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A Proof of the Gap Labeling Conjecture

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1. Introduction

The "gap labeling conjecture" as formulated by Bellissard [3] is a statement—about the possible gaps in the spectrum of certain Schrödinger operators—that arises in solid state physics. It has a reduction to a purely mathematical statement about the range of the trace on a certain crossed-product C^*-algebra (see [13]). By a *Cantor set* we mean a compact, totally disconnected metric space without isolated points. A group action is *minimal* if every orbit is dense.

Theorem 1.1. *Let Σ be a Cantor set and let $\Sigma \times \mathbb{Z}^n \to \Sigma$ be a free and minimal action of $\mathbb{Z}^n$ on Σ with invariant probability measure μ. Let $\mu\colon C(\Sigma) \to \mathbb{C}$ and $\tau_\mu\colon C(\Sigma) \rtimes \mathbb{Z}^n \to \mathbb{C}$ be the traces induced by μ and denote likewise the induced maps on K-theory. Then*

$$\mu(K_0(C(\Sigma))) = \tau_\mu(K_0(C(\Sigma) \rtimes \mathbb{Z}^n)).$$

Note that $K_0(C(\Sigma))$ is isomorphic to $C(\Sigma, \mathbb{Z})$, the group of integer-valued continuous functions on Σ, and that the image under μ is the subgroup of $\mathbb{R}$ generated by the measures of the clopen subsets of Σ.

We will give a proof of this conjecture in this paper. It was also proved independently by Bellissard, Benedetti, and Gambaudo [2] and by Benameur and Oyono-Oyono [4].

The strategy of the proof is to use Connes's index theory for foliations but in the form presented in the book by Moore and Schochet [11]. In fact, this approach underlies all three proofs [2; 4]. Thus, one may apply the index theorem to "foliated spaces", which are more general than foliations. These are spaces that have a cover by compatible flow boxes as in the case of genuine foliations, except that the transverse direction is not required to be $\mathbb{R}^n$. In the case at hand it is a Cantor set.

There are two steps in the proof. The main one is to show that

$$\tau_\mu(K_0(C(\Sigma) \rtimes \mathbb{Z}^n)) \subseteq \mu(K_0(C(\Sigma))). \tag{1.1}$$

This will be carried out in Section 4. The reverse containment is easier and is proved in Section 2. The authors would like to thank Ryszard Nest for several interesting discussions on this material.

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2. Index Theory for Foliated Spaces

We will work in a general framework based on the diagram below. Let Σ be a Cantor set provided with a free, minimal action of $\mathbb{Z}^n$ and an invariant measure μ, and let $X = \Sigma \times_{\mathbb{Z}^n} \mathbb{R}^n$ be its suspension—that is, the quotient of $\Sigma \times \mathbb{R}^n$ by the diagonal action of $\mathbb{Z}^n$. There is a free action of $\mathbb{R}^n$ on X defined by $[x, w] \cdot v = [x, w + v]$.

There is a Morita equivalence (see [14]) between the C^*-algebras associated to these group actions, $C(\Sigma) \rtimes \mathbb{Z}^n$ and $C(X) \rtimes \mathbb{R}^n$, which we will need. It is described in more detail in the proof of Proposition 2.1.

Consider the diagram

$$\begin{array}{ccccccccc} K_0(C(\Sigma)) & \xrightarrow{i_*} & K_0(C(\Sigma)\rtimes\mathbb{Z}^n) & \xrightarrow{\text{m.e.}} & K_0(C(X)\rtimes\mathbb{R}^n) & \xleftarrow{\phi_c} & K_n(C(X)) & \xrightarrow{\mathrm{ch}^{(n)}} & \check{H}^n(X;\mathbb{R}) \\ \big\downarrow{\scriptstyle \mu} & & \big\downarrow{\scriptstyle \tau_\mu} & & \big\downarrow{\scriptstyle \tilde{\tau}_\mu} & & & & \big\downarrow{\scriptstyle C_\mu} \\ \mathbb{R} & = & \mathbb{R} & = & \mathbb{R} & & = & & \mathbb{R}. \end{array}$$

Here, the first horizontal arrow is induced by the inclusion of $C(\Sigma)$ in $C(\Sigma)\rtimes\mathbb{Z}^n$, the second is provided by the strong Morita equivalence between $C(\Sigma) \rtimes \mathbb{Z}^n$ and $C(X) \rtimes \mathbb{R}^n$, the third is Connes's Thom isomorphism, and the fourth is the nth component of the Chern character. The first vertical arrow is the map induced by integration against the invariant measure, the second is the trace on $C(\Sigma) \rtimes \mathbb{Z}^n$ obtained from the invariant measure on Σ, the third is induced by the trace obtained from the associated invariant transverse measure on X, and C_μ is the homomorphism defined via evaluation on the associated Ruelle–Sullivan current. We claim that this diagram commutes. The left square commutes by definition of the trace, τ_μ. In what follows, the other two squares will be shown to commute: the second by looking at the strong Morita equivalence and the third by application of the index theory of foliated spaces.

PROPOSITION 2.1. *The diagram*

$$\begin{array}{ccc} K_0(C(\Sigma) \rtimes \mathbb{Z}^n) & \xrightarrow{\text{m.e.}} & K_0(C(X) \rtimes \mathbb{R}^n) \\ \big\downarrow{\scriptstyle \tau_\mu} & & \big\downarrow{\scriptstyle \tilde{\tau}_\mu} \\ \mathbb{R} & = & \mathbb{R} \end{array} \tag{2.1}$$

commutes.

Proof. This is a standard fact, and a proof is sketched in [1]. We indicate a different (but related) justification here.

The equivalence bimodule exhibiting the strong Morita equivalence between $C(\Sigma) \rtimes \mathbb{Z}^n$ and $C(X) \rtimes \mathbb{R}^n$ is obtained (see [14]) by completing $C_c(X \times \mathbb{R}^n)$. Denote the resulting bimodule by $\mathcal{E}$ and the associated linking algebra [5] by $\mathcal{A}$. Recall that $\mathcal{A}$ can be viewed as being made up of 2×2 matrices of the form

$$\begin{bmatrix} a & x \\ \tilde{y} & b \end{bmatrix},$$

where $a \in C(\Sigma) \rtimes \mathbb{Z}^n$, $b \in C(X) \rtimes \mathbb{R}^n$, $x \in \mathcal{E}$, and $\tilde{y} \in \mathcal{E}^{\mathrm{op}}$. This can be completed to a C^*-algebra, where the multiplication on the generators is given by

$$\begin{bmatrix} a & x \\ \tilde{y} & b \end{bmatrix}\begin{bmatrix} a' & x' \\ \tilde{y}' & b' \end{bmatrix} = \begin{bmatrix} aa' + \langle x, \tilde{y}' \rangle_{C(\Sigma)\rtimes\mathbb{Z}^n} & ax' + xb' \\ \tilde{y}a' + b\tilde{y}' & bb' + \langle \tilde{y}, x' \rangle_{C(X)\rtimes\mathbb{R}^n} \end{bmatrix}.$$

The algebra $\mathcal{A}$ contains both $C(\Sigma) \rtimes \mathbb{Z}^n$ and $C(X) \rtimes \mathbb{R}^n$ as full hereditary subalgebras, and hence the inclusions $i_1\colon C(\Sigma) \rtimes \mathbb{Z}^n \to \mathcal{A}$ and $i_2\colon C(X) \rtimes \mathbb{R}^n \to \mathcal{A}$ induce isomorphisms on K-theory. The given traces on the subalgebras give rise to a trace on $\mathcal{A}$ via $\tau\left(\left[\begin{smallmatrix} a & x \\ \tilde{y} & b \end{smallmatrix}\right]\right) = \tau_\mu(a) + \tilde{\tau}_\mu(b)$. The verification that this is in fact a trace requires checking that

$$\begin{aligned} &\tau_\mu(aa' + \langle x, \tilde{y}' \rangle_{C(\Sigma)\rtimes\mathbb{Z}^n}) + \tilde{\tau}_\mu(bb' + \langle \tilde{y}, x' \rangle_{C(X)\rtimes\mathbb{R}^n}) \\ &\qquad = \tau_\mu(a'a + \langle x', \tilde{y} \rangle_{C(\Sigma)\rtimes\mathbb{Z}^n}) + \tilde{\tau}_\mu(b'b + \langle \tilde{y}', x \rangle_{C(X)\rtimes\mathbb{R}^n}). \end{aligned}$$

This, in turn, comes down to showing that

$$\begin{aligned} \tau_\mu(\langle x, \tilde{y}' \rangle_{C(\Sigma)\rtimes\mathbb{Z}^n}) &= \tilde{\tau}_\mu(\langle \tilde{y}', x \rangle_{C(X)\rtimes\mathbb{R}^n}) \\ \tau_\mu(\langle x', \tilde{y} \rangle_{C(\Sigma)\rtimes\mathbb{Z}^n}) &= \tilde{\tau}_\mu(\langle \tilde{y}, x' \rangle_{C(X)\rtimes\mathbb{R}^n}). \end{aligned}$$

Each of these is a direct computation from the definitions of the pairings and the map $\tilde{\tau}_\mu$.

It is easy to check that $\tau(i_{1*}(a)) = \tau_\mu(a)$ and $\tau(i_{2*}(b)) = \tilde{\tau}_\mu(b)$. Since the isomorphism on K-theory induced by the strong Morita equivalence is given by $i_{2*}^{-1} i_{1*}$, the result follows. □

Since we will be using the theory of foliated spaces in the sense of Moore and Schochet [11], we make the following observation about the suspension, X.

PROPOSITION 2.2. *The suspension X, provided with its canonical $\mathbb{R}^n$-action, is a compact foliated space with transversal a Cantor set and invariant transverse measure obtained from μ.*

We will have need of Connes's Thom isomorphism theorem for $C(X) \rtimes \mathbb{R}^n$. It follows from the work of Fack and Skandalis [9] that the isomorphism is induced by Kasparov product with a KK-element obtained from the Dirac operator along the leaves of the foliated space X. Denoting Connes's Thom isomorphism by $\phi_c\colon K_0(C(X) \rtimes \mathbb{R}^n) \to K_n(C(X))$, one has the following description.

PROPOSITION 2.3. *The map ϕ_c is given by Kasparov product with the element*

$$[\not{\partial}] \in KK^n(C(X), C(X) \rtimes \mathbb{R}^n)$$

obtained from the Dirac operator along the leaves of the foliated space. Thus, for an element $[E] \in K_0(C(X))$,

$$\phi_c([E]) = \mathrm{Index}^{an}([\not{\partial} \otimes E]) \in K_n(C(X) \rtimes \mathbb{R}^n).$$

Proof. This follows from [9]. □

Finally, we shall use the version of Connes's foliation index theorem as presented by Moore and Schochet in [11]. The theorem provides a topological formula for the result of pairing the analytic index of a leafwise elliptic operator with the trace associated to a holonomy-invariant transverse measure. The topological side is obtained by pairing a tangential cohomology class with the Ruelle–Sullivan current associated to the invariant transverse measure.

The Ruelle–Sullivan current may be viewed as a homomorphism

$$C_\mu \colon H^*_\tau(X) \to \mathbb{R},$$

where $H^*_\tau(X)$ is tangential cohomology [11]. This is essentially de Rham cohomology constructed from forms that are smooth in the leaf direction yet are continuous only transversally. It is related to the Čech cohomology of X by a natural map $r \colon \check{H}^*(X) \to H^*_\tau(X)$, which in general is neither injective nor surjective. However, this allows one to extend C_μ to $\check{H}(X)$ as $C_\mu \circ r$. Moreover, for a foliated space such as X, there is a tangential Chern character $\mathrm{ch}_\tau \colon K_*(C(X)) \to H^*_\tau(X)$ obtained by applying Chern–Weil to a leafwise connection. It is related to the usual Chern character via $r \circ \mathrm{ch} = \mathrm{ch}_\tau$. With this notation at our disposal, we have the following result.

PROPOSITION 2.4. *Let C_μ be the Ruelle–Sullivan current associated to the invariant transverse measure μ, and let* $\mathrm{ch}^{(n)}$ *denote the component of the Chern character in $\check{H}^n(X)$. Then*

$$\widetilde{\tau_\mu}(\mathrm{Index}^{an}([\partial\!\!\!/ \otimes E])) = C_\mu \circ r \circ \mathrm{ch}^{(n)}([E]).$$

Proof. This is an application of the foliation index theorem [6; 11]. By that theorem it is sufficient to show that the right-hand side is what one obtains by pairing the index cohomology class with the Ruelle–Sullivan current. In general, the index class is represented by the tangential form

$$\mathrm{ch}(E) \wedge \mathrm{ch}(\sigma(\partial\!\!\!/)) \wedge \mathcal{T}d(T\mathcal{F} \otimes \mathbb{C}) = \mathrm{ch}(E) \wedge \hat{A}(T\mathcal{F}).$$

Note that $\hat{A}(T\mathcal{F})$ is a polynomial in the Pontryjagin forms, obtained from a connection that can be chosen to be flat along the leaves and hence is equal to 1. We know that $r \circ \mathrm{ch}(E) \in H^*_\tau(X)$ and so, taking into account that the homomorphism induced by the Ruelle–Sullivan current is zero except in degree n, we have

$$C_\mu\big(\mathrm{ch}(E) \wedge \mathrm{ch}(\sigma(\partial\!\!\!/)) \wedge \mathcal{T}d(T\mathcal{F} \otimes \mathbb{C})\big) = C_\mu \circ r \circ \mathrm{ch}^{(n)}(E),$$

as required. □

Given the foregoing results, the commutativity of the main diagram follows easily. Indeed, the commutativity of the right-hand rectangle is precisely the statement in Proposition 2.4.

We record the following fact, observed previously.

PROPOSITION 2.5. $\mu(K_0(C(\Sigma))) \subseteq \tau_\mu(K_0(C(\Sigma) \rtimes \mathbb{Z}^n))$.

It remains to verify the other containment,

$$\tau_\mu(K_0(C(\Sigma) \rtimes \mathbb{Z}^n)) \subseteq \mu(K_0(C(\Sigma))), \tag{2.2}$$

which will be done in the next section.

3. Construction of a Transfer Map

In this section we will provide the tool that enables the verification of (2.2). To accomplish this we will use the map (described in Connes's book [7 p. 120]) that associates—to a clopen set in a transversal to a foliation—a projection in its foliation algebra:

$$\alpha\colon K_0(C(\Sigma)) \to K_0(C(X) \rtimes \mathbb{R}^n).$$

The modifications necessary to apply to the foliated space in question are routine. It will be used to relate Bott periodicity for $C(\Sigma)$ to Connes's Thom isomorphism for $C(X) \rtimes \mathbb{R}^n$.

Consider the transversal $\Sigma \times \left\{\left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right)\right\} \subseteq X$. Let U be a clopen set of Σ and let χ_U be its characteristic function. We recall the description of the associated projection in $C(X) \rtimes \mathbb{R}^n$.

We may define a function

$$e_U\colon \Sigma \times [0,1] \times \mathbb{R}^n \to \mathbb{R},$$

which will yield an element of $C(X) \rtimes \mathbb{R}^n$. Toward this end, let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a continuous function with support in the cube of side $\frac{1}{4}$ centered at $\left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right)$ and satisfying

$$\int_{\mathbb{R}^n} f(x)^2\, dx = 1. \tag{3.1}$$

Set

$$e_U(x,t,s) = \chi_U(x) f\left(t - \tfrac{1}{2}\right) f\left(t - \tfrac{1}{2} - s\right). \tag{3.2}$$

Then it is easy to check that e_U descends to a function on $X \times \mathbb{R}^n$ that yields an element of $C(X) \rtimes \mathbb{R}^n$ satisfying $e_U = e_U^2 = e_U^*$. We then set

$$\alpha(\chi_U) = e_U. \tag{3.3}$$

Proposition 3.1. *The function α induces a homomorphism,*

$$\alpha\colon K_0(C(\Sigma)) \to K_0(C(X) \rtimes \mathbb{R}^n), \tag{3.4}$$

for which the following diagram commutes:

$$\begin{array}{ccc} K_0(C(\Sigma)) & \xrightarrow{\alpha} & K_0(C(X) \rtimes \mathbb{R}^n) \\ \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\tilde{\tau}_\mu} \\ \mathbb{R} & = & \mathbb{R}. \end{array} \tag{3.5}$$

Proof. For the relation with the traces, we note that

$$\begin{aligned}\tilde{\tau}_\mu(e_U) &= \int_{\mathbb{R}^n} e_U(x,t,0)\,d\mu(x)\,dt \\ &= \int_{\mathbb{R}^n} \chi_U(x) f(t) f(t)\,dt\,d\mu(x) = \mu(U). \end{aligned} \tag{3.6}$$

Showing that α provides a well-defined homomorphism is straightforward. □

The main property of α is provided by the following result. Let $\pi\colon \Sigma \times \mathbb{R}^n \to X$ be the quotient map. Let $\mathcal{L}$ be the union of all hyperplanes parallel to the coordinate axis and going through points of $\mathbb{Z}^n$, and set $A = \pi(\Sigma \times \mathcal{L})$. Let $j\colon X \setminus A \to X$ be the inclusion of the open set $X \setminus A$, which will induce a homomorphism $j_*\colon C_0(X \setminus A) \to C(X)$. Note that $C_0(X \setminus A) \cong C_0(\Sigma \times (0,1)^n) \cong C_0(\Sigma \times \mathbb{R}^n)$. We can now relate the map α to Bott periodicity and Connes's Thom isomorphism.

PROPOSITION 3.2. *There is a commutative diagram,*

$$\begin{array}{ccc} K_0(C(\Sigma)) & \xrightarrow{\alpha} & K_0(C(X) \rtimes \mathbb{R}^n) \\ \Big\downarrow{\scriptstyle\beta} & & \Big\uparrow{\scriptstyle\phi_c} \\ K_n(C_0(X \setminus A)) & \xrightarrow{j_*} & K_n(C(X)), \end{array} \tag{3.7}$$

where ϕ_c is Connes's Thom isomorphism and β is the Bott periodicity map.

Proof. We will deform the action $\Phi\colon X \times \mathbb{R}^n \to X$ as follows. Let $\theta_r\colon \mathbb{R}^n \to [0,1]$ be a family of continuous functions that (i) are periodic with respect to translation by $\mathbb{Z}^n$, (ii) have fundamental domain $[0,1]^n$, and (iii) on that fundamental domain satisfy:

(a) $\theta_r(\vec{v}) = 1$ on $\left[\frac{1}{4}, \frac{3}{4}\right]^n$;
(b) $\theta_r(\vec{v})$ decreases to r on $\partial[0,1]^n$ for $\vec{v} \in [0,1]^n \setminus \left[\frac{1}{4}, \frac{3}{4}\right]^n$; and
(c) $\theta_r(\vec{v}) > 0$ if $\vec{v} \notin \mathcal{L}$.

Set $\Phi^r([z,\vec{v}],\vec{w}) = [z, \vec{v} - \theta_r(\vec{v})\vec{w}]$. Here $[z,\vec{v}]$ denotes a point in $\Sigma \times_{\mathbb{Z}^n} \mathbb{R}^n$. Then it is easy to check that the family Φ^r has the following properties:

(i) Φ^1 is the given translation flow on X;
(ii) Φ^r is the given translation flow on $\left[\frac{1}{4}, \frac{3}{4}\right]^n \subseteq X$ for all $0 \le r \le 1$;
(iii) Φ^0 leaves the subset $A \subseteq X$ pointwise fixed; and
(iv) Φ^0 on $X \setminus A$ is conjugate to $1 \times$ translation on $\Sigma \times \mathbb{R}^n$.

It will be shown that the map α constructed with the action Φ^0 agrees with $\phi_c j_* \beta$, which will prove the proposition.

The family Φ^r can be used to define an action $([0,1] \times X) \times \mathbb{R}^n \to [0,1] \times X$ via the formula

$$\Phi(x,r) = (r, \Phi^r(x)).$$

Consider the following commutative diagram:

$$\begin{tikzcd}
K_0(C(\Sigma)) \arrow[d, "\beta"] & K_0(C(\Sigma \times [0,1])) \arrow[l, "e_0"'] \arrow[r, "e_1"] \arrow[d, "\beta"] & K_0(C(\Sigma)) \arrow[d, "\beta"] \\
K_0(C_0(\Sigma \times (0,1)^n)) \arrow[d, "j_*"] & K_0(C_0(\Sigma \times (0,1)^n \times [0,1])) \arrow[l, "e_0"'] \arrow[r, "e_1"] \arrow[d, "j_*"] & K_0(C_0(\Sigma \times (0,1)^n)) \arrow[d, "j_*"] \\
K_0(C(X)) \arrow[d, "\phi_{c,0}"] & K_0(C(X \times [0,1])) \arrow[l, "e_0"'] \arrow[r, "e_1"] \arrow[d, "\phi_c"] & K_0(C(X)) \arrow[d, "\phi_{c,1}"] \\
K_0(C(X) \rtimes_{\Phi^0} \mathbb{R}^n) & K_0(C(X \times [0,1]) \rtimes_{\Phi} \mathbb{R}^n) \arrow[l, "e_0"'] \arrow[r, "e_1"] & K_0(C(X) \rtimes_{\Phi^1} \mathbb{R}^n).
\end{tikzcd}$$

The horizontal maps are induced by evaluation at 0 and 1 and are all isomorphisms. Moreover, except for the bottom row, the compositions $\varepsilon_1 \varepsilon_0^{-1}$ are the identity homomorphism. The vertical maps $\phi_{c,0}$, ϕ_c, and $\phi_{c,1}$ are Connes's Thom isomorphism for the respective actions, and β denotes Bott periodicity.

Now, the composition on the left side takes an element $[\chi_U]$ to the element $\alpha([\chi_U]) = [e_U]_0$ for the action Φ^0. Further, since e_U is supported where the actions Φ^r all agree, we have $\varepsilon_1 \varepsilon_0^{-1}([e_{U0}]) = [e_{U1}]$. But then, by commutativity of the diagram, the result follows. $\square$

4. The Gap Labeling Theorem

In this section we will complete the proof of the main theorem. Recall that we must show the containment

$$\tau_\mu(K_0(C(\Sigma) \rtimes \mathbb{Z}^n)) \subseteq \mu(K_0(C(\Sigma))).$$

As a preliminary step, we look carefully at the following diagram:

$$\begin{tikzcd}
K_0(C_0(X \setminus A)) \arrow[r, "j^*"] \arrow[d, "\mathrm{ch}^{(n)}"] & K_0(C(X)) \arrow[d, "\mathrm{ch}^{(n)}"] \\
\check{H}^n(X/A) \arrow[r, "j^*"] & \check{H}^n(X).
\end{tikzcd} \tag{4.1}$$

We now proceed by first making two observations, as follows.

PROPOSITION 4.1. *The map* $\mathrm{ch}^{(n)}\colon K_0(C_0(X \setminus A)) \to \check{H}^n(X/A)$ *is an isomorphism.*

Proof. The space $(X \setminus A)^+ \cong (\Sigma \times \mathbb{R}^n)^+$ is the inverse limit of finite wedges of n-dimensional spheres. This is because Σ, as a Cantor set, is the inverse limit of finite sets. Since ch^n is an isomorphism on each of the finite wedges, passing to the limit yields the result. $\square$

PROPOSITION 4.2. *The map $j^*\colon \check{H}^n(X/A) \to \check{H}^n(X)$ is onto.*

Proof. The map j^* fits into the long exact sequence of the pair (X, A), and the next term is $\check{H}^n(A)$. Recall that the definition of cohomological dimension of a space X is

$$\dim_{\mathbb{R}}(X) = \sup\{k \mid \check{H}^k(X, B; \mathbb{R}) \neq 0 \text{ for some } B \subseteq X\}. \tag{4.2}$$

Thus, it will be sufficient to show that $\dim_{\mathbb{R}}(A) < n$ (see [8]).

We observe that $A = \bigcup_{i=1}^n \pi(\Sigma \times \mathbb{R}^{n-1}_{(i)})$, where $\pi : \Sigma \times \mathbb{R}^n \to X$ is the projection onto the quotient and $\mathbb{R}^{n-1}_{(i)}$ denotes the points with ith coordinate zero. Now, $\pi(\Sigma \times \mathbb{R}^{n-1}_{(i)})$ is the total space of a fiber bundle with base T^{n-1} and fiber a Cantor set C. This, in turn, is a finite union of compact sets, each homeomorphic to $D^{n-1} \times C$ (where D^{n-1} is an $n-1$ disk) and with $\dim_{\mathbb{R}}(D^{n-1} \times C) = n-1$ for each set. Thus, $\dim_{\mathbb{R}}(\pi(\Sigma \times \mathbb{R}^{n-1}_{(i)})) = n-1$ and hence $\dim_{\mathbb{R}}(A) = n-1$, since (again) the latter is a finite union of compact sets with that property. (See [8] for the properties of cohomological dimension needed in this argument.) □

Next we assemble a larger diagram that contains (3.7) and (4.1):

$$\begin{tikzcd}
K_0(C(\Sigma)) \arrow[r, "\alpha"] \arrow[d, "\beta"] & K_0(C(X) \rtimes \mathbb{R}^n) \arrow[r, "\tilde{\tau}_\mu"] & \mathbb{R} \arrow[d] \\
K^n(X, A) \arrow[r, "j^*"] \arrow[d, "\mathrm{ch}^{(n)}"] & K^n(X) \arrow[u, "\Phi_c"] \arrow[d, "\mathrm{ch}^{(n)}"] & \mathbb{R} \\
\check{H}^n(X, A) \arrow[r, "j^*"] & \check{H}^n(X) \arrow[r, "C_\mu \circ r"] & \mathbb{R}. \arrow[u]
\end{tikzcd} \tag{4.3}$$

The top left-hand square commutes by Proposition 3.2, and the bottom left one does so by naturality of the Chern character. The right-hand rectangle commutes by the results in Section 2. Note that both vertical maps on the left are isomorphisms and that the bottom j^* is onto. We will now use this to obtain a proof of the gap labeling conjecture.

THEOREM 4.3. *Let $\mathbb{Z}^n$ act minimally on a Cantor set Σ. Consider the diagram*

$$\begin{tikzcd}
K_0(C(\Sigma)) \arrow[r] \arrow[d, "\mu"] & K_0(C(\Sigma) \rtimes \mathbb{Z}^n) \arrow[d, "\tau_\mu"] \\
\mathbb{R} \arrow[r, equal] & \mathbb{R}.
\end{tikzcd} \tag{4.4}$$

Then we have

$$\mu(K_0(C(\Sigma))) = \tau_\mu(K_0(C(\Sigma) \rtimes \mathbb{Z}^n)).$$

Proof. It is sufficient to show that, if $\lambda = \tau_\mu(x)$ for some $x \in K_0(C(\Sigma) \rtimes \mathbb{Z}^n)$, then there exists a $y \in K_0(C(\Sigma))$ with $\tau_\mu(x) = \mu(y)$. Toward this end, we let x' be the element of $K_0(C(X) \rtimes \mathbb{R}^n)$ such that $\tilde{\tau}_\mu(x') = \tau_\mu(x)$. We will find a

$y \in K_0(C(\Sigma))$ with $\tilde{\tau}_\mu(x') = \mu(y)$. Because the bottom j^* is onto in (4.3), there is a $y \in K_0(C(\Sigma))$ such that $(C_\mu \circ r) j^* \operatorname{ch}^{(n)} \beta(y) = \tilde{\tau}_\mu(x')$. But by the commutativity of the diagram we must also have $\tilde{\tau}_\mu(\alpha(y)) = \tilde{\tau}_\mu(x')$. By the basic property of α this yields that $\mu(y) = \tilde{\tau}_\mu(x')$, which equals $\tau_\mu(x)$. □

5. A Remark on Tilings and Dynamics

Bellissard's original formulation of the gap labeling problem was for aperiodic tiling systems. However, we will show that the setting of the problem as addressed in this paper (i.e., free, minimal actions of $\mathbb{Z}^n$ on Cantor sets) is actually general enough to encompass many such tiling systems.

The following proof is based on two key ingredients: a result of Sadun and Williams, and an observation that arose during a stimulating conversation involving Nic Ormes, Charles Radin, and the second author. For the terminology, please refer to [10].

THEOREM 5.1. *Suppose that T is an aperiodic tiling that* (a) *satisfies the finite pattern condition and the property of repetitivity and* (b) *has only finitely many tile orientations. Suppose that Ω is the continuous hull associated with T, as described in* [10], *together with the natural action of $\mathbb{R}^n$. Then there is a Cantor set Σ, with a minimal action of $\mathbb{Z}^n$ on it, such that $C(\Omega) \rtimes \mathbb{R}^n$ and $C(\Sigma) \rtimes \mathbb{Z}^n$ are strongly Morita equivalent.*

Proof. The result of Sadun and Williams [15] states that there is a Cantor set Σ provided with a minimal $\mathbb{Z}^n$-action such that the space of the suspended action, $\Sigma \times_{\mathbb{Z}^n} \mathbb{R}^n$, is homeomorphic to Ω. Unfortunately, this homeomorphism is not a conjugacy of the $\mathbb{R}^n$ actions. In order to get around this we will bring in the fundamental groupoids (cf. [12]) of each of these spaces. It is easy to see that the homeomorphism between the spaces induces an isomorphism between the C^*-algebras of their fundamental groupoids.

Consider the fundamental groupoid of Ω, which we denote by $\Pi(\Omega)$. There is a map of the groupoid $\Omega \times \mathbb{R}^n$ into $\Pi(\Omega)$ defined by sending a pair (T, x) to the homotopy class of the path $\alpha(t) = T + tx$ for $t \in [0, 1]$. It follows from the structure of the space Ω that this map is an isomorphism of topological groupoids and hence induces an isomorphism between their C^*-algebras. An analogous argument shows that the same result holds for $\Sigma \times_{\mathbb{Z}^n} \mathbb{R}^n$. Thus, we have that $C(\Sigma) \rtimes \mathbb{Z}^n$ is strong Morita equivalent to $C(\Sigma \times_{\mathbb{Z}^n} \mathbb{R}^n) \rtimes \mathbb{R}^n$, which is isomorphic to $C(\Omega) \rtimes \mathbb{R}^n$. □

6. Final Remarks

The three proofs of the gap labeling theorem have similarities. In particular, they all make use of index theory for foliated spaces in various guises. There is even a stronger parallel between the present proof and that of Benameur and Oyono-Oyono [4]. Indeed, the fundamental difference appears when proving the existence of an element of $K_0(C(\Sigma))$ whose trace has the required value. We do this via noncommutative topological methods, while in [4] an analysis based on more traditional algebraic topology is used. The latter has the potential of providing more detailed information, but this is not necessary for the present result.

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