(i = 1, \ldots, r)$ is a tree in the sense of graphs consisting of $\mathbb{CP}(1)$s that correspond to sucessive blow-up maps. This means that the graph with vertices represented by the $\mathbb{CP}(1)$s of $\mathcal{E}^i$ and edges by the points where two of these $\mathbb{CP}(1)$s intersect has no loop.*
2. *All the singularities of $\tilde{\mathcal{F}}^r$ are simple.*

A *separatrix* for a singular foliation $\mathcal{F}$ is an irreducible analytic curve that contains the singularity $(0, 0) \in \mathbb{C}^2$ and is invariant under $\mathcal{F}$. We say that $\mathcal{F}$ is *dicritical* (or that (0, 0) is a dicritical singularity of $\mathcal{F}$) if $\mathcal{F}$ has infinitely many separatrizes.

A simple singularity having exactly one eigenvalue different from 0 is said to be a saddle node. It is well known (cf. [D]) that a saddle-node singularity is given by a vector field Y admitting the *Dulac's normal form*

$$[x(1 + \lambda y^p) + yR(x, y)]\partial/\partial x + y^{p+1}\partial/\partial y, \tag{2}$$

where $\lambda \in \mathbb{C}$ and $p \in \mathbb{N}^*$.

We are now able to state a useful lemma.

LEMMA 2.2. *Assume that $\mathcal{F}$ is a (local) singular holomorphic foliation with only finitely many leaves accumulating on the origin. Let $(\tilde{\mathcal{F}}^r, \mathcal{E}^r)$ be the reduction tree of $\mathcal{F}$. Then*:

1. *$\mathcal{E}^r$ consists of $\mathbb{CP}(1)$s that are invariant under $\tilde{\mathcal{F}}^r$ (if $\mathcal{F}$ is simple, i.e., $r = 0$, then this item is empty);*
2. *if $\tilde{p} \in \mathcal{E}^r$ is a singularity of $\tilde{\mathcal{F}}^r$, then $\tilde{\mathcal{F}}^r$ possesses two eigenvalues λ_1, λ_2 different from 0 at $\tilde{p}$. Furthermore, $\lambda_1/\lambda_2 \in \mathbb{R}_-$ (if $\mathcal{F}$ is simple, this item means that $\mathcal{F}$ has two nonvanishing eigenvalues whose quotient belongs to $\mathbb{R}_-$).*

Proof. First we suppose that $\mathcal{F}$ is not simple so that $\mathcal{E}^r$ is not empty. To check statement 1, suppose for a contradiction that there is a $\mathbb{CP}(1)$ in $\mathcal{E}^r$ that is not invariant by $\tilde{\mathcal{F}}^r$. Denoting this $\mathbb{CP}(1)$ by D_0, it follows that (except for finitely many points) the leaves of $\tilde{\mathcal{F}}$ intersect D_0 transversally. Since the "blow-down" map is proper, the images of these leaves provide infinitely many separatrizes for $\mathcal{F}$. This clearly contradicts our assumption.

The proof of the lemma is now reduced to checking that, at a simple singularity, a local foliation must have two nonvanishing eigenvalues with ratio in $\mathbb{R}_-$ provided that this foliation possesses only finitely many leaves accumulating on the origin.

Therefore we denote these eigenvalues by λ_1, λ_2 ($\lambda_1 \neq 0$). We claim that $\lambda_2 \neq 0$. In fact, if $\lambda_2 = 0$ then the foliation is a saddle node and, by using Dulac's normal form (2), it is easy to ensure the existence of infinitely many leaves accumulating on the origin. The resulting contradiction shows that $\lambda_2 \neq 0$.

Next suppose that $\lambda_1/\lambda_2 \in \mathbb{C} \setminus \mathbb{R}_-$ but that neither λ_1/λ_2 nor λ_2/λ_1 is a positive integer. In this case, Poincaré's theorem asserts that the foliation is linearizable on a neighborhood of the origin. By explicit integration we conclude that the origin is again accumulated by infinitely many leaves of the foliation.

Finally suppose that the quotient of the eigenvalues (say, λ_1/λ_2) belongs to $\mathbb{N}$. Here we have two possibilities: either the vector field is linearizable (and hence possesses infinitely many separatrizes) or it is conjugate to a Poincaré–Dulac vector field given by

$$X = (nx + y^n)\partial/\partial x + y\partial/\partial y. \qquad (3)$$

Again, an explicit integration shows that the foliation associated to a Poincaré–Dulac vector field has infinitely many leaves accumulating on the origin. The lemma is proved. □

To close this section, we shall give some basic statements about meromorphic semi-complete vector fields. Proofs will be left to the reader whenever they are straightforward generalizations of the corresponding statements in [R1] and [GR].

If X is a meromorphic vector field and L is a regular leaf (or orbit) of X (i.e., L is a leaf of $\mathcal{F}$ restricted to the complement of the poles and singularities of X), then L is naturally equipped with a *holomorphic* 1-form dT whose value on X equals 1. The form dT is a "foliated" 1-form and is called *time-form*. When L contains poles or singularities in its closure, dT may behave as a singular or as a meromorphic form on a neighborhood of these points. Our first statement is a simple but important lemma concerning integrals of dT over open curves.

LEMMA 2.3 [R1]. *Assume that X is a meromorphic semi-complete vector field defined on a neighborhood U of $(0, 0) \in \mathbb{C}^n$. Let L be a regular orbit of X and* *consider an embedded (one-to-one) curve* $c\colon [0,1] \to L$. *Then the integral of* dT *on* c *is different from* 0.

REMARK 2.4. We want to point out the nonexistence of strictly meromorphic semi-complete singularity in complex dimension 1. For example, the vector field $x^{-k}\partial/\partial x$ defined on $\mathbb{C}^*$ is not semi-complete around $0 \in \mathbb{C}$ for every $k \in \mathbb{N}^*$. Indeed, the corresponding time-form is $x^k\,dx$, whose integral on approriate embedded curves vanishes. The general case of a vector field $f\partial/\partial x$, where f has a pole at $0 \in \mathbb{C}$, can easily be derived from this fact (formally, one can also apply Corollary 2.7).

Next we show a simple lemma concerning (open) Riemann surfaces that are given by polynomial equations on the affine space. In other words, we shall consider a quasi-projective (algebraic, irreducible) curve over $\mathbb{C}$. If S is one such curve, we take its closure in the corresponding projective space and denote by $\hat{S}$ the nonsingular model of this closure (i.e., its normalization).

LEMMA 2.5. *Let* S *and* $\hat{S}$ *be as before, and assume that* S *is endowed with a meromorphic semi-complete vector field. Then* $\hat{S}$ *is either a rational curve or an elliptic curve. In fact,* X *is a holomorphic vector field with a holomorphic extension to* $\hat{S}$.

Proof. First we observe that S can be identified with a Zariski-open set of $\hat{S}$. To prove the lemma is clearly sufficient to show that X has a holomorphic extension to $\hat{S}$. According to Remark 2.4, X does not have poles on S because it is semi-complete. Similarly, a point p in $\hat{S} \setminus S$ cannot behave as a pole of X for the same reason. Thus we need only show that such a point p cannot be an essential singularity of X, either.

Thus we suppose for a contradiction that $p \in \hat{S} \setminus S$ is an essential singularity of X. The time-form dT associated to X is given in local coordinates around p as the inverse of X. Hence p is an essential singularity of dT as well. Now Picard's theorem promptly implies the existence of an open curve over which the integral of dT must vanish. The resulting contradiction proves the lemma. □

The following proposition actually holds in any dimension, but we shall state it only for dimension 2. Fix an open domain $U \subseteq \mathbb{C}^2$ and suppose that we are given a sequence of meromorphic semi-complete vector fields $\{X_i\}_{i\in\mathbb{N}}$ on U.

PROPOSITION 2.6 [GR]. *Assume that* $\{X_i\}$ *and* U *are as before. Suppose that the pole divisors* $\mathcal{D}_i$ *of* X_i *converge in the Haussdorff topology to some divisor* $\mathcal{D}$. *Suppose also that the order of the poles of* $\{X_i\}$ *is uniformly bounded and that* $\{X_i\}$ *converges on compact sets of* $U \setminus \mathcal{D}$ *toward a vector field* X. *Then* X *is a meromorphic semi-complete vector field on* U.

Consider a meromorphic vector field $X = gY/h$, where Y is a holomorphic vector field with an isolated singularity at $(0,0) \in \mathbb{C}^2$ and where g, h are holomorphic functions. Denote by Y^k (resp. g^r, h^s) the first nontrivial homogeneous component of the Taylor series of Y (resp. g, h) centered at $(0, 0) \in \mathbb{C}^2$ whose degree is supposed to be $k \in \mathbb{N}$ (resp. $s, r \in \mathbb{N}$). The vector field $X^{\text{ho}} = g^r Y^k / h^s$ will be called the *first homogeneous component of X*.

COROLLARY 2.7. *Assume that X as just described is semi-complete on a neighborhood of* $(0, 0) \in \mathbb{C}^2$. *Then* X^{ho} *is semi-complete on the whole* $\mathbb{C}^2$.

Proof. Fix $\varepsilon > 0$ such that X is semi-complete on the ball $B(\varepsilon)$ of radius $\varepsilon > 0$ centered at $(0, 0) \in \mathbb{C}^2$. Given $\rho > 0$, let $i_0 \in \mathbb{N}$ be such that $\rho/2^{i_0} < \varepsilon$. For every $i \geq i_0$, we consider the vector field X_i defined on $B(r)$ by

$$X_i(x, y) = 2^{i(r+k-s)} X(x/2^i, y/2^i).$$

Note that X_i converges to X^{ho} on $B(\rho)$ when $i \to \infty$ satisfies all the assumptions of Proposition 2.6. Since all the X_i are clearly semi-complete on $B(\rho)$, it follows that X^{ho} is also semi-complete on $B(\rho)$. Since $\rho > 0$ is arbitrary, we conclude that X^{ho} is semi-complete on $B(\rho)$ for all $\rho \in \mathbb{R}_+$.

In order to show that X^{ho} is semi-complete on the entire $\mathbb{C}^2$, we consider the semi-global flows $\Phi_n \colon \Omega_n \subseteq \mathbb{C} \times B(n) \to B(n)$ associated to X^{ho} on the ball $B(n)$ of radius n, $n = 1, 2, \dots$. Clearly there is no loss of generality in supposing that all the Ω_n are connected. Then we set $\Omega = \bigcup_{n \in \mathbb{N}} \Omega_n$ and define a map $\Phi \colon \Omega \subset \mathbb{C} \times \mathbb{C}^2 \to \mathbb{C}^2$ in the following way: given $(T, p) \in \Omega$, let n_0 be such that $(T, p) \in \Omega_{n_0}$. Next let $\Phi(T, p) = \Phi_{n_0}(T, p)$. It is immediate to check that Φ is well-defined and constitutes a meromorphic semi-global flow to X^{ho}. □

Another result that will be employed several times in the sequel is the following proposition.

PROPOSITION 2.8. *Let M be a complex manifold and let* $\mathcal{D} \subset M$ *be an analytic set* (*strictly contained in M*). *Assume that X is a meromorphic vector field on M whose restriction to* $M \setminus \mathcal{D}$ *is semi-complete. Then M is semi-complete on the whole of M.*

Proof. Let $\mathcal{D}_0$ be an irreducible component of $\mathcal{D}$. We just need to prove that the restriction of X to $\mathcal{D}_0$ is semi-complete. We can suppose that X is regular when restricted to $\mathcal{D}_0$ (i.e., $\mathcal{D}_0$ is not contained in the divisor of poles or zeros of X). In general, $\mathcal{D}_0$ may contain singularities of the foliation $\mathcal{F}$ associated to X; let us denote these singularities by $p_1, \dots, p_l$. If L_K is a compact part of $\mathcal{D}_0 \setminus \{p_1, \dots, p_l\}$, then it is sufficient to prove that the restriction of X to L_K is semi-complete. Since L_K is a compact part of a regular leaf, we can trivialize $\mathcal{F}$ on a neighborhood of L_K by using finitely many coordinates. This provides an identification of (the restriction of) leaves of $\mathcal{F}$ in this neighborhood with L_K itself. Under this identification, the restrictions of X to the leaves in question converge (as in Proposition 2.6) to the restriction of X to L_K. It follows that X restricted to L_K is semi-complete, proving the proposition. □

In particular we are now able to prove that the vector field X constructed in Example 2 of Section 1 is semi-complete. This is the content of our next corollary.

COROLLARY 2.9. *Let M be an elliptic surface and consider the meromorphic vector field X on M constructed in Example 2. Then X is semi-complete on M.*

Proof. We consider the singular foliation $\mathcal{F}$ associated to X. Clearly the regular leaves L of $\mathcal{F}$ are nothing but elliptic curves coinciding with the regular fibers of $\mathcal{P}$. Denote by $\mathcal{D}$ the divisor of M consisting of the singular fibers. Also consider the foliation $\mathcal{F}$ restricted to $M \setminus \mathcal{D}$. We first observe that the restriction of X to a leaf $L = \mathcal{P}^{-1}(q)$ in $M \setminus \mathcal{D}$ is complete (and hence semi-complete). Since L is an elliptic curve and is, in particular, compact, it is sufficient to show that the restriction of X to L is a globally defined holomorphic vector field. However, this is clear because we can choose the meromorphic form η that does not have poles on these regular fibers (i.e., D_M is invariant by the fibration).

Next we shall prove that X restricted to $M \setminus \mathcal{D}$ is semi-complete (indeed complete). We must therefore define a semi-global flow associated to X on $M \setminus \mathcal{D}$. In order to do so, we consider the mapping

$$\Phi_{sg,X} \colon \mathbb{C} \times (M \setminus \mathcal{D}) \to M \setminus \mathcal{D}$$

defined by $\Phi_{sg,X}(T, q) = \Phi_{X,L_q}(T, q)$, where L_q stands for the leaf of $\mathcal{F}$ containing q and Φ_{X,L_q} is the flow induced on L_q by the restriction of X. It is obvious that $\Phi_{sg,X}$ fulfills all the conditions required to be a semi-global (or global) flow on $M \setminus \mathcal{D}$. Finally, thanks to Proposition 2.8, this flow extends to a semi-global flow defined on the whole M. □

Corollary 2.10 is another immediate consequence of Proposition 2.8 and shows that semi-complete vector fields behave naturally with respect to blow-ups.

COROLLARY 2.10. *Assume that X is a meromorphic vector field that is semi-complete on a neighborhood U of the origin in $\mathbb{C}^2$. Then the blow-up $\tilde{X}$ of X is semi-complete on $\tilde{U} = \pi^{-1}(U)$.*

To finish this section we shall state two additional specific propositions concerning semi-complete vector fields. The corresponding proofs are easy adaptations of the original arguments appearing respectively in [R2] and [GR], and we have included a brief sketch of them. Observe, however, that these propositions are automatic in the presence of a holomorphic first integral. In particular, they would be automatic if we wanted to consider only applications to elliptic fibrations.

First suppose that we are given a vector field X of the form

$$X = x^a y^b f(x, y)(mx(1 + \text{h.o.t.})\partial/\partial x - ny(1 + \text{h.o.t.})\partial/\partial y),$$

where $f(0, 0) \neq 0$ and where $a, b \in \mathbb{Z}$ and $m, n \in \mathbb{N}$. Denote by $\mathcal{F}$ the singular foliation associated to X. Since the axes $\{x = 0\}$ and $\{y = 0\}$ are both invariant under $\mathcal{F}$, we can consider their local holonomies. Estimates carried out in [R2] then show the following.

PROPOSITION 2.11 [R2]. *Assume that X is semi-complete and that $am - bn \neq 0$. Then the local holonomy of $\{y = 0\}$ (resp., $\{x = 0\}$) is of finite order.*

Sketch of proof. Let $B_\varepsilon \subset \mathbb{C}$ be a small disc centered at $0 \in \mathbb{C}$ for $\varepsilon > 0$ very small. Also let $B_{\varepsilon,\varepsilon/2}$ be the annulus of radii $\varepsilon/2$ and ε. There is a holomorphic diffeomorphism $R\colon (B_{\varepsilon,\varepsilon/2} \setminus \mathbb{R}_-) \times B_\varepsilon \to (B_{\varepsilon,\varepsilon/2} \setminus \mathbb{R}_-) \times \mathbb{C}$ of the form $R(x, y) = (x, r(x, y))$, with $r(x, 0) = 0$, which takes the foliation $\mathcal{F}$ associated to X to the horizontal foliation of $(B_{\varepsilon,\varepsilon/2} \setminus \mathbb{R}_-) \times B_\varepsilon$ (i.e., the foliation whose leaves are of the form $(B_{\varepsilon,\varepsilon/2} \setminus \mathbb{R}_-) \times \{y\}$).

The push-forward R_*X of X by R is semi-complete on $(B_{\varepsilon,\varepsilon/2} \setminus \mathbb{R}_-) \times B_\varepsilon$. This vector field can also be written as $y^l F(x, r(x, y))$ for some $l \in \mathbb{Z}$. In particular, the vector field $F(x, r(x, 0))\partial/\partial x$ must be semi-complete on $B_{\varepsilon,\varepsilon/2} \setminus \mathbb{R}_-$. However, the function $F(x, r(x, 0))$ admits a holomorphic extension to $B_\varepsilon \setminus \mathbb{R}_-$ and its asymptotic order at $0 \in \mathbb{C}$ turns out to be $1 + a - bn/m$. Since $am - bn \neq 0$, the integral of the corresponding time-form over a curve that makes m turns around $0 \in \{y = 0\}$ vanishes. Hence the corresponding holonomy local diffeomorphism must coincide with the identity. □

Next consider the vector field $X_{111} = x(x - 2y)\partial/\partial x + y(y - 2x)\partial/\partial y$ (resp. X_{112}, X_{123}). Let $X^{\geq 3}_{111f}$ be a vector field of the form

$$X^{\geq 3}_{111f} = f(x, y)(X_{111} + \text{h.o.t.}),$$

where f is a meromorphic function. We define the vector fields $X^{\geq 3}_{112f}$ and $X^{\geq 3}_{123f}$ analogously. Finally, let $\mathcal{F}^{\geq 3}_{111}$ (resp. $\mathcal{F}^{\geq 3}_{112}$, $\mathcal{F}^{\geq 3}_{123}$) denote the singular foliation associated to $X^{\geq 3}_{111f}$ (resp. $X^{\geq 3}_{112f}$, $X^{\geq 3}_{123f}$). As explained in [GR], the proof of the next proposition uses a theorem due to Birkhoff and rediscovered later by Loray [L].

PROPOSITION 2.12 [GR]. *Suppose that $X^{\geq 3}_{111f}$ (resp. $X^{\geq 3}_{112f}$, $X^{\geq 3}_{123f}$) is semi-complete. Suppose also that the divisor of poles/zeros of f is contained in the union of the separatrizes of $\mathcal{F}^{\geq 3}_{111}$ (resp. $\mathcal{F}^{\geq 3}_{112}$, $\mathcal{F}^{\geq 3}_{123}$). If, in addition, the local holonomies of the separatrizes of $\mathcal{F}^{\geq 3}_{111}$ (resp. $\mathcal{F}^{\geq 3}_{112}$, $\mathcal{F}^{\geq 3}_{123}$) are trivial, then $\mathcal{F}^{\geq 3}_{111}$ (resp. $\mathcal{F}^{\geq 3}_{112}$, $\mathcal{F}^{\geq 3}_{123}$) is holomorphically conjugate to the foliation $\mathcal{F}_{111}$ (resp. $\mathcal{F}_{112}$, $\mathcal{F}_{123}$) associated to X_{111} (resp. X_{112}, X_{123}).*

Sketch of proof. We consider only the case of the vector field (resp. foliation) $X^{\geq 3}_{111}$ (resp. $\mathcal{F}^{\geq 3}_{111}$). Denote by $\tilde{\mathcal{F}}^{\geq 3}_{111}$ the blow-up of $\mathcal{F}^{\geq 3}_{111}$ and note that $\tilde{\mathcal{F}}^{\geq 3}_{111}$ has exactly three singularities p_1, p_2, p_3 on the exceptional divisor $\pi^{-1}(0)$. These singularities correspond to the intersections with $\pi^{-1}(0)$ of each of the (proper transform of the) separatrizes of $\mathcal{F}^{\geq 3}_{111}$. Furthermore, the foliation $\tilde{\mathcal{F}}^{\geq 3}_{111}$ possesses eigenvalues equal to 1 and 3 at each of the singularities p_1, p_2, p_3. For fixed $i \in \{1, 2, 3\}$, $\pi^{-1}(0)$ defines a separatrix for $\tilde{\mathcal{F}}^{\geq 3}_{111}$ at p_i. We claim that the local holonomy of this separatrix has order 3. Indeed, $\tilde{\mathcal{F}}^{\geq 3}_{111}$ has another separatrix at p_i—namely, the proper transform of the corresponding separatrix of $\mathcal{F}^{\geq 3}_{111}$, whose holonomy is trivial by assumption. The claim then follows from the fact that the eigenvalues of $\tilde{\mathcal{F}}^{\geq 3}_{111}$ at p_i are 1 and 3.

We now consider the global leaf L_0 of $\tilde{\mathcal{F}}_{111}^{\geq 3}$ given by $L_0 = \pi^{-1}(0)\backslash\{p_1, p_2, p_3\}$. The holonomy of this leaf is a homomorphism from the fundamental group of L_0 to $\mathrm{Diff}(\mathbb{C}, 0)$ that factors through the group $G = \{a, b, c : a^3 = b^3 = c^3 = abc = \mathrm{id}\}$. We claim that the image of the fundamental group of L_0 in $\mathrm{Diff}(\mathbb{C}, 0)$ is abelian and therefore cyclic of order 3. Before proving this claim, we point out that it implies our statement because it allows one to conjugate the foliations $\tilde{\mathcal{F}}_{111}^{\geq 3}$ and the blow-up $\tilde{\mathcal{F}}_{111}$ of $\mathcal{F}_{111}$ on a neighborhood of $\pi^{-1}(0)$. In fact, the construction of this conjugacy follows standard techniques of foliation theory complemented by a lemma of [MMo] that is necessary to extend it to neighborhoods of the singular points.

Finally, to prove the preceding claim we use the aforementioned result of Birkhoff (cf. [L]). This result classifies all homomorphisms from the group G (defined previously) to $\mathrm{Diff}(\mathbb{C}, 0)$. Other than those with abelian image, these homomorphisms are ramified coverings of the well-known representation of G obtained through the tiling of $\mathbb{C}$ by equilateral triangles. Now we assume that the subgroup of $\mathrm{Diff}(\mathbb{C}, 0)$ determined by the holonomy of L_0 is not abelian. Therefore, it must be given as just indicated. It is now easy to check the existence of a leaf L of $\mathcal{F}_{111}^{\geq 3}$ with a nontrivial *period,* that is, a loop $c \subset L$ on which the integral of the corresponding time-form equals $T_0 \neq 0$. It can also be checked that this leaf L accumulates on $\pi^{-1}(0)$. On the other hand, if Φ were the global flow associated to $X_{111}^{\geq 3}$ then the equation $\Phi(T_0, x) = x$ should define a proper analytic set containing L. This is impossible, since L accumulates on $\pi^{-1}(0)$. The proposition is proved. □

3. Reduction of Meromorphic Semi-Complete Singularities

Throughout this section, X stands for a meromorphic semi-complete vector field defined on a neighborhood of the origin in $\mathbb{C}^2$. In other words, X is given by $X = f_\alpha Y/f_\beta$, where Y is a holomorphic vector field for which the origin is either a regular point or an isolated singularity. Furthermore f_α, f_β are (nonidentically zero) holomorphic functions. Let $\mathcal{F}$ denote the singular holomorphic foliation associated to X. In this section and the next, we shall prove the theorem stated in the Introduction.

Let us say that a rational function $P = P_\alpha/P_\beta$ is *homogeneous* if both P_α and P_β are homogeneous polynomials (possibly with different degrees).

LEMMA 3.1. *Consider the linear vector field $Z = x\partial/\partial x + \lambda y\partial/\partial y$, where $\lambda \in \mathbb{R}_-$. Suppose that $P = P_\alpha/P_\beta$ is a nonconstant homogeneous rational function. Suppose also that PZ is semi-complete. Then*:

1. *λ is rational—that is, $\lambda = -n/m$ for appropriate relatively prime positive integers m, n; and*
2. *$P = x^c y^d$, where $mc - nd = 0$ or ± 1 if $\lambda \neq -1$. For $\lambda = -1$, P may also have the form $(xy)^a(x - y)$ for some $a \in \mathbb{Z}$.*

Proof. Since there is no meromorphic semi-complete vector field in dimension 1 (cf. Remark 2.4), the *zero set* of P_β must be invariant under Z. Since $\lambda \in \mathbb{R}_-$, the only separatrizes of Z are the axes $\{x = 0\}$ and $\{y = 0\}$. Thus P_β is of the form $x^a y^b$ for some $a, b \in \mathbb{N}$. Hence P has the form $x^c y^d Q(x, y)$, where $\mathbb{Q}$ is a homogeneous polynomial that is not divisible by x or y.

Observe that the orbit L of Z passing through the point (x_1, y_1), $x_1 y_1 \neq 0$, is parameterized by $\mathsf{A}\colon T \mapsto (x_1 e^T, y_1 e^{\lambda T})$. The restriction to L of the vector field PZ is given in the coordinate T by $P(x_1 e^T, y_1 e^{\lambda T})\partial/\partial T$.

We now assume for a contradiction that λ is not rational. Hence the parameterization A is a one-to-one map from $\mathbb{C}$ to L. It follows that the one-dimensional vector field $x_1^c y_1^d e^{(c+\lambda d)T} Q(x_1 e^T, y_1 e^{\lambda T})\partial/\partial T$ is semi-complete on $\mathbb{C}$, and we note that $e^{(c+\lambda d)T} Q(x_1 e^T, y_1^{\lambda T})$ is an entire function on $\mathbb{C}$. Since λ is not rational and Q is a polynomial, we conclude that this function has an essential singularity at infinity. The same argument of Lemma 2.5, which is based on Picard's theorem, provides a contradiction in this case. Hence λ must be rational.

Now set $\lambda = -n/m$ for relatively prime positive integers m, n. The orbit L is isomorphic to $\mathbb{CP}(1)$ minus two points. It follows from Lemma 2.5 that the non-singular model of its closure is a rational curve. In particular, X can have at most two singularities on L. Now suppose that $\lambda \neq -1$. Then the polynomial Q must be constant, for otherwise $\{Q = 0\}$ intersects L more than twice. This would give rise to more than two singularities for X on L, which is impossible. Thus the vector field PZ has the form $x^a y^b Z$ and, by explicit integration, we obtain the further condition $mc - nd = 0$ or ± 1.

On the other hand, if $\lambda = 1$ then the analogous argument shows that Q is either constant or linear. When Q is not constant modulo a linear change of coordinates, we have $Q = x - y$. Hence PZ becomes $x^c y^d (x - y) Z$. Again explicit integration ensures that we must have $c = d$. The lemma is proved. □

The rest of this section is devoted to discussing the nature of a meromorphic semi-complete vector field X defined on a neighborhood of the origin $(0, 0) \in \mathbb{C}^2$ and satisfying the following assumptions.

1. $X = f_\alpha Y / f_\beta$, where Y is a holomorphic vector field with an isolated singularity at $(0, 0) \in \mathbb{C}^2$ and f_α, f_β are holomorphic functions.
2. The foliation $\mathcal{F}$ associated to X (or to Y) has vanishing eigenvalues at $(0, 0) \in \mathbb{C}^2$.
3. The blow-up $\tilde{\mathcal{F}}$ of $\mathcal{F}$ is such that every singularity $\tilde{p} \in \pi^{-1}(0)$ of $\tilde{\mathcal{F}}$ has two eigenvalues different from 0 whose quotient belongs to $\mathbb{R}_-$. Moreover, $\pi^{-1}(0)$ is left invariant by $\tilde{\mathcal{F}}$.

Let $\{\tilde{p}_1, \ldots, \tilde{p}_k\} \subset \pi^{-1}(0)$ be the singularities of $\tilde{\mathcal{F}}$. Denote by $\tilde{X}$ the blow-up of X and recall that $\tilde{X}$ is semi-complete on a neighborhood of $\pi^{-1}(0)$ (cf. Lemma 2.10). As in Corollary 2.7, denote by $\operatorname{ord}(Y)$ (resp. $\operatorname{ord}(f_\alpha)$, $\operatorname{ord}(f_\beta)$) the order of Y (resp. f_α, f_β) at the origin.

By assumption 3 we know that $\pi^{-1}(0) \setminus \{\tilde{p}_1, \ldots, \tilde{p}_k\}$ is a regular leaf of $\tilde{\mathcal{F}}$. The order of $\tilde{X}$ on $\pi^{-1}(0)$ is equal to

$$\operatorname{ord}_{\pi^{-1}(0)}(\tilde{X}) = \operatorname{ord}(Y) + \operatorname{ord}(f_\alpha) - \operatorname{ord}(f_\beta) - 1. \tag{4}$$

In other words, given a point $\tilde{p} \in \pi^{-1}(0)$, we have coordinates (x, t) ($\{x = 0\} \subset \pi^{-1}(0)$) around $\tilde{p}$ such that

$$\tilde{X} = x^{(\mathrm{ord}_{\pi^{-1}(0)}(\tilde{X}))} \tilde{X}^1,$$

where $\tilde{X}^1$ is a meromorphic vector field whose divisor of zeros/poles does not contain $\pi^{-1}(0)$. Clearly $\tilde{X}$ has a curve of zeros (resp. poles) on $\pi^{-1}(0)$ if and only if $\mathrm{ord}_{\pi^{-1}(0)}(\tilde{X}) > 0$ (resp. $\mathrm{ord}_{\pi^{-1}(0)}(\tilde{X}) < 0$). We say that $\tilde{X}$ is *regular* on $\pi^{-1}(0)$ if $\mathrm{ord}_{\pi^{-1}(0)}(\tilde{X}) = 0$. Our immediate goal is to prove Proposition 3.2.

PROPOSITION 3.2. *Assume that X satisfies assumptions 1, 2, and 3 and that $\pi^{-1}(0)$ is invariant by $\tilde{\mathcal{F}}$. Then $\tilde{\mathcal{F}}$ possesses exactly three singularities. More precisely, if Y^{k_0} denotes the first nontrivial homogeneous component of Y* (*as in Corollary 2.7*) *then, up to a multiplicative function, Y^{k_0} is conjugate by a linear change of coordinates to X_{111}, X_{112}, or X_{123}.*

We begin by showing that $\mathrm{ord}_{\pi^{-1}(0)}(\tilde{X})$ is different from 0. This is the content of our next lemma.

LEMMA 3.3. *Let X be a meromorphic semi-complete vector field that satisfies the preceding assumptions. Then* $\mathrm{ord}_{\pi^{-1}(0)}(\tilde{X}) \neq 0$.

Proof. Suppose for a contradiction that $\mathrm{ord}_{\pi^{-1}(0)}(\tilde{X}) = 0$. Because $\tilde{X}$ is semi-complete, so is its restriction $\tilde{X}|_{\pi^{-1}(0)}$ to $\pi^{-1}(0)$. This implies that $\tilde{X}|_{\pi^{-1}(0)}$ actually extends to the entire $\pi^{-1}(0)$ as a holomorphic vector field. In particular, $\tilde{X}|_{\pi^{-1}(0)}$ has at most two singularities.

On the other hand, we know that $k \geq 3$ and so there is a singularity $\tilde{p}_1 \in \mathrm{Sing}\,\tilde{\mathcal{F}}$ such that

$$\tilde{X} = t^{-1}h_1(m_1x(1 + \text{h.o.t.})\partial/\partial x - n_1t(1 + \text{h.o.t.})\partial/\partial t$$

on a neighborhood of $\tilde{p}_1$ endowed with suitable coordinates (x, t). Again m_1, n_1 are positive integers and (6) ensures that $m_1/n_1 \neq 1$. In particular, $h_1(0, 0) \neq 0$. Thus Lemma 3.1 asserts that $-n_1/m_1 = 0, -1$, or 1. Since $m_1, n_1 \in \mathbb{N}^*$ we see that the unique possibility is $n_1/m_1 = 1$, which is impossible. □

As the referee has pointed out, the subsequent proof of Proposition 3.2 may be explained as follows. To each singularity $\tilde{p}_i$ of $\tilde{\mathcal{F}}$ in $\pi^{-1}(0)$ we associate its multiplicity n_i, namely, the order of the local holonomy of $\pi^{-1}(0)$ around $\tilde{p}_i$. Then $\pi^{-1}(0)$ can be viewed as an orbifold, and the statement of Proposition 3.2 asserts that this orbifold is of elliptic type in the sense that its universal covering is $\mathbb{C}$. Note that not all orbifolds of elliptic type appear in the list of the proposition in question, since we have an additional constraint furnished by the fact that the self-intersection of $\pi^{-1}(0)$ is -1. For example, the orbifold consisting of four singular points each with multiplicity 2 is elliptic as well. However, it is excluded from the present case because the corresponding self-intersection would be -2 (nonetheless, it will appear later on). Clearly, orbifolds of rational type are also compatible with the structure of semi-complete flows. These are excluded from the statement of Proposition 3.2 because the number of singularities is supposed to be at least three. However, they will also appear later in the discussion of the singular fibers of elliptic fibrations.

Fix a singularity $\tilde{p}_i$ $(i = 1, \dots, k)$ of $\tilde{\mathcal{F}}$. In view of our assumptions, we have coordinates (x, t) around $\tilde{p}_i$ as before such that $\tilde{X}$ has the form

$$\tilde{X} = x^{(\mathrm{ord}_{\pi^{-1}(0)}(\tilde{X}))} \frac{F_1(x,t)}{F_2(x,t)} \tilde{Y},$$

where F_1, F_2 are holomorphic functions that do not vanish identically on $\{x = 0\}$ and where $\tilde{Y} = \lambda_1 x(1+\text{h.o.t.})\partial/\partial x + (\lambda_2 t + \text{h.o.t.})\partial/\partial t$. However, since $\lambda_2/\lambda_1 \in \mathbb{R}_-$, it follows that $\tilde{Y}$ admits two smooth transverse separatrizes. Thus $\tilde{Y}$ has indeed the more precise form

$$\tilde{Y} = \lambda_1 x(1 + \text{h.o.t.})\partial/\partial x + \lambda_2 t(1 + \text{h.o.t.})\partial/\partial t.$$

In view of Lemma 3.1, the foregoing discussion may be summarized as follows.

LEMMA 3.4. *In appropriate local coordinates (x, t) that satisfy the preceding conditions we have $F_1/F_2 = t^d h(x,t)$, where $d \in \mathbb{Z}$ and $\lambda_2/\lambda_1 \in \mathbb{Q}_-$. Either $h(0,0) \neq 0$ or the set $\{h = 0\}$ coincides with $\{x = t\}$, in which case $\lambda_1 = -\lambda_2$.*

Fix a singularity $\tilde{p}_i \in \operatorname{Sing} \tilde{\mathcal{F}}$ $(i = 1, \dots, k)$ and consider a neighborhood U_i together with coordinates (x_i, t_i) $(\{x_i = 0\} \subset \pi^{-1}(0))$ such that $\tilde{X}$ has the form

$$\tilde{X}(x_i, t_i) = x_i^{(\mathrm{ord}_{\pi^{-1}(0)}(\tilde{X}))} t_i^{d_i} h_i [m_i x_i (1 + \text{h.o.t.})\partial/\partial x_i - n_i t_i (1 + \text{h.o.t.})\partial/\partial t_i], \quad (5)$$

where $d_i \in \mathbb{Z}$ and $m_i, n_i \in \mathbb{N}$. The following elementary relations are general (i.e., they do not depend on the semi-completeness of X) and can be verified by an explicit calculation:

$$k = \mathrm{ord}\, Y + 1; \qquad \sum_{i=1}^{k} \frac{m_i}{n_i} = 1. \quad (6)$$

It follows in particular that m_i/n_i in (5) is always different from 1 because $k \geq 3$ (recall that all the eigenvalues of $\mathcal{F}$ at the origin are 0). We conclude that $h_i(0,0) \neq 0$ and hence that $(\mathrm{ord}_{\pi^{-1}(0)}(\tilde{X}))m_i - n_i d_i = -1$, 0, or 1.

LEMMA 3.5. *The integers n_i $(i = 1, \dots, k)$ satisfy the equation*

$$\sum_{i=1}^{k} \left(1 - \frac{\varepsilon_i}{n_i}\right) = 2 \quad (7)$$

for appropriate $\varepsilon_i \in \{-1, 0, 1\}$.

Proof. Define $\varepsilon_i = (\mathrm{ord}_{\pi^{-1}(0)}(\tilde{X}))m_i - n_i d_i$ so that $\varepsilon_i \in \{-1, 0, 1\}$. Next consider the Taylor expansion of f_α (resp. f_β) in terms of homogeneous components f_α^j (resp. f_β^j) of degree $j \in \mathbb{N}$, namely, $f_\alpha = \sum_{j=r}^{\infty} f_\alpha^j$ $\left(\text{resp. } f_\beta = \sum_{j=s}^{\infty} f_\beta^j\right)$. Thus $F_1(x,t) = f_\alpha^r(1,t) + r_\alpha(x,t)$ (resp. $F_2(x,t) = f_\beta^s(1,t) + r_\beta(x,t)$), where r and s stand respectively for $\mathrm{ord}(f_\alpha)$ and $\mathrm{ord}(f_\beta)$ and where r_α and r_β are holomorphic functions. It promptly results that

$$\sum_{i=1}^{k} d_i = \mathrm{ord}(f_\alpha) - \mathrm{ord}(f_\beta). \tag{8}$$

By definition, $d_i = (\mathrm{ord}_{\pi^{-1}(0)}(\tilde{X}))m_i/n_i - \varepsilon_i/n_i$, so

$$\begin{aligned}
&\mathrm{ord}(f_\alpha) - \mathrm{ord}(f_\beta)\\
&\quad = \sum_{i=1}^{k} d_i = (\mathrm{ord}_{\pi^{-1}(0)}(\tilde{X})) \sum_{i=1}^{k} \frac{m_i}{n_i} - \sum_{i=1}^{k} \frac{\varepsilon_i}{n_i}\\
&\quad = -\sum_{i=1}^{k} \frac{\varepsilon_i}{n_i} + (\mathrm{ord}(Y) - 1 + \mathrm{ord}(f_\alpha) - \mathrm{ord}(f_\beta)) \quad \text{(by (8) and (6)).}
\end{aligned}$$

Using (6) again, we obtain $\sum_{i=1}^{k}(1 - \varepsilon_i/n_i) = 2$. The lemma is proved. □

Recall that m_i, n_i are positive integers. Note that $1 + \varepsilon_i/n_i \geq 1/2$ provided that $n_i \geq 2$. It is easy to determine all possible solutions of equation (7), and this will lead to the proof of Proposition 3.2.

Proof of Proposition 3.2. According to Lemma 3.3, the order of $\tilde{X}$ on $\pi^{-1}(0)$ never vanishes. We seek nonzero positive integers m_i, n_i such that

$$\sum_{i=1}^{k} \frac{m_i}{n_i} = 1; \qquad \sum_{i=1}^{k} \left(1 - \frac{\varepsilon_i}{n_i}\right) = 2 \quad \text{for } \varepsilon_i = -1, 0, 1.$$

Because $k \geq 3$, we can see that $n_i \neq 1$ for all $i \in \{1, \ldots, k\}$; otherwise, the sum $\sum_{i=1}^{k}(m_i/n_i)$ would be strictly greater than unity, which is impossible. For $n_i \geq 2$ we have $1 - \varepsilon_i/n_i \geq 1/2$, so one can easily check that all the possible solutions of $\sum_{i=1}^{k}(1 - \varepsilon_i/n_i) = 2$ are: $n_1 = n_2 = n_3 = 3$; $n_1 = n_2 = 4$ and $n_3 = 2$; $n_1 = 2$, $n_2 = 3$, and $n_3 = 6$; and $n_1 = n_2 = n_3 = n_4 = 2$. However,

$$1 = \sum_{i=1}^{k} \frac{m_i}{n_i} \geq \sum_{i=1}^{k} \frac{1}{n_i}.$$

Thus the solution $n_1 = n_2 = n_3 = n_4 = 2$ can be discarded. Furthermore, the other three possible solutions imply that all the m_i involved are equal to 1. In other words, the foliation $\tilde{\mathcal{F}}^{k_0}$ associated to the blow-up $\tilde{Y}^{k_0}$ of Y^{k_0} has exactly three singularities on $\pi^{-1}(0)$. The eigenvalues of these singularities are proportioned as $1:1:1$ or $1:1:2$ or $3:2:1$. By blowing down these foliations we obtain the desired normal forms corresponding to X_{111}, X_{112}, and X_{123}. The proposition is proved. □

4. Classification of Meromorphic Semi-Complete Singularities

In this section, the proof of our Main Theorem (see Section 1) will be finished. The first step is to strengthen Proposition 3.2 by proving the following theorem.

THEOREM 4.1. *Assume that X satisfies the assumptions of Proposition 3.2. Then, up to a multiplicative function, X is holomorphically conjugate to Y^{k_0}; that is, the foliation associated to X is $\mathcal{F}_{111}$, $\mathcal{F}_{112}$, or $\mathcal{F}_{123}$.*

By Proposition 3.2 we know that X may be written as

$$\frac{f_\alpha}{f_\beta}(Y^{k_0} + \text{h.o.t.}).$$

Denote by f_α^r and f_β^s the first nontrivial homogeneous component of f_α and f_β, respectively. In what follows we shall treat only the case $Y^{k_0} = X_{111} = x(x-2y)\partial/\partial x + y(y-2x)\partial/\partial y$. However, the other two cases are totally analogous. With the preceding notation we may state our next lemma as follows.

LEMMA 4.2. *$f_\alpha^r/f_\beta^s = (xy(x-y))^a$ for some $a \in \mathbb{Z}$.*

Proof. By definition, f_α^r/f_β^s is the quotient of two homogeneous polynomials. Furthermore, Corollary 2.7 shows that the vector field $f_\alpha^r Y^{k_0}/f_\beta^s$ is semi-complete on the entire $\mathbb{C}^2$.

On the other hand, it may immediately be verified that the generic orbit of Y^{k_0} is a torus minus a finite number of points. The restriction of Y^{k_0} to such an orbit L induces a constant vector field in the corresponding parameterization of L. Actually, L supports only these constant vector fields. In the same parameterization of L, the restriction of the vector field $f_\alpha^r Y^{k_0}/f_\beta^s$ must be constant, too (cf. Lemma 2.5). Hence f_α^r/f_β^s is constant on L. Since L is a generic orbit, we conclude that f_α^r/f_β^s is a first integral of Y^{k_0}. Therefore $f_\alpha^r/f_\beta^s = (xy(x-y))^a$, establishing the lemma. □

Recall that $\mathcal{F}$ (resp. $\mathcal{F}^{k_0}$) is the singular foliation associated to X (resp. Y^{k_0}). Moreover, $\tilde{\mathcal{F}}$ (resp. $\tilde{\mathcal{F}}k_0$) stands for the blow-up of $\mathcal{F}$ (resp. $\mathcal{F}^{k_0}$). Clearly $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}^{k_0}$ share the same singularities $\tilde{p}_1, \tilde{p}_2, \tilde{p}_3$ in $\pi^{-1}(0)$. For every $i \in \{1, 2, 3\}$, the eigenvalues of $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}^{k_0}$ at $\tilde{p}_i$ are 1 and -3.

We also observe that $\tilde{L}_0 = \pi^{-1}(0) \setminus \{\tilde{p}_1, \tilde{p}_2, \tilde{p}_3\}$ is a regular leaf of $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}^{k_0}$. Hence it is possible to consider the holonomy groups $G, G^{k_0} \subset \text{Diff}(\mathbb{C}, 0)$ of $\tilde{L}_0$ with respect to $\tilde{\mathcal{F}}, \tilde{\mathcal{F}}^{k_0}$. Because $\mathcal{F}^{k_0}$ is given by a homogeneous vector field, it is preserved by homotheties. This implies that G^{k_0} commutes with homotheties and thus is itself constituted by homotheties. Considering the eigenvalues of $\tilde{\mathcal{F}}^{k_0}$ at $\tilde{p}_i$, we conclude that G^{k_0} is cyclic of order 3. In particular, G also contains at least three elements.

Proof of Theorem 4.1. Clearly we can suppose that the three separatrizes of $\mathcal{F}$ coincide with the lines $\{x = 0\}$, $\{y = 0\}$, and $\{x = y\}$. Next we claim that the union of zeros/poles of f_α/f_β is contained in the union of these three lines. This is clear for the poles, as pointed out in Remark 2.4. On the other hand, suppose for a contradiction that f_α vanishes on leaves of $\mathcal{F}$ other than the separatrizes. Consider one such leaf L. Since the projective holonomy of $\mathcal{F}$ consists of at least three elements, it follows that L intersects $\{f_\alpha = 0\}$ in three or more points. However, this contradicts the fact that the restriction of X to L is semi-complete.

By the foregoing observation and Lemma 4.2, we conclude that X has the form

$$X = (xy(x-y))^a f[X_{111} + \text{h.o.t.}]$$

with $f(0,0) \neq 0$. We want to prove that $X_{111} + \text{h.o.t.}$ is conjugate to X_{111} by a diffeomorphism that is tangent to the identity. In order to do this, consider the blow-up $\tilde{\mathcal{F}}$ of $\mathcal{F}$ and a singularity $\tilde{p}_i \in \pi^{-1}(0)$. As mentioned, the eigenvalues of $\tilde{\mathcal{F}}$ at $\tilde{p}_i$ are 1 and -3. The normal form just displayed then implies that the corresponding vector field $\tilde{X}$ is given on a neighborhood of $\tilde{p}_i$ by

$$x^{3a+1} t^a \tilde{f}(x\partial/\partial x - 3t\partial/\partial t + \text{h.o.t.}),$$

for appropriate coordinates (x, t) with $\tilde{f}(\tilde{p}_i) \neq 0$. Proposition 2.11 shows that the local holonomies of the separatrizes of $\tilde{\mathcal{F}}$ at $\tilde{p}_i$ are of finite order. Since the linear part of the holonomy of the separatrix transverse to the exceptional divisor is 1, it follows that this holonomy is, in fact, trivial. Now Proposition 2.12 establishes the theorem. □

Assume next that $\mathcal{F}$ is a singular holomorphic foliation on a complex surface M. Suppose that $S \subseteq M$ is a smooth compact Riemann surface left invariant by $\mathcal{F}$ and denote by $p_1, \ldots, p_l$ the singularities of $\mathcal{F}$ belonging to S. Given p_i, $i = 1, \ldots, l$, consider local coordinates (x, y) around p_i, where $\mathcal{F}$ is given by the vector field $F\partial/\partial x + G\partial/\partial y$ and where $G = yg$ and $\{y = 0\} \subset S$. The *index of $\mathcal{F}$ with respect to S at p_i* is defined as

$$\text{Ind}_{p_i}(\mathcal{F}, S) = \text{Res}\, \frac{\partial}{\partial y}\left(\frac{G}{F}\right)(x, 0)\, dx.$$

According to [CSa], one has

$$\sum_{=1}^{l} \text{Ind}_{p_i}(\mathcal{F}, S) = S.S, \tag{9}$$

where $S.S$ denotes the self-intersection of S. Extensions of this formula to the case of singular separatrizes can be found in [Su].

Now let us consider again the vector fields X_{111}, X_{112}, X_{123} and Z_{112}, Z_{123}, Z_{32}. Note that each of the vector fields X_{111}, X_{112}, X_{123} possesses exactly three separatrizes that are smooth. Their indices are

$$\begin{aligned} X_{111}&: \ -2, -2, -2; \\ X_{112}&: \ -1, -3, -3; \\ X_{123}&: \ -1, -2, -5. \end{aligned}$$

The vector fields Z_{112} and Z_{123} each have two separatrizes, which are smooth and tangent to each other. The indices of these separatrizes are

$$\begin{aligned} Z_{112}&: \ -2, -2; \\ Z_{123}&: \ -1, -4. \end{aligned}$$

Finally, Z_{32} has a unique separatrix that is singular and has index 0. In particular, all the smooth separatrizes of these vector fields have negative integer indices.

Proof of Main Theorem. Consider a reduction tree for $\mathcal{F}$ as in (1) (cf. Theorem 2.1). We shall argue by induction on the length of this reduction tree (i.e., by induction on r). For $r = 0$ the statement is a consequence of Lemma 2.2, Corollary 2.7, and Lemma 3.1. We suppose that the statement holds for any singularity whose reduction tree has length $r_0 \geq 1$. We shall prove that it also holds for a singularity $\mathcal{F}$ with reduction tree of length $r_0 + 1$.

Therefore, let $\mathcal{F}$ be a foliation whose reduction tree has length $r_0 + 1$. By the induction hypothesis, the singularities of $\tilde{\mathcal{F}}^1$ on $\mathcal{E}^1$ are of type A, B, or C (see Section 1). Note, however, that $\mathcal{E}^1$ consists of a single rational curve whose self-intersection is -1. Furthermore, if all of these singularities are of type C then the induction step follows again from Theorem 4.1.

Thus we suppose the existence of a singularity $\tilde{p}_1$ of type A or B. Note also that a separatrix of a singularity of type A, B, or C is strictly negative. As mentioned before, indices of separatrizes of singularities of form A or B are strictly negative integers. Thus (9) implies that $\tilde{p}_1$ is the unique singularity of $\tilde{\mathcal{F}}^1$ in $\mathcal{E}^1 = \pi^{-1}(0)$. In addition, $\pi^{-1}(0)$ must define a smooth separatrix of index -1 for $\tilde{\mathcal{F}}^1$ at $\tilde{p}_1$. By inspecting the lists of indices we see that, around $\tilde{p}_1$, $\tilde{\mathcal{F}}^1$ is given by one of the following vector fields: X_{112}, X_{123}, or Z_{123}. For each of these cases the foliation $\tilde{\mathcal{F}}^1$ is uniquely determined on a neighborhood of $\mathcal{E}^1 = \pi^{-1}(0)$. In fact, consider two foliations as above having a unique singularity in $\mathcal{E}^1 = \pi^{-1}(0)$ whose model is (say) X_{112}. Then a local conjugacy between these singularities around $\tilde{p}_1$ (which we may suppose to be the same modulo a linear change of coordinates) can immediately be globalized to a neighborhood of the entire exceptional divisor, since its holonomy is trivial. Hence each of the described cases gives rise, by blow-down, to a unique foliation defined around $(0, 0) \in \mathbb{C}^2$ (up to change of coordinates). These foliations are (respectively) Z_{112}, Z_{123}, or Z_{32}. This completes the induction step and proves the theorem. □

5. Elliptic Fibrations and Complements

In this last section we present an alternative treatment of Kodaira's description of elliptic fibrations. Our point of view also leads to some partial generalizations of these results (for example, similar results apply to semi-complete vector fields whose orbits need not be elliptic curves).

We place ourselves in the classical setting: M is a complex surface together with a proper holomorphic map $\mathcal{P}\colon M \to D \subset \mathbb{C}$ (where D stands for the unit disc) such that:

1. $\mathcal{P}$ defines a (regular) fibration of $M \setminus \mathcal{P}^{-1}(0)$ over $D \setminus \{0\}$; and
2. $\mathcal{P}^{-1}(p)$ is an elliptic curve for every $p \in D \setminus \{0\}$.

The purpose of Kodaira's theorem is to characterize the possible structures of a singular fiber $\mathcal{P}^{-1}(0)$. The advantage of our method is that it gives the structure of the *singularities of the foliation associated with* $\mathcal{P}$, and this cannot be obtained *a priori* from such standard techniques as the classification of root systems of affine type. With this information about the structure of the singularities in question, we are able to immediately describe the *singular foliation defined by* $\mathcal{P}$ *on* *a neighborhood of the singular fiber.* In what follows we assume that $\mathcal{P}^{-1}(0)$ is not a regular elliptic curve. For the convenience of the reader, we first restate Kodaira's theorem so as to include the description of the singularities of the foliation associated to $\mathcal{P}$; we then derive the description of the neighborhood of each singular fiber. Recall that we have supposed that our singular fibers do not contain an elliptic curve (in which case the fiber would be a multiple of this elliptic curve). We also assume that the singular fibers do not contain smooth rational curves of self-intersection -1.

THEOREM 5.1. *Let M and $\mathcal{P}$ be as before. The structure of a singular fiber of $\mathcal{P}$ is necessarily one of the following types. Note also that, except for Types I_1 and III, every irreducible component of the fiber is a smooth rational curve with self-intersection -2.*

Fibers of Type I_1: A rational curve with a node singularity. This node singularity corresponds to a singularity with eigenvalues $(1, -1)$ *for the fibration viewed as a singular foliation.*

Fibers of Type II: A rational curve with a cusp singularity. The foliation is that associated to Z_{32}.

Fibers of Type III: Two smooth rational curves with quadratic tangential intersection. Similarly to the preceding case, the foliation in question is that associated to Z_{112}.

Fibers of Type IV: Constituted by three smooth rational curves meeting transversely at a common point. More precisely, the foliation associated to $\mathcal{P}$ is the same foliation $\mathcal{F}_{111}$ associated to the vector field X_{111} (*the separatrizes of $\mathcal{F}_{111}$ being the singular fiber*).

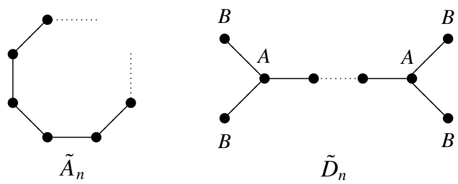

Figure 1 Fibers of Types $\tilde{A}_n$ and $\tilde{D}_n$

Fibers of Type $\tilde{A}_n$ (*or I_n, $n \geq 2$*)*: A "loop" of smooth rational curves. The holonomy of the fibration around each irreducible component is trivial* (*but the monodromy is not necessarily trivial, as this can be a multiple fiber*)*. Finally, the fibration viewed as a foliation has eigenvalues* $(1, -1)$ *at each point of intersection of two irreducible components.*

Fibers of Type $\tilde{D}_n$ (*or I_n^**)*: The arrangement consists of smooth rational curves. The rational curves indicated by A contain three singularities of the fibration whose* *respective eigenvalues are* $(-1, 2)$, $(-1, 2)$, *and* $(-1, 1)$. *The rational curves indicated by B contain only one singularity having eigenvalues* $(2, -1)$. *The remaining curves possess two singularities, each with eigenvalues* $(-1, 1)$. *The holonomy of the fibration has order* 2 *along the components A and is trivial around the other components. A degenerate case of this fiber is constituted by a single curve A and four curves B.*

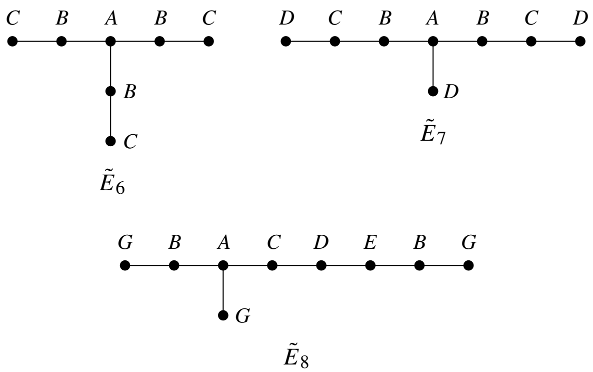

Figure 2 Fibers of Types $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$

Fibers of Type $\tilde{E}_6$ (*or IV**): *The arrangement of smooth rational curves can be described as follows. The rational curve A contains three singularities, each with eigenvalues* $(-2, 3)$ (*and the holonomy along A is* $\mathbb{Z}/3\mathbb{Z}$). *The curves B contain two singularities whose eigenvalues are* $(-3, 2)$ *and* $(-1, 2)$. *The holonomy along the curve B is* $\mathbb{Z}/2\mathbb{Z}$. *Finally, the curves C contain a unique singularity with eigenvalues* $(-2, 1)$. *Clearly, the holonomy along C is trivial.*

Fibers of Type $\tilde{E}_7$ (*or III**): *The description of this arrangement is as follows. The rational curve A contains three singularities, two with eigenvalues* $(-3, 4)$ *and the other with eigenvalues* $(-1, 2)$; *the holonomy along A is* $\mathbb{Z}/4\mathbb{Z}$. *The curves B contain two singularities with eigenvalues given* (*respectively*) *by* $(-4, 3)$ *and* $(-2, 3)$. *The holonomy along B is* $\mathbb{Z}/3\mathbb{Z}$. *The curves C have two singularities with eigenvalues* $(-3, 2)$ *and* $(-1, 2)$, *and the holonomy along C is therefore* $\mathbb{Z}/2\mathbb{Z}$. *Finally, the curve D has only one singularity, whose eigenvalues are* $(-2, 1)$ *and also have trivial holonomy.*

Fibers of Type $\tilde{E}_8$ (*or II**): *This last arrangement admits the following description. The rational curve A contains three singularities whose eigenvalues are respectively* $(-1, 2)$, $(-2, 3)$, *and* $(-5, 6)$. *The holonomy along A is* $\mathbb{Z}/6/Z$. *The curve B has two singularities with eigenvalues* $(-3, 2)$ *and* $(-1, 2)$. *The holonomy along B is then* $\mathbb{Z}/2\mathbb{Z}$. *The curve C* (*resp. D*, *E*) *has two singularities with eigenvalues* $(-6, 5)$, $(-4, 5)$ (*resp.* $(-5, 4)$, $(-3, 4)$, *and* $(-4, 3)$, $(-2, 3)$). *The holonomy along C* (*resp. D*, *E*) *is* $\mathbb{Z}/5\mathbb{Z}$ (*resp.* $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$). *Finally, the curve G has trivial holonomy and a unique singularity whose eigenvalues are* $(-2, 1)$.

REMARK 5.2. It follows from the preceding description that the order of the holonomy of a component of a singular fiber is equal to the multiplicity of this component.

Suppose now that we are given holomorphic maps $\mathcal{P}_1$ (resp. $\mathcal{P}_2$) from M_1 (resp. M_2) to $D \subset \mathbb{C}$ defining elliptic fibrations in M_1, M_2. Denote by X_1, X_2 the corresponding semi-complete vector fields and by $\mathcal{F}_1$, $\mathcal{F}_2$ their associated foliations. Theorem 5.3 follows as a consequence of this description. Whereas this theorem may have been known before, I was unable to find it explicitly stated in the literature.

THEOREM 5.3. *Suppose that* $\mathcal{P}_1^{-1}(0)$ *and* $\mathcal{P}_2^{-1}(0)$ *are singular fibers of type, say,* $\tilde{E}_6$. *Then there exist a neighborhood* $U_1 \subset M_1$ (*resp.* $U_2 \subset M_2$) *of* $\mathcal{P}_1^{-1}(0)$ (*resp.* $\mathcal{P}_2^{-1}(0)$) *that is invariant under* $\mathcal{F}_1$ (*resp.* $\mathcal{F}_2$) *as well as a* C^∞*-diffeomorphism* $h\colon U_1 \to U_2$ *conjugating the restrictions of these foliations to the sets in question. Moreover, h is transversely holomorphic.*

Proof. Since $\mathcal{P}_1^{-1}(0)$ and $\mathcal{P}_2^{-1}(0)$ have the same type ($\tilde{E}_6$), we consider their corresponding components C_i^1 and C_i^2. We also consider the holonomy of C_i^1 minus its singularities (resp. C_i^2 minus its singularities) relative to $\mathcal{F}_1$ (resp. $\mathcal{F}_2$). A consequence of the previous discussion is that this holonomy is a finite cyclic group whose order depends solely on the position of C_i^1 (resp. C_i^2) in $\mathcal{P}_1^{-1}(0)$ (resp. $\mathcal{P}_2^{-1}(0)$) and of the type of $\mathcal{P}_1^{-1}(0)$ (resp. $\mathcal{P}_2^{-1}(0)$). In other words, given our assumptions, the holonomies of C_i^1, C_i^2 minus their singularities are holomorphically conjugate. This enables us to conjugate the restrictions of the foliations $\mathcal{F}_1$, $\mathcal{F}_2$ to neighborhoods of C_i^1, C_i^2 (minus the corresponding singularities). However, a classical argument in [MMo] shows that this conjugacy actually extends to neighborhoods of the mentioned singularities. Therefore it extends to other components of the singular fibers and eventually to neighborhoods U_1, U_2 of $\mathcal{P}_1^{-1}(0)$, $\mathcal{P}_2^{-1}(0)$ as required. □

Obviously we have analogous results concerning the other possible types of singular fibers. These results show that the only way we can have different neighborhoods of a same singular fiber is by deforming the complex structure of the regular fibers. In particular, a neighborhood $\mathcal{P}^{-1}(D)$ of the singular fiber $\mathcal{P}^{-1}(0)$ is completely determined by the structure of $\mathcal{P}^{-1}(0)$ and by a meromorphic function h defined on D that "measures the complex structure" of the regular fibers.

To obtain concrete models for all these fibers (in particular showing that all the cases are realizable), we note that the vector fields X_{11}, Z_{112}, and Z_{32} induce pencils of $\mathbb{CP}(2)$ viewed as the compactification of $\mathbb{C}^2$. By performing an appropriate number of blow-ups at the singularities of these pencils contained in the "line at infinity", we can turn all of them into fibrations. These fibrations turn out to be elliptic, and the proper transform of the "line at infinity" together with the exceptional divisors introduced are (respectively) of type $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$. These fibrations also provide models for the fibers of Type IV, III, and II. Fibers of Type I_1 and of Type $\tilde{A}_n$ or $\tilde{D}_n$ (i.e., I_n or I_n^*) can easily be handled by a simple procedure of surgery. Finally, with the foregoing models we can reconstruct an elliptic surface by assembling the models involved with the singular fibers and the remaining regular fibration. With a little additional effort we then derive the complete description of all elliptic surfaces set forth by Kodaira [K].

With the notation used in the beginning of this section, we decompose the singular fiber $\mathcal{P}^{-1}(0)$ into irreducible components C_i of multiplicity m_i, $i = 1, \ldots, s$ (notation: $\mathcal{P}^{-1}(0) = \sum_{i=1}^{s} \mathsf{m}_i C_i$). Recall that we assume $\mathcal{P}^{-1}(0)$ does not contain smooth rational curves of self-intersection -1. Let us begin with a rather simple lemma borrowed from [F, p. 172].

LEMMA 5.4. *Given an irreducible component C_i, either $C_i = C_1$ is the unique component of $\mathcal{P}^{-1}(0)$ (and has arithmetic genus equal to 1) or C_i is a smooth rational curve with self-intersection -2.*

Nonetheless, we observe that if C_1 is the unique component of $\mathcal{P}^{-1}(0)$ then $C_1.C_1 = 0$, where $C_1.C_1$ stands for the self-intersection of C_1. Indeed, C_1 is homologically equivalent to a regular fiber.

LEMMA 5.5. *Suppose that C_1 has an irreducible singularity. Then $\mathcal{P}^{-1}(0) = \mathsf{m}_1 C_1$ and, in fact, we have Type II of Theorem 5.1.*

Proof. Assume that p_1 is an irreducible singularity of C_1. It follows from the Main Theorem that the foliation is given around p_1 by the vector field Z_{32}. Consider the meromorphic semi-complete vector field X defined on M, and suppose that X is regular on C_1. Then, up to normalization of C_1, X induces a semi-complete vector field on C_1 that has a singularity of order 2 (corresponding to p_1). Such a vector field is therefore holomorphic and so p_1 must be its unique singularity. Hence p_1 is the unique singularity of C_1 and we have Type II.

In general we suppose that X has poles (or zeros) on C_1. Again thanks to our Main Theorem, we can write X on a neighborhood of p_1 as $f(x^3 + y^2)^a Z_{32}$ for some holomorphic function f, $f(0, 0) \neq 0$, and some $a \in \mathbb{Z}$. Because X possesses a global first integral, it is clear that—up to multiplying X by a power of this first integral—we can eliminate the term $x^3 + y^2$. The new vector field (a reparametrization of X) is regular on C_1 and is still semi-complete, since it was obtained by multiplying X by a function that is constant on its orbits. Thus the preceding argument applies to prove the lemma. □

To complement the preceding discussion, we suppose now that C_1 has only reducible singularities.

LEMMA 5.6. *Suppose that C_1 has a reducible singularity. Then we have Type I_1.*

Proof. Assume that p_1 is a reducible singularity of C_1. We claim that p_1 is a linear singularity of the corresponding foliation. Indeed, suppose for a contradiction that it is of Type III or IV. The corresponding vector field would then be Z_{112} or X_{111}. Modulo multiplying by a suitable first integral, these vector fields induce a singularity of order 2 on each of their separatrizes. Hence C_1 would have more than one singularity of order 2, which is obviously impossible. We then conclude that p_1 is linear as desired. In particular, all singularities of C_1 are nodal. Since $C_1.C_1 = 0$, the index formula for reducible singularities (cf. [Su]) shows that the eigenvalues of these singularities are $(1, -1)$. Finally, there can exist only one such singularity because it provides a holomorphic vector field with already two singularities on the normalization of C_1 (by the same argument employed previously). This characterizes Type I_1 of Theorem 5.1. □

From now on we consider the most interesting cases of Theorem 5.1—namely, those in which all the components of $\mathcal{P}^{-1}$ are smooth rational curves of self-intersection -2. We denote by $\mathcal{F}$ the foliation given by the fibration and by X the corresponding semi-complete vector field.

LEMMA 5.7. *Assume that $\mathcal{F}$ has a singularity in $\mathcal{P}^{-1}(0)$ at which the eigenvalues of $\mathcal{F}$ vanish. Then we have Type III or Type IV of Theorem 5.1.*

Proof. Since the vector field Z_{32} is ruled out from our discussion (given that its separatrix is singular), it follows that the index of each separatrix of the vector fields in the Main Theorem is strictly negative.

Now suppose that $p_1 \in \mathcal{P}^{-1}(0)$ is a singularity at which the eigenvalues of $\mathcal{F}$ vanish. Clearly the separatrizes of p_1 are all contained in $\mathcal{P}^{-1}(0)$. Again our Main Theorem implies that the model of X around p_1 is X_{112}, X_{123}, Z_{112}, Z_{123}, or of Type IV. However, since the components of $\mathcal{P}^{-1}(0)$ have self-intersection -2, the fact that indices are strictly negative shows that the indices of $\mathcal{F}$ with respect to its separatrizes at p_1 cannot be less than -2. Hence X can only be the models X_{111} or Z_{112}. On the other hand, notice that the models X_{111} and Z_{112} have all separatrizes of index -2. In particular p_1 must be the unique singularity of the components of $\mathcal{P}^{-1}(0)$ passing through p_1. Therefore, $\mathcal{P}^{-1}(0)$ is constituted by these components (i.e., $\mathcal{P}^{-1}(0)$ is the union of the global separatrizes of $\mathcal{F}$ at p_1). If the form of X around p_1 is X_{111} then we obtain Type IV of Theorem 5.1. The other possibility leads to Type III. The lemma is proved. □

Thanks to Lemma 5.7, we suppose in the sequel that all the singularities of $\mathcal{F}$ have nontrivial eigenvalues. Since their quotient is negative rational, $\mathcal{F}$ will be given around a singularity by a vector field of the form $mx\partial/\partial x - ny\partial/\partial y$ (in suitable coordinates).

We shall need an extension of Lemma 3.5. Suppose that M is a complex surface and that $\mathrm{C} \subset M$ is a smooth rational curve of negative self-intersection. Suppose in addition that X is a meromorphic semi-complete vector field defined on a neighborhood of C whose associated foliation $\mathcal{F}$ leaves C invariant.

Denote by $p_1, \dots, p_k$ the singularities of $\mathcal{F}$ in C. Around each p_i, we assume the existence of local coordinates (x_i, t_i) ($p_i \simeq (0,0)$, $\{x_i = 0\} \subset \mathrm{C}$), where X becomes $X = m_i x_i \partial/\partial x_i - n_i t_i \partial/\partial t_i$ with $m, n \in \mathbb{N}$. Then we have the following lemma.

LEMMA 5.8. *Assume that X, $\mathcal{F}$, and C are as before. Then $\sum_{i=1}^{k}(1 - \varepsilon_i/n_i) = 2$ for appropriate integers $\varepsilon_i \in \{-1, 0, 1\}$.*

In other words, Lemma 5.8 extends Lemma 3.5 for smooth rational curves of arbitrary negative self-intersection. The idea of the proof is to reduce the statement to the case of Lemma 3.5 in which the self-intersection of $\mathcal{C}$ is -1. Let us first make a few comments regarding the proof. We want to induce from X, $\mathcal{F}$ a semi-complete vector field defined on a neighborhood of $\pi^{-1}(0) \subset \tilde{\mathbb{C}}^2$ (i.e., a smooth rational curve of self-intersection -1). In order to do this, it is not *a priori* sufficient to have X defined on a neighborhood of $\mathcal{C}$; rather, we shall need to have X defined "globally" (on a vector bundle) so as to use a well-known birational map.

Proof of Lemma 5.8. Let $-n$ be the self-intersection of $\mathcal{C}$. We denote by $N\mathcal{C}(n)$ the normal bundle of $\mathcal{C}$, which is uniquely determined by its Chern class because it is a line bundle over a rational curve. Moreover, a theorem due to Grauert (see [A]) ensures that a neighborhood of $\mathcal{C}$ in M is holomorphically equivalent to a neighborhood of the *zero-section* in $N\mathcal{C}(n)$. Hence we identify the zero-section of $N\mathcal{C}$ with $\mathcal{C}$ itself and consider a neighborhood $V \in N\mathcal{C}(n)$ of $\mathcal{C}$ equipped with the semi-complete vector field X. Also note that $N\mathcal{C}(n)$ can be compactified into a $\mathbb{CP}(1)$ bundle over $\mathbb{CP}(1)$ (a "projective line bundle") by adding the "section at infinity". The result of this compactification is the Hirzebruch surface F_n. In particular, F_1 is the compactification of $\tilde{\mathbb{C}}^2$ viewed as a line bundle over $\pi^{-1}(0)$.

CLAIM. *Without loss of generality, we can suppose that X is a meromorphic vector field on the whole of F_n.*

Proof. The proof is similar to the argument used in Corollary 2.7. Let V be a neighborhood of $\mathcal{C}$ in $N\mathcal{C}(n)$ where X is semi-complete. We consider the automorphism $\Lambda_r : N\mathcal{C}(n) \to N\mathcal{C}(n)$ consisting of multiplying vectors on the fibers of $N\mathcal{C}(n)$ by $1/2^r$, $r \in \mathbb{N}$. In coordinates (x, y) for $\{y = 0\} \subset \mathcal{C}$, the vector field X is given as $y^a h(x, y)Y$, where $a \in \mathbb{Z}$, $\{y = 0\}$ does not constitute a divisor of zeros/poles of h, and Y is a holomorphic vector field with isolated singularities.

The vector field $X_r = 2^{ra}\Lambda_r^* X$ is semi-complete on $\Lambda_r^{-1}(V)$. Using Proposition 2.6, we conclude that X_r converges to a vector field defined and semi-complete on $N\mathcal{C}$ when r goes to infinity. From the preceding construction, it is clear that this limit is a vector field with a meromorphic extension to F_n fulfilling the required conditions. □

Recall that F_1 and F_n are birationally equivalent. Since X is now defined on the whole of F_n, it induces a vector field $\tilde{X}_1$ on F_1. Denote by $\tilde{\mathcal{F}}_1$ the foliation associated to $\tilde{X}_1$. The foliation $\tilde{\mathcal{F}}_1$ has $k + n - 1$ singularities on $\pi^{-1}(0)$, denoted by $\{\tilde{p}_1, \dots, \tilde{p}_k, q_1, \dots, q_{n-1}\}$. The singularities $\tilde{p}_i$ of $\tilde{X}_1$ are equal to the singularities p_i of the original vector field X. On the other hand, the singularities q_j are introduced by the birational equivalence between F_1 and F_n. They are of *radial* type. More precisely, there are coordinates (x_j, t_j) ($\{x_j = 0\} \subset \pi^{-1}(0)$) on a neighborhood of q_j in which $\tilde{X}_1$ becomes

$$\tilde{X}_1 = (x_j/t_j)^{(\mathrm{ord}_{\pi^{-1}(0)}\tilde{X}_1)} t_j^{-1} h_j (x_j \partial/\partial x_j + t_j \partial/\partial t_j) \quad \text{with } h_j(0,0) \neq 0.$$

The rest of the proof is now completely analogous to the proof of Lemma 3.5. □

Combining Lemma 5.8 with the index formula (stating that $\sum_{i=1}^{k}(m_i/n_i)$ is equal to the the absolute value of the self-intersection of C), we immediately derive our next lemma.

LEMMA 5.9. *Let M, X, $\mathcal{F}$, C be as in the statement of Lemma 5.8. Suppose that the self-intersection of C is -2. Then $k \le 4$. Besides, if $k = 4$ (resp. 3) then the 4-tuple (resp. 3-tuple) formed by the ratios of the eigenvalues of the corresponding singularities in C is one of the following*:

4sing. $(1, 1/3, 1/3, 1/3)$, $(1, 1/2, 1/4, 1/4)$, $(1, 1/2, 1/3, 1/6)$, *or* $(1/2, 1/2, 1/2, 1/2)$;

3sing. $(1, 1/2, 1/2)$, $(4/3, 1/3, 1/3)$, $(2/3, 2/3, 2/3)$, $(3/2, 1/4, 1/4)$, $(5/4, 1/2, 1/4)$, $(3/4, 3/4, 1/2)$, $(3/2, 1/3, 1/6)$, $(1/2, 4/3, 1/6)$, $(1/2, 1/3, 7/6)$, *or* $(1/2, 2/3, 5/6)$.

Proof. Denote by $k \in \mathbb{N}$ the number of the singularities in question and suppose that $k \ge 2$. Also consider the equations $\sum_{i=1}^{k}(1 - \varepsilon_i/n_i) = 2$ and $\sum_{i=1}^{k}(m_i/n_i)$.

We first deal with the case $n_1 = 1$. It immediately follows that $m_1 = 1$ and $\varepsilon_1 = 0, 1$. The case $\varepsilon_1 = 1$ reduces to the cases treated in the proof of Proposition 3.2. We see that there are at most four singularities whose eigenvalues are the first three 4-tuples in 4sing. The case $\varepsilon_1 = 0$ is also easy to treat and leads to one or two additional singularities; when this total is three singularities, one has $n_2 = n_3 = 2$ and $m_2 = m_3 = 1$ and so the case $\varepsilon_1 = 0$ belongs to the list of 3sing.

Now we can suppose that $n_i \ge 2$ for all i. Then $1 - \varepsilon_i/n_i \le 1/2$ and thus $k \le 4$. For $k = 4$, we have $n_i = 2$ and $m_i = 1$ for all $i \in \{1, 2, 3, 4\}$.

Finally we consider $k = 3$ and $n_i \ge 2$. Solutions for n_i verifying

$$\sum_{i=1}^{3}\left(1 - \frac{\varepsilon_i}{n_i}\right) = 2$$

are as in Proposition 3.2—namely, $(3, 3, 3)$, $(2, 4, 4)$, and $(2, 3, 6)$. It is now clear how to work out all the solutions in 3sing. The lemma is proved. □

Proof of Theorem 5.1. We keep the preceding notations. We can suppose without loss of generality that all singularities of $\mathcal{F}$ have nonvanishing eigenvalues. Also, $\mathcal{P}^{-1}(0) = \sum_{i=1}^{s} \mathsf{m}_i C_i$ does not contain smooth rational curves of self-intersection -1. In fact, all the components C_i of $\mathcal{P}^{-1}(0)$ are smooth rational curves of self-intersection -2.

The following remark will often be used in the sequel: consider $p \in C_i \cap C_j \subseteq \operatorname{Sing}\mathcal{F}$. If the index of $\mathcal{F}$ w.r.t. C_i at p is λ ($\lambda \in \mathbb{Q}_-$), then the index of $\mathcal{F}$ w.r.t. C_j at p is λ^{-1}. In particular, both λ and λ^{-1} must belong to $[-2, 0)$ because the self-intersection of C_i is -2 for every $i = 1, \dots, r$. Therefore Lemma 5.9 implies that if some C_i contains four singularities then the ratios are $(1/2, 1/2, 1/2, 1/2)$.

It is clear that the resulting fiber is a degenerate case of $\tilde{D}_n$ (which is also called I_n^*); namely, we have only one A-curve and four B-curves.

Hence we can suppose that each C_i $(i = 1, \dots, r)$ contains at most three singularities of $\mathcal{F}$. Furthermore, if some C_{i_0} effectively contains three singularities then the corresponding ratios of their eigenvalues are in the list

$$(1, 1/2, 1/2),\ (2/3, 2/3, 2/3),\ (3/4, 3/4, 1/2),\ (1/2, 2/3, 5/6). \qquad (10)$$

Let us examine the case when C_1 contains three singularities corresponding to $(1, 1/2, 1/2)$. This will lead us to the fiber $\tilde{D}_n$ (I_n^*). The other three cases follow the same pattern and are left to the reader (they lead respectively to the fibers $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$).

So denote by p_2, p_3, p_4 the singularities of $\mathcal{F}$ on C_1 whose respective eigenvalues are $1, -1$ and $1, -2$ and $1, -2$. Denote by C_2 (resp. C_3, C_4) the component of $\mathcal{P}^{-1}(0)$ containing the separatrix of p_2 (resp. p_3, p_4) that is transverse to C_1. The index of $\mathcal{F}$ w.r.t. C_3 (resp. C_4) at p_3 (resp. p_4) is -2, so C_3, C_4 does not contain other singularities (i.e., $\mathcal{P}^{-1}(0)$ "ends" at C_3, C_4). Nonetheless, the index of $\mathcal{F}$ w.r.t. C_2 at p_2 is -1 and hence C_2 must contain additional singularities. One has two possibilities: C_2 contains either two or three singularities.

If C_2 contains three singularities then these are as in $(1, 1/2, 1/2)$. Therefore $\mathcal{P}^{-1}(0)$ will "end" in the obvious way (it will consist of six components arranged as in the Type $\tilde{D}_n$). If C_2 contains only two singularities then the other singularity, denoted by p_5, also has eigenvalues $1, -1$. Denote by C_5 the component of $\mathcal{P}^{-1}(0)$ containing the separatrix of p_5 transverse to C_2. The index of $\mathcal{F}$ w.r.t. C_5 at p_5 is still -1 and hence C_5 must contain additional singularities. The preceding alternative is verified again; that is, C_5 contains either two or three singularities. When C_5 contains three singularities, they are as in $(1, 1/2, 1/2)$ and so $\mathcal{P}^{-1}(0)$ ends. Otherwise C_5 contains only two singularities and we continue inductively. Since $\mathcal{P}^{-1}(0)$ is compact, at some point we will meet a rational curve containing three singularities so that $\mathcal{P}^{-1}(0)$ will end. Clearly the resulting singular fiber is of Type $\tilde{D}_n$ (I_n^*).

Finally, it remains to analyze the case where all components of $\mathcal{P}^{-1}(0)$ contain only one or two singularities. This is rather easy and promptly leads to fibers of Type $\tilde{A}_n$ (also called I_n). The proof of the theorem is complete. □

References

[A] V. Arnold, *Chapitres supplémentaires de la théorie des équations différentielles ordinaires,* Mir, Moscow, 1980.

[CSa] C. Camacho and P. Sad, *Invariant varieties through singularities of holomorphic vector fields,* Ann. of Math. (2) 115 (1982), 579–595.

[CeM] D. Cerveau and J.-F. Mattei, *Formes intégrables holomorphes singulières,* Astérisque 97 (1982).

[CeSc] D. Cerveau and B. Scardua, *Complete polynomial vector fields in two complex variables,* preprint, Université de Rennes I.

[D] H. Dulac, *Recherches sur les points singuliers des équations différentielles,* J. École Polytechnique 9 (1904), 1–125.

[F] R. Friedman, *Algebraic surfaces and holomorphic vector bundles,* Springer-Verlag, New York, 1998.

[GR] E. Ghys and J. C. Rebelo, *Singularités des flots holomorphes II,* Ann. Inst. Fourier (Grenoble) 47 (1997), 1117–1174.

[GrH] P. Griffiths and J. Harris, *Principles of algebraic geometry,* Wiley, New York, 1978.

[K] K. Kodaira, *On compact analytic surfaces II,* Ann. of Math. (2) 77 (1963), 563–626.

[L] F. Loray, *Feuilletages holomorphes à holonomie résoluble,* Ph.D. thèse, Univ. Rennes I, 1994.

[MMo] J.-F. Mattei and R. Moussu, *Holonomie et intégrales premières,* Ann. Sci. École Norm. Sup. (4) 13 (1980), 469–523.

[R1] J. C. Rebelo, *Singularités des flots holomorphes,* Ann. Inst. Fourier (Grenoble) 46 (1996), 411–428.

[R2] ———, *Champs complets avec singularités non isolées sur les surfaces complexes,* Bol. Soc. Mat. Mexicana (3) 5 (1999), 359–395.

[S] A. Seidenberg, *Reduction of singularities of the differentiable equation* $A\,dy = B\,dx$, Amer. J. Math. 90 (1968), 248–269.

[Su] T. Suwa, *Indices of vector fields and residues of singular holomorphic foliations,* Hermann, Paris, 1998.

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