Michigan Math. J. 52 (2004)

Stable Commutator Length of a Dehn Twist

Mustafa Korkmaz

0. Introduction

The purpose of this paper is to prove that, in the mapping class group of an orientable surface, the stable commutator length of a Dehn twist about a simple closed curve not bounding a disc with punctures is positive for every g. We also give an upper bound for this length. The result gives an asymptotic estimate to Problem 2.13$(B)(C)(D)$ in Kirby's problem book [8].

The upper bound for the stable commutator length of a Dehn twist is given based on known results. In the nonseparating case, however, we derive a better upper bound by proving that the tenth power of such a Dehn twist is a product of two commutators, which is an interesting result itself.

It was shown by Farb, Lubotzky, and Minsky [6] that the growth rate of a Dehn twist on an orientable surface of genus at least 1 is linear, answering a question of Ivanov (cf. [8, Prob. 2.16]). As an application of our main result we give a new proof of this fact by extending it to the genus-0 case.

The positivity of the stable commutator length of a Dehn twist about a separating simple closed curve was proved by Endo and Kotschick [5]. They concluded from this that the mapping class groups are not uniformly perfect and that the natural map from the second bounded cohomology to the ordinary cohomology of the mapping class group is not injective, which verified two conjectures of Morita (cf. [14, Conj. 6.19, 6.21]). Endo and Kotschick use Seiberg–Witten theory in the proof of their main result.

The main idea of our proof of the main result in this paper is to use the handlebody decomposition of a 4-manifold admitting a Lefschetz fibration. This was suggested to the author by Stipsicz for a signature computation in [9]. In the computation we use the symplectic Parshin–Arakelov inequality of Li [12] to prove the nonexistence of certain Lefschetz fibrations.

Donaldson [3] proved that every symplectic 4-manifold admits a Lefschetz fibration after perhaps blowing up. Conversely, Gompf [7] showed that the total space of every genus-g Lefschetz fibration admits a symplectic structure provided $g \geq 2$. This gives a combinatorial approach to symplectic 4-manifolds through certain relations in mapping class groups. But understanding the relations in mapping class groups is not so easy. Usually, information in mapping class groups

Received August 20, 2002. Revision received October 16, 2002.

gives information about the corresponding 4-manifolds. Examples of such applications are given in [4; 9; 10]. The present paper, however, gives an application in the reverse direction.

This paper has grown from a question of András Stipsicz, who asked the author whether the mapping class groups were uniformly perfect. The author thanks him and Dieter Kotschick for their comments on the content of this paper and also thanks the referee for making helpful suggestions.

1. Preliminaries

For a compact orientable surface S of genus g with p marked points (which we call punctures) and q boundary components, we denote by $\mathrm{Mod}_{g,p}^q$ the mapping class group of S, the group of isotopy classes of orientation-preserving diffeomorphisms $S \to S$ that restrict to the identity on the boundary and preserve the set of punctures. The isotopies are also assumed to be the identity on the boundary and punctures. If p and/or q is zero then we omit it from the notation, so that, for example, Mod_g denotes $\mathrm{Mod}_{g,0}^0$.

Let S be an oriented surface. A simple closed curve a on S is called *trivial* if it bounds either a disc or a disc with one puncture. For every simple closed curve a on S, there is a well-known diffeomorphism called the (right) Dehn twist about a, denoted by t_a, obtained by cutting the surface along a and twisting one of the sides to the right by 2π and then gluing the two sides back. A diffeomorphism and its isotopy class are denoted by the same symbol, and similarly for simple closed curves. For a Dehn twist t_a, we always assume that a is nontrivial as the Dehn twist about a trivial simple closed curve is itself trivial in the mapping class group.

We first state the next lemma, which is elementary and will be used in the sequel. A proof of it can be found in [15, Thm. F, p. xiii].

LEMMA 1.1. *Let $\{a_n\}$ be a sequence of real numbers with nonnegative terms such that $a_{n+m} \leq a_n + a_m$ for every n and m. Then the limit $\lim_{n\to\infty}(a_n/n)$ exists.*

For a group G let $[G, G]$ denote the commutator subgroup, the subgroup of G generated by all commutators $[a, b] = aba^{-1}b^{-1}$ for $a, b \in G$. For $x \in [G, G]$, we define the commutator length $c(x)$ of x to be the minimum number of factors needed to express x as a product of commutators. Clearly, $c(x^{n+m}) \leq c(x^n) + c(x^m)$. Therefore, we can define

$$\|x\| = \lim_{n\to\infty} \frac{c(x^n)}{n},$$

which is called the *stable commutator length of x*.

Recall that for a group G, the first homology group $H_1(G)$ of G with integer coefficients is isomorphic to the derived quotient group $G/[G, G]$.

The next theorem and its corollary will be useful to us.

THEOREM 1.2 [2]. *Let G be a group and let $u, v \in G$. Then $[u, v]^k$ can be written as a product of $E\left(\frac{k}{2}\right) + 1$ commutators, where $E\left(\frac{k}{2}\right)$ denotes the integer part of $\frac{k}{2}$.*

COROLLARY 1.3 [1]. *Let G be a group and let $u_1, v_1, u_2, v_2, \ldots, u_r, v_r$ be elements of G. Then $([u_1, v_1][u_2, v_2] \cdots [u_r, v_r])^k$ can be written as a product of $k(r-1) + E\left(\frac{k}{2}\right) + 1$ commutators.*

Proof. The proof follows from $(uv)^k = (uvu^{-1})(u^2vu^{-2}) \cdots (u^kvu^{-k})u^k$ and Theorem 1.2. □

2. The Main Result

It is well known that the mapping class group Mod_g is perfect when $g \geq 3$ (cf. [16]). Hence, every element, in particular each Dehn twist, is a product of commutators. In the case of $g = 2$, $H_1(\mathrm{Mod}_2)$ is isomorphic to the cyclic group of order 10 and is generated by the class of any Dehn twist about a nonseparating simple closed curve. A Dehn twist t_a in Mod_2 is not contained in the commutator subgroup, but t_a^{10} is. Hence, we can talk about $\|t_a^{10}\|$ in this case.

Our main result is the following theorem.

THEOREM 2.1. *Let S be a closed connected oriented surface of genus $g \geq 2$ and let a be a nontrivial simple closed curve on S. Then $\|t_a\| \geq \frac{1}{18g-6}$ if $g \geq 3$ and $\|t_a^{10}\| \geq \frac{1}{3}$ if $g = 2$.*

Proof. Suppose first that $g \geq 3$. We assume to the contrary that $\|t_a\| < \frac{1}{18g-6}$. Choose a rational number r with $\|t_a\| < r < \frac{1}{18g-6}$. Then there exists an arbitrarily large positive integer n such that rn is an integer and t_a^n can be written as a product of rn commutators. This gives a relatively minimal genus-g Lefschetz fibration over a closed orientable surface Σ of genus rn with the vanishing cycle a repeated n times as follows (we refer the reader to [7] for the details of the theory of Lefschetz fibrations). Consider $D^2 \times S$, where D^2 is the 2-disc. Attach n 2-handles to $\partial D^2 \times S$ along the simple closed curve a with -1 framing relative to the product framing. This gives a relatively minimal genus-g Lefschetz fibration $X_1 \to D^2$ with monodromy t_a^n along the boundary ∂D^2. Since t_a^n is a product of rn commutators, there is a surface bundle X_2 with fibers S over an orientable surface of genus rn with one boundary component such that the monodromy along the boundary is t_a^n. The boundary of X_1 and X_2 are genus-g surface bundles over S^1 with monodromy t_a^n. Now glue X_1 and X_2 via a fiber-preserving, orientation-reversing diffeomorphism between boundaries to obtain a relatively minimal Lefschetz fibration $X \to \Sigma$ over a closed connected orientable surface Σ with the generic fiber S.

The Euler characteristic of X is easily computed to be

$$\chi(X) = 4(g-1)(rn-1) + n = 4grn - 4rn - 4g + 4 + n.$$

Also, it follows from the construction that $b_1(X) \leq 2g + 2rn$.

We now give a lower bound for $b_2^-(X)$. For each $i = 1, 2, \ldots, n-1$, the cores of the ith and $(i+1)$th 2-handles attached along a give a sphere S_i whose self-intersection is -2. If $[S_i]$ denotes the homology class in $H_2(X; \mathbb{R})$ of S_i, then

$[S_i][S_{i+1}] = \pm 1$ and $[S_i][S_j] = 0$ for $|i - j| \geq 2$, since the attaching regions of 2-handles can be chosen to be disjoint. We can orient S_i so that $[S_i][S_{i+1}] = -1$. It follows that the homology classes $[S_1], \dots, [S_{n-1}]$ are linearly independent and form a basis for an $(n-1)$-dimensional subspace V of $H_2(X; \mathbb{R})$. Hence, the matrix of the intersection form restricted to V in this basis is the matrix $-A$, where

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 2 & 1 & \cdots & 0 & 0 \\ . & . & . & . & \cdots & . & . \\ 0 & 0 & 0 & 0 & \cdots & 2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 2 \end{pmatrix}.$$

It is easy to check that the matrix A is positive definite. Therefore, the restriction of the intersection form to V is negative definite. It follows that

$$b_2^-(X) \geq n - 1.$$

An easy computation gives an upper bound for $b_2^+(X)$:

$$\begin{aligned} n - 1 + b_2^+(X) &\leq b_2(X) \\ &= \chi(X) + 2b_1(X) - 2 \\ &\leq 4grn - 4rn - 4g + 4 + n + 2(2g + 2rn) - 2 \\ &\leq 4grn + n + 2. \end{aligned}$$

Hence,

$$b_2^+(X) \leq 4grn + 3.$$

From these inequalities we obtain an upper bound for the signature of X,

$$\sigma(X) \leq 4grn - n + 4.$$

On the other hand, following an argument of Kotschick [11], Li proved in [12] that

$$2(g-1)(rn-1) \leq c_1^2(X).$$

Hence, we obtain

$$\begin{aligned} 2(g-1)(rn-1) &\leq c_1^2(X) \\ &= 3\sigma(X) + 2\chi(X) \\ &\leq 3(4grn - n + 4) + 2(4grn - 4rn - 4g + 4 + n) \\ &= 20grn - 8rn - n + 20 - 8g. \end{aligned}$$

As a result, we conclude that there exists an arbitrarily large n such that

$$0 \leq [(18g-6)r - 1]n + 18 - 6g.$$

Since $(18g-6)r - 1$ is negative, this is a contradiction and so proves the theorem for $g \geq 3$.

If S is a closed surface of genus 2, then repeating the foregoing proof by taking n as a multiple of 10 completes the proof. $\square$

REMARK 2.2. I had originally proved that the stable commutator length of a Dehn twist in the theorem is greater than or equal to $\frac{1}{20g-8}$, using the inequality $c_1^2(X) \geq 0$ proved in [17]. The improvement was kindly suggested by Kotschick and Stipsicz.

COROLLARY 2.3. *Let S be a connected orientable surface of genus $g \geq 0$ with p punctures and q boundary components such that $g + q \geq 2$. Let a be a simple closed curve on S not bounding a disc with punctures. Suppose that t_a^k is in the commutator subgroup of* $\mathrm{Mod}_{g,p}^q$. *Then* $\|t_a^k\| > 0$.

Proof. Let us glue a torus with one boundary component along each boundary component of S. By forgetting the punctures, we get a closed surface R of genus $g + q$. The circle a is now nontrivial on R. In this way we have a map F from the mapping class group of S to that of R. Clearly, $c(t_a^k) \geq c(F(t_a)^k)$. Since $F(t_a)$ is a Dehn twist about the nontrivial simple closed curve a on R, the corollary follows from $\|t_a^k\| \geq \|F(t_a)^k\|$ and Theorem 2.1. $\square$

It was shown in [10] that the commutator length of a Dehn twist is 2. Theorem 2.1 and Theorem 1.2 give the corollary.

COROLLARY 2.4. *Let S be a closed orientable surface of genus $g \geq 2$, and let a be a nontrivial simple closed curve on S. Then the element t_a^k of* Mod_g *cannot be a commutator if $k > 9g - 3$.*

Proof. Assume that t_a^k is a commutator. Then t_a^{kn} is a product of $E\left(\frac{n}{2}\right) + 1$ commutators, that is, $c(t_a^{nk}) \leq E\left(\frac{n}{2}\right) + 1$. Dividing both sides by kn and taking the limit as n tends to infinity gives the desired contradiction $k \leq 9g - 3$. $\square$

3. An Upper Bound for $\|t_a\|$

In this section we give an upper bound for $\|t_a\|$ for a simple closed curve a. In the case where a is nonseparating, we obtain a better upper bound by proving that the tenth power t_a^{10} of a Dehn twist t_a is a product of two commutators.

The next lemma is well known.

LEMMA 3.1. *Let a and b be two simple closed curves on an oriented surface S.*

(a) *If a is disjoint from b, then t_a commutes with t_b.*
(b) *If a intersects b transversely at only one point, then $t_a t_b t_a = t_b t_a t_b$.*

LEMMA 3.2. *Let S be a connected oriented surface, and let a, b, c, d be four simple closed curves on S such that there exists an orientation-preserving diffeomorphism of S mapping a and b to d and c, respectively. Then $t_a t_b^{-1} t_c t_d^{-1}$ is a commutator.*

Proof. Let g be the isotopy class of a diffeomorphism mapping a and b to d and c. Then

$$t_a t_b^{-1} t_c t_d^{-1} = t_a t_b^{-1} t_{g(b)} t_{g(a)}^{-1} = t_a t_b^{-1} g t_b t_a^{-1} g^{-1} = [t_a t_b^{-1}, g]. \qquad \square$$

THEOREM 3.3. *Let S be a connected oriented surface of genus $g \geq 2$ and let a be a nonseparating simple closed curve on S. Then t_a^{10} can be written as a product of two commutators.*

Proof. Since Dehn twists about two nonseparating simple closed curves are conjugate and since a conjugate of a commutator is again a commutator, it suffices to prove the theorem for some nonseparating simple closed curve.

Let a_1, a_2, a_3 be three nonseparating simple closed curves on S such that a_2 intersects a_1 and a_3 transversely only once, a_1 is disjoint from a_3, and $a_1 \cup a_3$ does not disconnect S. A regular neighborhood $a_1 \cup a_2 \cup a_3$ is a torus with two boundary components, say a_4 and a_5, that are nonseparating on S. Clearly, a_4 and a_5 are disjoint from a_1, a_2, a_3 and also from each other. Let us denote t_{a_i} by t_i. It is well known that $t_4 t_5 = (t_1 t_2 t_3)^4$. Using Lemma 3.1, we obtain

$$\begin{aligned} t_4 t_5 &= (t_1 t_2 t_3)(t_1 t_2 t_3)(t_1 t_2 t_3)(t_1 t_2 t_3) \\ &= (t_1 t_2 t_1)(t_3 t_2 t_3)(t_1 t_2 t_1)(t_3 t_2 t_3) \\ &= (t_2 t_1 t_2)(t_2 t_3 t_2)(t_2 t_1 t_2)(t_2 t_3 t_2). \end{aligned}$$

Conjugating with t_2^{-1} yields

$$\begin{aligned} t_4 t_5 &= t_1 t_2 t_2 t_3 t_2 t_2 t_1 t_2 t_2 t_3 t_2 t_2 \\ &= t_1 (t_2^2 t_3 t_2^{-2}) t_2^4 t_1 t_2^{-1} (t_2^3 t_3 t_2^{-3}) t_2^{-1} t_2^6. \end{aligned}$$

If we let $\alpha = t_2^2(a_3)$ and $\beta = t_2^3(a_3)$ then we obtain the equality

$$(t_4 t_\alpha^{-1} t_5 t_1^{-1}) = t_2^4 (t_1 t_2^{-1} t_\beta t_2^{-1}) t_2^6.$$

The curves a_4 and a_5 do not intersect α and a_1. Because the complements of $a_4 \cup \alpha$ and $a_5 \cup a_1$ are connected, there is an orientation-preserving diffeomorphism taking a_4 and α to a_1 and a_5, respectively. The curve a_2 intersects a_1 and β transversely at one point. Hence, there is an orientation-preserving diffeomorphism mapping a_1 and a_2 to a_2 and β, respectively. By Lemma 3.2, each parenthesis is a commutator. Since the conjugate of a commutator is again a commutator, the proof is complete. $\square$

THEOREM 3.4. *Let S be a connected oriented surface of genus $g \geq 2$ and let a be a simple closed curve on S.*

(a) *If a is nonseparating, then $\|t_a\| \leq \frac{3}{20}$ when $g \geq 3$ and $\|t_a^{10}\| \leq \frac{3}{2}$ when $g = 2$.*
(b) *If a is separating, then $\|t_a\| \leq \frac{3}{4}$ when $g \geq 3$.*

Proof. If a is nonseparating then, by Theorem 3.3 and Corollary 1.3, the element t_a^{10n} can be written as a product of $n + E\left(\frac{n}{2}\right) + 1$ commutators. Part (a) follows from this.

For (b), if a is separating then t_a^2 is a product of two commutators [10]. It follows now from Corollary 1.3 that $\|t_a\| \leq \frac{3}{4}$. □

REMARK 3.5. In the case of an oriented surface of genus 2, if a is a separating simple closed curve such that one of the components of the complement of a has genus $g \geq 2$, then it is easy to conclude from the proof of [10, Prop. 10] that t_a^2 can be written as a product of two commutators. It follows that $\|t_a\| \leq \frac{3}{4}$.

If the genera of both components of the complement of a are 1, then t_a^5 is in the commutator subgroup of Mod_2. There are two nonseparating simple closed curves b, c on S intersecting each other at one point such that $t_a = (t_b t_c)^6$. It can be shown from this that t_a^5 is a product of 21 commutators. It follows that $\|t_a^5\| \leq \frac{41}{2}$.

4. An Application: Growth Rate of Dehn Twists

It is well known that the mapping class groups are finitely presented (cf. [18]). For a finite generating set A of $\mathrm{Mod}_{g,p}^q$, let d denote the corresponding word metric on $\mathrm{Mod}_{g,p}^q$. That is, if $f, g \in \mathrm{Mod}_{g,p}^q$ then

$$d(f, g) = \min\{n \mid g^{-1}f = x_1 x_2 \cdots x_n,\ x_i \in A\}.$$

In [6], Farb, Lubotzky, and Minsky proved that if $g \geq 1$ then Dehn twists in the mapping class group $\mathrm{Mod}_{g,p}^q$ have linear growth rate. That is, the limit

$$\lim_{n\to\infty} \frac{d(t_a^n, 1)}{n}$$

is positive. Here, we give another proof of this fact and extend it to the genus-0 case. Note that this limit depends on the choice of the generating set, but the positivity of it does not.

THEOREM 4.1. *Let S be a connected oriented surface of genus g with p punctures and q boundary components. Suppose that $g + q \geq 2$ and $(g, q) \neq (2, 0)$. If a is a simple closed curve on S not bounding a disc with punctures, then for any finite generating set of the mapping class group the limit*

$$\lim_{n\to\infty} \frac{d(t_a^n, 1)}{n}$$

is positive; *that is, the growth rate of $d(t_a^n, 1)$ is linear.*

Proof. Suppose first that S is a closed surface so that $g \geq 3$. Let A be a finite generating set for the mapping class group Mod_g. Since Mod_g is perfect, every element in Mod_g is a product of commutators. Let k be a positive integer such that each element of A can be written as a product of k commutators. Thus, t_a^n can be written as a product of $kd(t_a^n, 1)$ commutators for any positive integer n. Hence $c(t_a^n) \leq kd(t_a^n, 1)$. It follows that

$$\lim_{n\to\infty} \frac{d(t_a^n, 1)}{n} \geq \frac{1}{k}\|t_a\|$$

and so, by Theorem 2.1, the limit is positive.

In the general case, let us glue a surface of genus 2 with one boundary to S along each boundary component of S and forget the punctures. Let A be a finite generating set for $\mathrm{Mod}_{g,p}^q$. As in the proof of Corollary 2.3, this gives a homomorphism $F\colon \mathrm{Mod}_{g,p}^q \to \mathrm{Mod}_{g+q}$. Extend $F(A)$ to a finite generating set B for Mod_{g+q}. Clearly, we have $d(t_a^n, 1) \geq d(F(t_a)^n, 1)$ with respect to the generating sets A and B. Since $F(t_a)$ is a Dehn twist about a nontrivial simple closed curve on a closed surface, the proof follows from the closed case. □

5. A Question

We end with a question, which arises from the topology of the Stein fillings of a contact 3-manifold. Let S be an oriented surface with one boundary component, and let $a_1, a_2, \ldots$ be a sequence of nonseparating simple closed curves on S. We ask whether there exists a positive integer N such that

$$c(t_{a_1} t_{a_2} \cdots t_{a_n}) \geq Nn.$$

More generally, does the limit

$$\lim_{n\to\infty} \frac{c(t_{a_1} t_{a_2} \cdots t_{a_n})}{n}$$

exist? If it does, is it positive (for any choice of a_i)? Note that, if all a_n are equal, then we have already proved that there exists such an N.

References

[1] C. Bavard, *Longueur stable des commutateurs,* Enseign. Math. (2) 37 (1991), 109–150.

[2] M. Culler, *Using surfaces to solve equations in free groups,* Topology 20 (1981), 133–145.

[3] S. Donaldson, *Lefschetz fibrations in symplectic geometry,* Proc. Internat. Cong. Math. (Berlin, 1998), Doc. Math., Extra Vol. II (1998), 309–314.

[4] H. Endo, M. Korkmaz, D. Kotschick, B. Ozbagci, and A. I. Stipsicz, *Commutators, Lefschetz fibrations and the signatures of surface bundles,* Topology 41 (2002), 961–977.

[5] H. Endo and D. Kotschick, *Bounded cohomology and non-uniform perfection of mapping class groups,* Invent. Math. 144 (2001), 169–175.

[6] B. Farb, A. Lubotzky, and Y. N. Minsky, *Rank-1 phenomena of mapping class groups,* Duke Math. J. 106 (2001), 581–597.

[7] R. E. Gompf and A. I. Stipsicz, *4-manifolds and Kirby calculus,* Grad. Stud. Math., 20, Amer. Math. Soc., Providence, RI, 1999.

[8] R. Kirby, *Problems in low-dimensional topology,* Geometric topology (Athens, GA, 1993), pp. 35–473, Amer. Math. Soc., Providence, RI, 1997.

[9] M. Korkmaz, *Noncomplex smooth 4-manifolds with Lefschetz fibrations,* Internat. Math. Res. Notices 2001 (2001), 115–128.

[10] M. Korkmaz and B. Ozbagci, *Minimal number of singular fibers in a Lefschetz fibration,* Proc. Amer. Math. Soc. 129 (2001), 1545–1549.

[11] D. Kotschick, *Signatures, monopoles and mapping class groups,* Math. Res. Lett. 5 (1998), 227–234.
[12] T. J. Li, *Symplectic Parshin–Arakelov inequality,* Internat. Math. Res. Notices 2000 (2000), 941–954.
[13] S. Matsumoto and S. Morita, *Bounded cohomology of certain groups of homeomorphisms,* Proc. Amer. Math. Soc. 94 (1985), 539–544.
[14] S. Morita, *Structure of the mapping class groups of surfaces: A survey and a prospect,* Proceedings of the Kirbyfest (Berkeley, 1998), Geom. Topol. Mongr., 2, pp. 349–406, Geom. Topol. Publ., Coventry, 1999.
[15] M. Pollicott and M. Yuri, *Dynamical systems and ergodic theory,* London Math. Soc. Stud. Texts, 40, Cambridge Univ. Press, Cambridge, U.K., 1998.
[16] J. Powell, *Two theorems on the mapping class group of a surface,* Proc. Amer. Math. Soc. 68 (1978), 347–350.
[17] A. I. Stipsicz, *Chern numbers of certain Lefschetz fibrations,* Proc. Amer. Math. Soc. 128 (2000), 1845–1851.
[18] B. Wajnryb, *An elementary approach to the mapping class group of a surface,* Geom. Topol. 3 (1999), 405–466.

Department of Mathematics
Middle East Technical University
06531 Ankara
Turkey

korkmaz@arf.math.metu.edu.tr