Michigan Math. J. 52 (2004)

Fatou Sets for Rational Maps of $\mathbb{P}^k$

Kazutoshi Maegawa

1. Introduction

In this paper we study the dynamics of a rational self-map of $\mathbb{P}^k$. We deal with the Fatou set and give a rough classification of the dynamics on the Fatou components for a rational map whose indeterminacy set is nonempty. Our result is just the first step toward a complete classification of Fatou components, but it shows us a new dynamical phenomenon.

For an algebraically stable (AS) rational self-map f of degree ≥ 2 of $\mathbb{P}^k$, the Fatou set is defined to be the set of all the Lyapunov stable points for f. The connected components of the Fatou set are called Fatou components. The indeterminacy set I (and, moreover, the extended indeterminacy set E) are automatically disjoint from the Fatou set. Let $\operatorname{Orb}(x)$ denote the forward orbit of $x \in \mathbb{P}^k$. The main purpose of this paper is to show the following theorem. Keep in mind that it is not a direct consequence of the definition of the Fatou set because $\operatorname{codim}_{\mathbb{C}} I \geq 2$.

Theorem 1.1. *Let f be an AS rational self-map of degree ≥ 2 of $\mathbb{P}^k$, and let U be a Fatou component for f. If there exists at least one point $x \in U$ such that $\overline{\operatorname{Orb}(x)} \cap I \neq \emptyset$, then $\overline{\operatorname{Orb}(y)} \cap I \neq \emptyset$ for any $y \in U$.*

From this theorem it follows that there are two types of Fatou components in terms of the relationship between their respective limit maps and I. We call a Fatou component that contains no point whose forward orbit accumulates at I a *regular* Fatou component. We will show that any regular Fatou component is Stein. Further, we will give a formula that relates the Green $(1, 1)$ current to the union of all regular Fatou components. This is an improvement on the theorem of Fornæss and Sibony in [FS].

Acknowledgment. This study was done as part of the author's dissertation. He thanks his advisor, Professor T. Ueda, for useful conversations as well as Professor J. E. Fornæss for his hospitality while the author visited the University of Michigan during the Fred and Lois Gehring special year in complex analysis.

Received May 28, 2002. Revision received May 6, 2003.
Partially supported by JSPS research fellowships for young scientists.

2. Preliminaries

In this section we lay the groundwork for the results of Section 3. We begin by recalling some definitions and basic facts concerning rational self-maps of $\mathbb{P}^k$.

Let F be a polynomial self-map of $\mathbb{C}^{k+1}$ such that all the component polynomials are homogeneous, have the same degree, and have no common factor. Let O be the origin of $\mathbb{C}^{k+1}$. Because F maps any complex line through O that is not contained in $F^{-1}(O)$ onto a complex line through O, it follows that the map F induces a map f from $\mathbb{P}^k \setminus \pi(F^{-1}(O))$ to $\mathbb{P}^k$ such that $\pi \circ F = f \circ \pi$ in $\mathbb{C}^{k+1} \setminus F^{-1}(O)$, where π is the canonical projection from $\mathbb{C}^{k+1} \setminus \{O\}$ to $\mathbb{P}^k$. Let I be the algebraic set in $\mathbb{P}^k$ given by the common zeros of all the component polynomials of F, that is, $I := \pi(F^{-1}(O))$. Then f is a holomorphic map outside I and cannot extend to be continuous at I, since $\operatorname{codim}_{\mathbb{C}} I \geq 2$. The obtained map f is called a *rational* self-map of $\mathbb{P}^k$, and F is called the *minimal defining polynomial* map for f. We call I the *indeterminacy* set for f. The degree of f is defined to be the degree of F.

DEFINITION 2.1. We set $f(p) := \bigcap_{r>0} \overline{f(U_r(p) \setminus I)}$ for $p \in I$, where $U_r(p)$ is the ball centered at p and of radius r with respect to the Fubini–Study metric. Then, $f(p)$ is an algebraic set of dimension ≥ 1.

Set $I_1 := I$ and $I_n := \pi(F^{-n}(O))$ for $n \geq 2$. Then $I_1 \subset I_2 \subset \cdots$. Let us recall the definition of algebraically stable rational self-maps. See also [S].

DEFINITION 2.2. Let f be a rational self-map of degree ≥ 2 of $\mathbb{P}^k$. We say that f is *algebraically stable* (AS) if $\operatorname{codim}_{\mathbb{C}} I_n \geq 2$ for any $n \geq 1$.

REMARK 2.3 (Iteration of AS rational self-map). Definition 2.2 is equivalent to stating that there is no common factor of all the component polynomials of F^n for each $n \geq 1$. Hence, F^n induces a rational self-map of $\mathbb{P}^k$ whose indeterminacy set is I_n $(n \geq 2)$. We denote the induced rational map by f^n $(n \geq 2)$. Obviously, such f^n are also algebraically stable. Outside I_n, the map f^n is equal to $f \circ \cdots \circ f$ (n times), where $\circ$ denotes a composition of maps in the usual sense. Let $\operatorname{Orb}(x)$ denote the forward orbit $\{x, f(x), f^2(x), \dots\}$ for $x \in \mathbb{P}^k$.

We will use the following definitions and notation concerning dynamics of AS rational self-maps of $\mathbb{P}^k$.

DEFINITION 2.4. Let f be an AS rational self-map of degree ≥ 2 of $\mathbb{P}^k$.

(a) We define the *extended indeterminacy set* E by $E := \overline{\bigcup_{n\geq 1} I_n}$.

(b) We define the *regular domain* $\mathcal{R}$ by

$$\mathcal{R} := \{p \in \mathbb{P}^k \mid p \in \exists V : \text{an open set},\ I \subset \exists W : \text{an open set} \quad \text{s.t. } f^n(V) \cap W = \emptyset\ \forall n \geq 0\}.$$

We call points in $\mathcal{R}$ *regular points* and the others *irregular* points. It is easy to show that $\mathcal{R} \subset \mathbb{P}^k \setminus E$.

(c) Usually, a Lyapunov stable point is defined to be a point at which the sequence of the iterates is equicontinuous. In the present case, however, the meaning of this is not so clear because $\dim f^n(p) \geq 1$ for $p \in I_n$. Here we define Lyapunov stability by saying that $p \in \mathbb{P}^k$ is a *Lyapunov stable point* for f if, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $\operatorname{diam} f^n(U_\delta(p)) < \varepsilon$ for all $n \geq 1$.

If $p \in E$, then p is not Lyapunov stable. Let us show this. Suppose to the contrary that $p \in E$ is Lyapunov stable. Set ε to be so small that any ball in $\mathbb{P}^k$ that is of radius $\varepsilon > 0$ is Stein. Take $\delta > 0$ for ε. There is a positive integer m such that $U_\delta(p) \cap I_m = \emptyset$, so $f^m(U_\delta(p))$ contains an algebraic set of dimension ≥ 1. Yet this is impossible, since $f^m(U_\delta(p))$ is contained in an ε-ball (recall that no algebraic set of dimension ≥ 1 has a Stein neighborhood).

We define the *Fatou set* Ω by

$$\Omega := \{p \in \mathbb{P}^k \mid p \text{ is a Lyapunov stable point}\}.$$

A connected component of Ω is called a *Fatou component*. The complement of the Fatou set is called the *Julia set* J. From the preceding consideration, $E \subset J$. By a normal family argument, it follows that Ω is an open subset in $\mathbb{P}^k$. Actually,

$$\Omega = \{p \in \mathbb{P}^k \setminus E \mid p \in \exists V : \text{an open set s.t. } \{f^n\}_{n\geq 1} \text{ is a normal family in } V\}.$$

(d) Let U be a Fatou component. We say that $g\colon U \to \mathbb{P}^k$ is the *limit map* in U if there is a subsequence $\{f^{n_j}\}$ that converges locally uniformly to g in U as $j \to \infty$.

Remark 2.5. Both $\mathcal{R}$ and Ω are backward invariant.

Remark 2.6. Regular points are also called *normal points* by several authors (see e.g. [S]). In this paper, to avoid the confusion with points at which $\{f^n\}_{n\geq 1}$ is normal, we use “regular”.

The following Propositions 2.7, 2.9, 2.12, 2.13, and 2.15 are contained in [FS] or [U]. Although the assumptions in some of them differ from the original presentation, the original proofs are still valid under our assumptions.

Let $\|\cdot\|$ denote the Euclidian norm in $\mathbb{C}^{k+1}$. Let f be an AS rational self-map of degree $d \geq 2$ of $\mathbb{P}^k$, and let F be the minimal defining polynomial map for f.

Proposition 2.7. *Let* $M := \max_{\|x\|=1}\|F(x)\|$. *Then* $M > 0$ *and*

$$\|F(x)\| \leq M\|x\|^d$$

for all $x \in \mathbb{C}^{k+1}$.

Definition 2.8. We define the *Green function* for f by

$$G(x) = \lim_{n\to\infty} \frac{1}{d^n} \log\|F^n(x)\|, \quad x \in \mathbb{C}^{k+1}.$$

Proposition 2.9. (i) $G \equiv -\infty$ *or* G *is plurisubharmonic in* $\mathbb{C}^{k+1}$ *such that* $G(F(x)) = dG(x)$ *and* $G(\lambda x) = \log|\lambda| + G(x)$ *for all* $\lambda \in \mathbb{C}$ *and all* $x \in \mathbb{C}^{k+1}$.

(ii) *If $h(x)$ is a function on $\mathbb{C}^{k+1} \setminus \{O\}$ such that $h(x) - \log\|x\|$ is bounded, then*

$$\lim_{n\to\infty} \frac{1}{d^n} h(F^n(x)) = G(x).$$

Let A be a sufficiently large positive number. Replacing $h(x)$ by $h(x) + A$, the sequence turns out to be decreasing.

REMARK 2.10. Observe that it follows from (i) that $G < 0$ in a neighborhood of O. This will be used later.

REMARK 2.11. We say that an AS rational map f is *dominating* if f is generically of maximal rank k. For such f, the Green function G is plurisubharmonic (see [S]).

When $G \not\equiv -\infty$, we denote by T the (1, 1) current in $\mathbb{P}^k$ such that $\pi^* T = dd^c G$.

PROPOSITION 2.12. *Suppose $G \not\equiv -\infty$. Then the Fatou set $\Omega \subset \mathbb{P}^k \setminus \operatorname{supp} T$.*

Proof. Let $p \in \Omega$. We can take a neighborhood V of p and a subsequence $\{f^{n_j}\}_{j\geq 1}$ that converges uniformly in V. We can assume that all $f^{n_j}(V)$ are contained in an open set Z such that (a) there is a function $h_Z(x)$ in $\mathbb{C}^{k+1} \setminus \{O\}$ that is pluriharmonic in $\pi^{-1}(Z)$ and (b) $h_Z(x) - \log\|x\|$ is bounded in $\mathbb{C}^{k+1} \setminus \{O\}$. By Proposition 2.9(ii), $\lim_{n\to\infty}(1/d^n)h_Z(F^n(x)) = G(x)$. We can assume that the sequence on the left-hand side is decreasing, so Harnack's theorem applies to it in $\pi^{-1}(V)$. As a result, $G(x)$ is pluriharmonic in $\pi^{-1}(V)$. □

Set $B_G := \{G = 0\}$.

PROPOSITION 2.13. *Let W be a neighborhood of I. Set*

$$Q := Q(W) := \{p \in \mathbb{P}^k \mid f^n(p) \in \mathbb{P}^k \setminus W \ \forall n \geq 0\},$$

$$\tilde{Q} := \pi^{-1}(Q).$$

Then $f(Q) \subset Q$, the sequence defining G converges uniformly on $\tilde{Q}$, and $B_G \cap \tilde{Q}$ is contained in $\{x \in \mathbb{C}^{k+1} \mid r < \|x\| < R\}$ for some $0 < r, R < +\infty$.

Proof. The forward invariance of Q is obvious. Set $\tilde{W} := \pi^{-1}(W)$, and let $m = \min_{\|x\|=1,\, x\in\mathbb{C}^{k+1}\setminus\tilde{W}} \|F(x)\|$. Then $m > 0$ and $\|F(x)\| \geq m\|x\|^d$ for all $x \in \mathbb{C}^{k+1}\setminus\tilde{W}$. Let $M = \max_{\|x\|=1}\|F(x)\|$. Then $M > 0$ and $\|F(x)\| \leq M\|x\|^d$ for all $x \in \mathbb{C}^{k+1}$ (Proposition 2.7). Hence, there is a constant $0 < C < +\infty$ such that

$$\left| \frac{1}{d} \log\|F^{n+1}(x)\| - \log\|F^n(x)\| \right| < C$$

for all $x \in \tilde{Q}$ and all $n \geq 0$. The uniform convergence of $\{(1/d^n)\log\|F^n(x)\|\}$ in $\tilde{Q}$ follows and so $G(x) = \log\|x\| + v(x)$ in $\tilde{Q}$, where $v(x)$ is a bounded continuous function in $\tilde{Q}$ such that $v(\lambda x) = v(x)$ for all $x \in \tilde{Q}$ and all $\lambda \in \mathbb{C}$. Thus, we can take the desired $0 < r, R < +\infty$. □

REMARK 2.14. Let p be a regular point. By definition, there exist a neighborhood V of p and a neighborhood W of I such that $f^n(V) \cap W = \emptyset$ for all $n \geq 0$. So $p \in Q(W)$.

Proposition 2.15 was stated for the Green functions of holomorphic self-maps of $\mathbb{P}^k$ by Ueda [U], although his proof doesn't concern the dynamics.

PROPOSITION 2.15. *Let P be a plurisubharmonic function in $\mathbb{C}^{k+1}$ such that $P(\lambda x) = \log|\lambda| + P(x)$ for $\lambda \in \mathbb{C}$ and $x \in \mathbb{C}^{k+1}$. Then the following statements hold.*

(i) *Let p be a point in $\mathbb{P}^k$, and let $\tilde{p}$ be a point in $\mathbb{C}^{k+1} \setminus \{O\}$ such that $\pi(\tilde{p}) = p$. Then P is pluriharmonic at $\tilde{p}$ if and only if there exist a neighborhood D of p and a holomorphic map $s\colon D \to \mathbb{C}^{k+1} \setminus O$ such that $s(D) \subset B_P := \{P = 0\}$ and $\pi \circ s = \mathrm{id}$.*

(ii) *The holomorphic map s in* (i) *is unique up to a constant factor with absolute value* 1.

3. Fatou Sets and Indeterminacy Sets

In this section we demonstrate the main theorem (Theorem 3.2).

LEMMA 3.1. *Let f be an AS rational self-map of degree ≥ 2 of $\mathbb{P}^k$, and let p be a point in the Fatou set Ω for f. Then p is a regular point if and only if $\overline{\mathrm{Orb}(p)} \cap I = \emptyset$.*

Proof. It is obvious that, by definition, the regularity implies no accumulation of the forward orbit at I. The inverse is also easy to see by considering the equicontinuity of $\{f^n\}$ at p. □

We will show a fundamental theorem about the forward orbits of points in the Fatou set.

THEOREM 3.2. *Let f be an AS rational self-map of degree ≥ 2 of $\mathbb{P}^k$, and let U be a Fatou component for f. If there exists at least one point $x \in U$ such that $\overline{\mathrm{Orb}(x)} \cap I \neq \emptyset$, then $\overline{\mathrm{Orb}(y)} \cap I \neq \emptyset$ for any $y \in U$.*

REMARK 3.3. This theorem is not a direct consequence of Lyapunov stability (i.e., the local equicontinuity of $\{f^n\}_{n\geq 1}$ in the Fatou set) because $\mathrm{codim}_{\mathbb{C}}\, I \geq 2$.

REMARK 3.4. Note that we don't assume that U is periodic. Thus, if there exists a wandering domain then it has this property.

Proof of Theorem 3.2. By Lemma 3.1 we need only prove the case $\mathcal{R} \neq \emptyset$. For this case, by Proposition 2.13 and Remark 2.14 it follows that $G \not\equiv -\infty$. Hence, by Proposition 2.12, $U \subset \mathbb{P}^k \setminus \mathrm{supp}\, T$. Suppose that the forward orbit of $p \in U$ accumulates at I. We will show that the forward orbit of any point in some neighborhood of p also accumulates at I.

By Proposition 2.15, we can take a small neighborhood D of p and a holomorphic section s such that $s(D) \subset B_G$. We can take a subsequence $\{f^{n_j}\}_{j\geq 1}$ such that $\{f^{n_j}\}_{j\geq 1}$ converges uniformly in D as $j \to \infty$ and $\{f^{n_j}(p)\}_{j\geq 1}$ converges to a point q in I as $j \to \infty$. Without loss of generality, we can assume that there is an open set W, which is a polydisk in a local chart, such that all $f^{n_j}(D)$, $j = 1, 2, \dots$, are contained in W.

Set $W^* := \{x \in \mathbb{C}^{k+1} \mid r < \|x\|\} \cap \pi^{-1}(W)$, where r is a sufficiently small positive number. We regard $\mathbb{P}^{k+1}$ as the compactification of $\mathbb{C}^{k+1}$ in the usual way. Then W^* is hyperbolically imbedded in $\mathbb{P}^{k+1}$ and $F^{n_j}(s(D)) \subset W^*$ for all $j \geq 1$ because $f^{n_j}(D) \subset W$ and $F^{n_j}(s(D)) \subset B_G \subset \{x \in \mathbb{C}^{k+1} \mid r < \|x\|\}$ for all $j \geq 1$ (recall that $G < 0$ in a neighborhood of O). Hence $\{F^{n_j} \circ s\}_{j\geq 1}$ is a normal family as a sequence of maps from D to $\mathbb{P}^{k+1}$.

Take a locally uniformly convergent subsequence $\{F^{n_{j(m)}} \circ s\}_{m\geq 1}$. By shrinking D, we assume that the convergence is uniform in D. Let ψ be the limit map of this sequence, and consider $\psi(D)$. We will check that $\psi(p) \in \mathbb{P}^{k+1} \setminus \mathbb{C}^{k+1}$. Suppose that $\psi(p) \in \mathbb{C}^{k+1}$. By $G(F^{n_{j(m)}} \circ s(p)) = 0$ for all $m \geq 1$ and by the upper semicontinuity of G, it follows that $G(\psi(p)) \geq 0$. However, $G = -\infty$ in $\pi^{-1}(q)$ and $\psi(p) \in \pi^{-1}(q)$. This is a contradiction. It therefore follows that $\psi(p) \in \mathbb{P}^{k+1} \setminus \mathbb{C}^{k+1}$ and hence $\psi^{-1}(\mathbb{P}^{k+1} \setminus \mathbb{C}^{k+1}) \neq \emptyset$. We set $A := \psi^{-1}(\mathbb{P}^{k+1} \setminus \mathbb{C}^{k+1})$; then A is an analytic subvariety in D. If A is a proper subvariety in D then, by applying the maximum principle to the holomorphic functions that are the components of $F^{n_{j(m)}} \circ s$ $(m \geq 1)$ and using the charts of D and $\mathbb{C}^{k+1}$, we obtain a contradiction. Hence, $A = D$. To show that the forward orbit of any point in D accumulates at I, we have only to show that any point in D is an irregular point (Lemma 3.1). But this now follows immediately from Proposition 2.13, Remark 2.14, and the equality $A = D$.

The set of points in U whose forward orbits accumulate at I is an open set in U, as we have shown. On the other hand, by Lemma 3.1 the set of points in U whose forward orbits do not accumulate at I is also an open set in U. Since U is connected, the forward orbit of any point in U accumulates at I if there is at least one point in U whose forward orbit accumulates at I. □

By Lemma 3.1 and Theorem 3.2, any Fatou component consists either of only regular points or of only irregular points.

DEFINITION 3.5. A Fatou component U is *regular* if all points in U are regular. Denote by Ω_R the union of all regular Fatou components, and let $\Omega_{IR} := \Omega \setminus \Omega_R$.

There is an AS rational self-map of $\mathbb{P}^2$ for which $\Omega_R \neq \emptyset$ and $\Omega_{IR} \neq \emptyset$.

EXAMPLE 3.6. Consider an AS polynomial self-map of $\mathbb{C}^2$ $(\subset \mathbb{P}^2)$,

$$f[x : y : t] = [x^2 t : y^3 : t^3].$$

For f, the indeterminacy set is $I = \{[1 : 0 : 0]\}$. $X = \{[0 : 1 : 0]\}$ is a super-attracting fixed point. A domain $\{|y| > |t|\}$ is the basin of attraction for X and a regular Fatou component. A domain $\{|x| < |t|, |y| < |t|\}$ is another regular Fatou component. A domain $\{|x| > |t|, |y| < |t|\}$ is an irregular Fatou component such that any point in it goes to I in the forward time.

Ueda [U] proved that all the Fatou components for holomorphic self-maps of degree ≥ 2 of $\mathbb{P}^k$ are Kobayashi hyperbolic. Here we will generalize his result to the case of AS rational self-maps. In general, Fatou components are not necessarily Kobayashi hyperbolic—for example, the basin of attraction of an attracting fixed point for a Hénon map (this is a Fatou–Bieberbach domain). So here we will give a sufficient condition.

THEOREM 3.7. *Let f be an AS rational self-map of degree ≥ 2 of $\mathbb{P}^k$, and let U be a Fatou component for f. If each point in ∂U is regular, then U is hyperbolically imbedded in $\mathbb{P}^k$.*

Proof. It is obvious in light of Theorem 3.2 that U is a regular Fatou component. Since U is relatively compact, we can take an open set W in Proposition 2.13 such that $U \subset Q(W)$. Thus, by the same method [U] as in the case of holomorphic maps, the hyperbolic imbeddedness of U follows. □

Suppose $G \not\equiv -\infty$. In [FS] it is shown that $\mathcal{R} \setminus \operatorname{supp} T \subset \Omega$. Here we will show that $\partial(\mathcal{R} \setminus \operatorname{supp} T) \subset J$ and $\mathcal{R} \setminus \operatorname{supp} T$ is Stein.

THEOREM 3.8. *Let f be an AS rational self-map of degree ≥ 2 of $\mathbb{P}^k$ such that $G \not\equiv -\infty$. Then*:

(i) $\Omega_R = \mathcal{R} \setminus \operatorname{supp} T$;
(ii) Ω_R *is Stein.*

Proof. (i) $\mathcal{R} \setminus \operatorname{supp} T \subset \Omega$ was shown in [FS]. By Theorem 3.2, any Fatou component intersecting $\mathcal{R} \setminus \operatorname{supp} T$ is regular, so $\mathcal{R} \setminus \operatorname{supp} T \subset \Omega_R$. By the definition, $\Omega_R \subset \mathcal{R}$. By Proposition 2.12, $\Omega_R \subset \mathbb{P}^k \setminus \operatorname{supp} T$. Hence, $\Omega_R \subset \mathcal{R} \setminus \operatorname{supp} T$.

(ii) We now show the pseudoconvexity of a connected component U of Ω_R. (Since $U \neq \mathbb{P}^k$, the pseudoconvexity implies the Steinness of U.) Let U_1 be the connected component of $\mathbb{P}^k \setminus \operatorname{supp} T$, which contains U. Let us recall Ueda's method used in [U] to show the Kobayashi hyperbolicity of Fatou components in the case of holomorphic maps. Proposition 2.15 applies to points in U_1, so let p be a point in U_1 and choose D and s as in Proposition 2.15(i). By Proposition 2.15(ii), the section s can be continued analytically along all curves in U_1. Therefore, the analytic continuation of s defines a covering manifold $\alpha\colon \widetilde{U}_1 \to U_1$ of U_1 as well as a holomorphic map $\tilde{s}\colon \widetilde{U}_1 \to \mathbb{C}^{k+1}$ such that $\pi \circ \tilde{s} = \alpha$, where $\tilde{s}$ is injective and $\tilde{s}(\widetilde{U}_1)$ is in B_G.

We set a polydisc Δ and a Hartogs figure H in Δ as follows:

$$\Delta := \{(z, w) \in \mathbb{C}^k \mid z \in \mathbb{C},\ w \in \mathbb{C}^{k-1},\ |z| < 1,\ |w| < 1\},$$

$$H := \{(z, w) \in \mathbb{C}^k \mid |z| < 1,\ |w| < r\} \cup \{(z, w) \mid r_1 < |z| < 1,\ |w| < 1\}, \quad 0 < r, r_1 < 1.$$

Let $\psi\colon \Delta \to \mathbb{P}^k$ be an injective holomorphic map such that $\psi(H) \subset U$. We will show that $\psi(\Delta) \subset U$. Since U_1 is pseudoconvex and $U \subset U_1$, we have $\psi(\Delta) \subset U_1$. Since Δ is simply connected, there is a holomorphic map $\psi_1\colon \Delta \to \widetilde{U}_1$ such that $\alpha \circ \psi_1 = \psi$. Set $\psi_2 := \tilde{s} \circ \psi_1$; then $\pi \circ \psi_2 = \psi$ and $\psi_2(\Delta) \subset B_G$. We can assume without loss of generality that $\psi(H) \subset\subset U$. Since each point in U is a regular point and $\psi(H) \subset\subset U$, there is a neighborhood V of $\psi(H)$ contained in U and a neighborhood W of I such that $f^n(V) \cap W = \emptyset$ for all $n \geq 0$. Then, $V \subset Q = Q(W)$. (See Proposition 2.13 and Remark 2.14.)

Set $S := \{r < \|x\| < R\}$, where r and R are positive constants appearing in Proposition 2.13. It follows that $F^n \circ \psi_2(H)$ is contained in S for all $n \geq 0$, since $\psi(H) \subset Q$ and $\psi_2(H) \subset B_G$. By applying the maximal principle to the component functions of $F^n \circ \psi_2$ in Δ, it follows that $F^n \circ \psi_2(\Delta)$ is contained in $\{\|x\| < R\}$ for all $n \geq 0$. Hence, $\{F^n \circ \psi_2\}_{n\geq 0}$ is a normal family in Δ. This implies that $\{f^n\}_{n\geq 0}$ is a normal family in $\psi(\Delta)$, since $F^n \circ \psi_2(\Delta) \subset B_G$ and B_G is disjoint from a neighborhood of O. Therefore, $\psi(\Delta) \subset U$. □

In the rest of this section we will see some cases in which all the Fatou components are Stein.

Let us regard $\mathbb{P}^k$ as the compactification of $\mathbb{C}^k$; that is, $\mathbb{P}^k = \mathbb{C}^k \cup \Pi$, where Π is a hyperplane. We say that a polynomial self-map f of $\mathbb{C}^k$ is *weakly regular* if $\overline{f(\Pi \setminus I)} \cap I = \emptyset$, where the latter f means the extended map onto $\mathbb{P}^k$ and I the indeterminacy set (see [GS]). Note that weakly regular polynomial self-maps are algebraically stable. We set $X := \overline{f(\Pi \setminus I)}$. Here X is an attractor: there is a neighborhood V of X in $\mathbb{P}^k$ such that $f(V) \subset\subset V$ and $\bigcap_{n\geq 0} f^n(V) \subset X$.

PROPOSITION 3.9. *Let f be a weakly regular polynomial self-map of degree ≥ 2 of $\mathbb{C}^k$, and assume* $\dim I = 0$. *Then the Fatou set Ω is Stein.*

Proof. Let $W \subset \mathbb{P}^k$ be the basin of attraction for X, that is, $W = \bigcup_{n\geq 0} f^{-n}(V)$. Obviously $\partial W \subset J$, so there are two types of Fatou components: one type is a subset of W and the other type is a subset of the interior of $\mathbb{P}^k \setminus W$. Fatou components of the first type are Stein because they are regular. We will show the Steinness of the second type of Fatou component U. Clearly U is in $\mathbb{C}^k$, so the pseudoconvexity of U implies the Steinness of U. Let us use Δ and H as in the proof of Theorem 3.8. Let $\psi\colon \Delta \to \mathbb{P}^k$ be an injective holomorphic map such that $\psi(H) \subset U$. We will show that $\psi(\Delta) \subset U$. We may assume that $\psi(H) \subset\subset U$. Let $\{f^{n_j}\}$ be any subsequence of $\{f^n\}$ that converges locally uniformly in U, and let ϕ be the limit map.

If there is a $p \in \psi(H)$ such that $\{f^{n_j}(p)\}$ is bounded in $\mathbb{C}^k$, then $\phi(\psi(H)) \subset \mathbb{C}^k$. By the maximal principle, $\{f^{n_j}\}$ converges locally uniformly in $\psi(\Delta)$.

If there is a $p \in \psi(H)$ such that $\{f^{n_j}(p)\}$ is unbounded, then $\phi(p)$ is a point q in I. By the maximal principle and the connectedness of $\phi(\psi(H))$, it follows that $\phi(\psi(H)) = \{q\}$. Hence there is a hyperplane $L \subset \mathbb{P}^k$ such that, for sufficiently large j, the image $f^{n_j}(\psi(H))$ is disjoint from L and q is also disjoint from L. By the pseudoconvexity of $\mathbb{P}^k \setminus L$, it follows that $f^{n_j}(\psi(\Delta))$ is also disjoint from L. We can take advantage of the coordinate in $\mathbb{P}^k \setminus L$ (note that $\mathbb{P}^k \setminus L$

is biholomorphic to $\mathbb{C}^k$, apparently). In this coordinate, we may regard q as the origin O of $\mathbb{C}^k$. Again by the maximal principle, it follows that $\{f^{n_j}\}$ converges locally uniformly in $\psi(\Delta)$ to O.

We have shown that $\psi(\Delta) \subset U$. Hence, U is pseudoconvex. □

REMARK 3.10. We can also prove the following assertion in the same manner as for the preceding proposition.

Let f be a polynomial automorphism of degree ≥ 2 of $\mathbb{C}^k$. Suppose that f, f^{-1} are weakly regular and that $\dim(I_+ \cap I_-) = 0$. *Then Ω_+ and Ω_- are Stein.* (*Here $I_\pm$ and $\Omega_\pm$ are the indeterminacy sets and the Fatou sets for $f_\pm$, respectively.*)

Let X_+ be the attractor for f_+ and let W_+ be the basin of attraction for X_+. It is sufficient to show that, for any point in $\mathbb{C}^k \setminus W_+$, the forward orbit does not accumulate at $\Pi \setminus (I_+ \cap I_-)$. This has been shown in Proposition 1.7 of [GS].

References

[FS] J. E. Fornæss and N. Sibony, *Complex dynamics in higher dimension II,* Ann. of Math. Stud., 137, pp. 135–187, Princeton Univ. Press, Princeton, NJ, 1995.

[GS] V. Guedj and N. Sibony, *Dynamics of polynomial automorphisms of* $\mathbb{C}^k$, Ark. Mat. 40 (2002), 207–243.

[S] N. Sibony, *Dynamique des applications rationnells de* $\mathbb{P}^k$, Dynamique et géométrie complexes (Lyon, 1997), Panor. Synthèses, 8, pp. 97–185, Soc. Math. France, Paris, 1999.

[U] T. Ueda, *Fatou sets in complex dynamics on projective spaces,* J. Math. Soc. Japan 46 (1994), 545–555.

Mathematical Science Group
Faculty of Integrated Human Studies
Kyoto University
Yoshida Nihonmatsu-cho
Sakyo-ku, Kyoto
Japan, 606-8501

km@math.h.kyoto-u.ac.jp