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The Third Cauchy–Fantappiè Formula of Leray

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Introduction

In the third paper [L1] in his famous series *Problème de Cauchy,* Jean Leray founded the modern theory of residues. This paper is reprinted in Leray's *Œuvres scientifiques* [L2] with an introduction by G. M. Henkin. In it, almost as an aside, Leray presents three representation formulas for holomorphic functions on a domain in affine space, calling them the first, second, and third Cauchy–Fantappiè formulas. The first formula, nowadays often called the Cauchy–Leray formula, is truly fundamental: most integral representation formulas can be derived from it in one way or another. For an introduction to this area of complex analysis, see [B] and [K]. The second and third formulas, obtained from the first one using residue theory, have received little attention in the literature. The proof of the second formula is straightforward, but the third one turns out to be quite subtle. We will show by means of examples that it does not hold without some additional conditions not mentioned by Leray. We present and study sufficient conditions and a necessary condition for the third formula to hold and, in the case of a contractible domain, characterize it cohomologically.

We assume that the reader is familiar with approximately the first half of Leray's paper [L1], including the coboundary map and the associated long exact homology sequence, absolute and relative residues, the residue formula, and the interaction of the residue map with several natural cohomology maps. To establish a context and notation, we begin by reviewing Leray's derivation of the three formulas.

1. The Cauchy–Fantappiè Formulas

We start with a number of definitions, following Leray. Let X be a domain in $\mathbb{C}^n$ $(n \geq 1)$, and let $Y = \mathbb{P}^n \times X$. Leray assumes that X is convex, but we do not. We define

$$Q = \{(\xi, x) \in Y : \xi \cdot x = \xi_0 + \xi_1 x_1 + \cdots + \xi_n x_n = 0\}$$

and

$$P_z = \{(\xi, x) \in Y : \xi \cdot z = 0\} = \text{hyperplane} \times X$$

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for $z \in X$, which will be thought of as a fixed base point throughout. These are smooth hypersurfaces in Y intersecting transversely. Here, $\xi_0, \dots, \xi_n$ are homogeneous coordinates for a point $\xi \in \mathbb{P}^n$. Define also

$$\omega(x) = dx_1 \wedge \cdots \wedge dx_n,$$

$$\omega'(\xi) = \sum_{k=1}^{n} (-1)^{k-1} \xi_k \, d\xi_1 \wedge \cdots \wedge \widehat{d\xi_k} \wedge \cdots \wedge d\xi_n,$$

and

$$\omega^*(\xi) = \sum_{k=0}^{n} (-1)^{k} \xi_k \, d\xi_0 \wedge \cdots \wedge \widehat{d\xi_k} \wedge \cdots \wedge d\xi_n.$$

Then we have that $(\xi \cdot z)^{-n}\omega'(\xi)$ is a well-defined holomorphic $(n-1)$-form and $(\xi \cdot z)^{-n-1}\omega^*(\xi)$ is a well-defined holomorphic n-form on $\mathbb{P}^n \setminus \{\xi : \xi \cdot z = 0\}$. Let

$$\Phi_z = \frac{\omega'(\xi) \wedge \omega(x)}{(\xi \cdot z)^n} \quad \text{and} \quad \Psi_z = \frac{\omega^*(\xi) \wedge \omega(x)}{(\xi \cdot z)^{n+1}}.$$

Then Φ_z is a holomorphic $(2n-1)$-form and Ψ_z is a holomorphic $2n$-form on $Y \setminus P_z$, with poles of order n and $n+1$ along P_z respectively, and

$$d\Phi_z = n\Psi_z.$$

We usually omit the subscript z.

We will work with homology and cohomology groups throughout the paper. Leray uses integral homology with the torsion part removed; we might as well use homology with complex coefficients. Our cohomology groups will be complex de Rham groups, both absolute and relative.

The projection $Q \setminus P \to X \setminus \{z\}$ is a fibration with contractible fiber $\mathbb{C}^{n-1}$, so it is a homotopy equivalence and induces isomorphisms on all homology and cohomology groups. The boundary map $H_{2n}(X, X \setminus \{z\}) \to H_{2n-1}(X \setminus \{z\})$ is injective with a 1-dimensional source. Its image (i.e., the kernel of the map $H_{2n-1}(X \setminus \{z\}) \to H_{2n-1}(X)$) is generated by the image of a unique integral class α_z in $H_{2n-1}(Q \setminus P)$ on which $i^{-n}\Phi > 0$. The image of α in $H_{2n-1}(X \setminus \{z\})$ is represented by the boundary of any smoothly bounded relatively compact subdomain of X containing z, with the appropriate orientation. It may be checked that the orientation is given by the outward normal when $n \equiv 1, 2 \pmod 4$ and by the inward normal when $n \equiv 0, 3 \pmod 4$.

With these definitions we can now state the first formula. For the proof, see [L1, Sec. 56].

THE FIRST CAUCHY–FANTAPPIÈ FORMULA. *If f is a holomorphic function on a domain X in $\mathbb{C}^n$ and if $z \in X$, then*

$$f(z) = \frac{(n-1)!}{(2\pi i)^n} \int_{\alpha_z} f\Phi_z.$$

Here, by $f\Phi_z$ we mean Φ multiplied by f precomposed with the projection $Y \to X$. Note that $f\Phi$ is closed on $Q \setminus P$ (since it is holomorphic of degree equal to the dimension of $Q \setminus P$) and so the integral makes sense.

For $n = 1$, this is the Cauchy formula. In most applications, α is represented by a cycle that is a graph over the boundary of a subdomain containing z. This case is well described in Berndtsson's survey [B], as are variants with weight factors and related solution formulas for the $\bar{\partial}$ operator. For an example where α is represented by a cycle that is not a graph, see [K, 4.5].

Now we move on to the second formula. We wish to express α as the image of a class in $H_{2n-2}(Q \cap P)$ by Leray's coboundary map $\delta\colon H_{2n-2}(Q \cap P) \to H_{2n-1}(Q \setminus P)$. Let us represent α by the $(2n-1)$-real-dimensional smooth submanifold

$$M = \{([x \cdot \bar{z} - |x|^2, \bar{x}_1 - \bar{z}_1, \ldots, \bar{x}_n - \bar{z}_n], x) : x \in \partial B\}$$

of Y with the orientation specified previously, where $B \Subset X$ is the open ball of radius $\varepsilon > 0$ centered at z. Note that $\xi \cdot x = 0$ and $\xi \cdot z = -\varepsilon^2$ for $(\xi, x) \in M$, so $M \subset Q \setminus P$. Using the same formula on $B \setminus \{z\}$ and taking closure in Q, we obtain a $2n$-real-dimensional smooth submanifold of Q, isomorphic to $\bar{B}$ with z blown up, with boundary M. It intersects P transversely in the common fiber $\mathbb{P}^{n-1}$ of P and Q over z. This fiber represents a class $\beta_z \in H_{2n-2}(Q \cap P)$, oriented so that $\delta\beta = -\alpha$. Applying the residue formula (which says that the residue map is the dual of $\delta/2\pi i$) to the first formula, we obtain immediately the second formula.

THE SECOND CAUCHY–FANTAPPIÈ FORMULA. *If f is a holomorphic function on a domain X in $\mathbb{C}^n$ and if $z \in X$, then*

$$f(z) = -\frac{(n-1)!}{(2\pi i)^{n-1}} \int_{\beta_z} \operatorname{Res}_{Q \cap P}(f\Phi_z).$$

By Leray's long exact sequence, the kernel of δ is the image of the map $H_{2n}(Q) \to H_{2n-2}(Q \cap P)$ that is induced by intersecting with P, which is injective because $H_{2n}(Q \setminus P) = 0$. Hence, β is the unique class with $\delta\beta = -\alpha$ if and only if $H_{2n}(Q) = 0$, which is the case if X is contractible.

The class β is never a boundary in P. To obtain the third formula we must express β as a relative boundary with respect to a new smooth hypersurface S in Y such that P, Q, and S are in general position. Consider the following diagram:

$$\begin{tikzcd}
 & & H_{2n-2}(P \cap S) \arrow[d] \\
 & \beta \in H_{2n-2}(Q \cap P) \arrow[r] \arrow[d, "p"'] & H_{2n-2}(P) \arrow[d] \\
H_{2n-1}(P, Q \cup S) \arrow[r, "\partial"] & H_{2n-2}(Q \cap P, S) \arrow[r] & H_{2n-2}(P, S).
\end{tikzcd}$$

Assume that $p\beta = \partial\gamma$ for some $\gamma \in H_{2n-1}(P, Q \cup S)$ (i.e., that β is homologous in P to a cycle in $P \cap S$). We will refer to this as the *topological condition* on S. It holds if the map $H_{2n-2}(P \cap S) \to H_{2n-2}(P)$ is surjective. It also holds if, but not only if, S contains a fiber of the projection $P \to X$: in Example C of Section 3, the inclusion $P \cap S \hookrightarrow P$ is a homotopy equivalence, but S does not contain a P-fiber. Note that γ is uniquely determined modulo the image of $H_{2n-1}(P, S)$.

Leray shows that the residue map Res commutes with p^* and anti-commutes with the dual ∂^* of the boundary map. Note that $f\Phi$ vanishes on $(Q \setminus P) \cap S$ for dimensional reasons, so it represents a class in $H^{2n-1}(Q \setminus P, S)$. Now, by the second formula,

$$\begin{aligned}
-&\frac{(2\pi i)^{n-1}}{(n-1)!} f(z) \\
&= \int_\beta \mathrm{Res}_{Q\cap P}\, f\Phi = \int_\beta \mathrm{Res}_{Q\cap P}\, p^*(f\Phi) = \int_\beta p^* \,\mathrm{Res}_{(Q\cap P, S)}\, f\Phi \\
&= \int_{p\beta} \mathrm{Res}_{(Q\cap P, S)}\, f\Phi = \int_{\partial\gamma} \mathrm{Res}_{(Q\cap P, S)}\, f\Phi = \int_\gamma \partial^* \,\mathrm{Res}_{(Q\cap P, S)}\, f\Phi \\
&= -\int_\gamma \mathrm{Res}_{(P, Q\cup S)}\, \partial^*(f\Phi).
\end{aligned}$$

The map ∂^* is the dual of the boundary map ∂ in *relative* homology. It is not simply the exterior derivative. To calculate $\partial^*(f\Phi) \in H^{2n}(Y \setminus P, Q \cup S)$, we extend $f\Phi$ to a smooth form on $Y \setminus P$, vanishing on $S \setminus P$, and differentiate. If $\Phi | S \setminus P = 0$, then the extension can be chosen to be $f\Phi$ itself, and $\partial^*(f\Phi)$ is represented by $d(f\Phi) = fd\Phi = nf\Psi$. Leray [L1, p. 155] seems to assume that this is so; ∂^* becomes d, and his third formula reads as follows.

THE THIRD CAUCHY–FANTAPPIÈ FORMULA. *Let f be a holomorphic function on a domain X in $\mathbb{C}^n$, and let $z \in X$. Let S be a smooth hypersurface in Y such that P_z, Q, and S are in general position. Then*

$$f(z) = \frac{n!}{(2\pi i)^{n-1}} \int_\gamma \mathrm{Res}_{(P, Q\cup S)}\, f\Psi_z$$

for every $\gamma \in \partial^{-1}(p\beta_z) \subset H_{2n-1}(P, Q \cup S)$.

Here we view $f\Psi$ as a class in $H^{2n}(Y \setminus P, Q \cup S)$, which the residue map takes to $H^{2n-1}(P, Q \cup S)$. At this point, the reader might find it helpful to consult Example A in Section 3 to see what the formula looks like in a simple case.

Without some additional conditions on S, *the third formula is false,* as shown by Examples A, D, and E in Section 3. The third formula fails in two different ways in these examples. In Examples D and E, $n = 2$ and the integral on the right-hand side is not independent of the choice of γ in $\partial^{-1}(p\beta)$ for $f = 1$. In Example A, $X = \mathbb{C}$ and $H_1(P, Q \cup S)$ is 1-dimensional and so there is only one choice of γ, but the formula fails for $f \in \mathcal{O}(\mathbb{C})$ with $f(1) \neq 0$. We do not have an example where the formula holds for $f = 1$ but fails for other f. In fact, we know nothing about the set of functions for which the formula holds beyond the obvious fact that it is a vector subspace of $\mathcal{O}(X)$.

An important special case in which the third formula does hold is when S is the preimage of a smooth hypersurface S_0 in X. Then $\Phi | S \setminus P$ clearly vanishes, the topological condition is satisfied, and it is easy to see that P_z, Q, and S are in general position if and only if $z \notin S_0$. Examples B and C in Section 3 show that the third formula holds not only when S is a preimage.

Before presenting our examples, we give various answers to the question of when the third formula holds, starting with a sufficient condition that is also necessary in the contractible case.

2. Necessary and Sufficient Conditions

Throughout this section, X is a domain in $\mathbb{C}^n$, $z \in X$, $f \in \mathcal{O}(X)$, S is a smooth hypersurface in Y such that P_z, Q, and S are in general position, and S satisfies the topological condition $\partial^{-1}(p\beta_z) \neq \emptyset$.

From our discussion in the previous section, it is clear that the third formula holds if and only if

$$\operatorname{Res}_{(P,Q\cup S)} \partial_Q^*(f\Phi) = \operatorname{Res}_{(P,Q\cup S)} d(f\Phi) \quad \text{on } \partial^{-1}(p\beta) \subset H_{2n-1}(P, Q \cup S),$$

where we have written ∂_Q for the boundary map

$$H_{2n}(Y \setminus P, Q \cup S) \to H_{2n-1}(Q \setminus P, S)$$

that previously was simply denoted ∂, because now we also want to consider the boundary map

$$\partial_S \colon H_{2n}(Y \setminus P, Q \cup S) \to H_{2n-1}(S \setminus P, Q).$$

For convenience, let us write φ for $f\Phi$. As already described, $\partial_Q^*\varphi$ is represented by $d\sigma$, where $\sigma | Q \setminus P = \varphi$ and $\sigma | S \setminus P = 0$. Now φ also represents a class in $H^{2n-1}(S \setminus P, Q)$, and $\partial_S^*\varphi$ is represented by $d\tau$, where $\tau | Q \setminus P = 0$ and $\tau | S \setminus P = \varphi$. Hence, $d\varphi - \partial_Q^*\varphi = \partial_S^*\varphi$.

Consider the following diagram:

$$\begin{tikzcd}
 & H^{2n-2}(P, Q) \arrow[r, "p^*"] \arrow[d, "i^*"'] & H^{2n-2}(P) \arrow[d, "i^*"'] \\
H^{2n-3}(P \cap Q \cap S) \arrow[r, "\partial^*"] \arrow[d, "\partial^*"'] & H^{2n-2}(P \cap S, Q) \arrow[r, "p^*"] \arrow[d, "\partial^*"'] & H^{2n-2}(P \cap S) \\
H^{2n-2}(P \cap Q, S) \arrow[r, "\partial^*"] \arrow[d, "p^*"'] & H^{2n-1}(P, Q \cup S) \arrow[d, "p^*"'] & \\
H^{2n-2}(P \cap Q) \arrow[r, "\partial^*"] & H^{2n-1}(P, Q). &
\end{tikzcd}$$

The rows and columns are parts of long exact sequences. The top and bottom squares commute. Let us verify that the middle square anti-commutes, pretending for a moment (to simplify the notation) that Q and S are hypersurfaces in P. Take a closed form ω_0 on $Q \cap S$ and extend it to a form ω on P. The upper part of the square works like this: first, $\partial^*\omega_0 = d\omega|S$; then we extend $d\omega|S$ to a form ω_1 on P such that $\omega_1|Q = 0$, and $\partial^*(d\omega|S) = d\omega_1$. The lower part of the square works like this: first, $\partial^*\omega_0 = d\omega|Q$; then we extend $d\omega|Q$ to a form ω_2 on P such that $\omega_2|S = 0$, and $\partial^*(d\omega|Q) = d\omega_2$. We must show that $d\omega_1 + d\omega_2 = 0$ in $H^{2n-1}(P, Q \cup S)$. This holds because $d\omega_1 + d\omega_2 = d(\omega_1 + \omega_2 - d\omega)$ and $\omega_1 + \omega_2 - d\omega = 0$ on $Q \cup S$.

Write $\rho = \operatorname{Res}_{(P,Q\cup S)} \partial_S^*\varphi$. We have seen that the third formula holds if and only if $\rho = 0$ on $\partial^{-1}(p\beta)$—that is, if and only if $\rho = 0$ on γ and on $\operatorname{Ker}\partial$, where γ is some element of $\partial^{-1}(p\beta)$. This is equivalent to $\rho \in \operatorname{Im}\partial^*$, say $\rho = \partial^*\sigma$ with $\sigma \in H^{2n-2}(P \cap Q, S)$, and $p^*\sigma(\beta) = \rho(\gamma) = 0$.

Let us now assume that X is contractible. Then $H_{2n-2}(P \cap Q)$ is generated by β, so $p^*\sigma(\beta) = 0$ implies $p^*\sigma = 0$. Hence, the third formula is equivalent to $\rho \in \partial^*(\operatorname{Ker} p^*)$, that is, to ρ coming from $H^{2n-3}(P \cap Q \cap S)$ (which is, of course, trivial when $n = 1$).

The anti-commuting square

$$\begin{array}{ccc} \varphi \in H^{2n-1}(S \setminus P, Q) & \xrightarrow{\partial_S^*} & H^{2n}(Y \setminus P, Q \cup S) \\ \Big\downarrow{\scriptstyle \mathrm{Res}} & & \Big\downarrow{\scriptstyle \mathrm{Res}} \\ H^{2n-2}(P \cap S, Q) & \xrightarrow{\partial^*} & H^{2n-1}(P, Q \cup S) \ni \rho \end{array}$$

shows that ρ comes from $H^{2n-2}(P \cap S, Q)$ with preimage $-\operatorname{Res}_{(P\cap S,Q)}\varphi$ there. By chasing the big diagram, we see that ρ comes from $H^{2n-3}(P \cap Q \cap S)$ if and only if $p^* \operatorname{Res}_{(P\cap S,Q)}\varphi = \operatorname{Res}_{P\cap S}\varphi$ comes from $H^{2n-2}(P, Q)$. Now the map $p^* \colon H^{2n-2}(P, Q) \to H^{2n-2}(P)$ is zero because the map $i^* \colon H^{2n-2}(P) \to H^{2n-2}(P \cap Q)$, which follows it in the long exact cohomology sequence, is an isomorphism. Hence, the third formula holds if and only if $\operatorname{Res}_{P\cap S}\varphi$ is zero in $H^{2n-2}(P \cap S)$ (here, we view φ as representing a class in $H^{2n-1}(S \setminus P)$). This is equivalent to the existence of a smooth $(2n-2)$-form σ on $S \setminus P$ such that $\varphi + d\sigma$ extends smoothly to S. In the general case, when X is not necessarily contractible, this is a sufficient condition for the formula to hold.

Let us now derive a necessary condition for the third formula to hold. This condition fails in Examples D and E in Section 3. The third formula implies that the integral on its right-hand side is independent of the choice of γ in $\partial^{-1}(p\beta)$, that is,

$$\int_{p\theta} \operatorname{Res}_{(P,Q\cup S)} d\varphi = 0 \quad \text{for every } \theta \in H_{2n-1}(P, S).$$

Viewing $d\varphi$ as also representing a class in $H^{2n}(Y \setminus P, S)$ that is the image by p^* of the class represented by $d\varphi$ in $H^{2n}(Y \setminus P, Q \cup S)$, we see that this condition is equivalent to $\operatorname{Res}_{(P,S)} d\varphi$ vanishing in $H^{2n-1}(P, S)$. This is equivalent to $d\varphi$ coming from $H^{2n}(Y, S)$, which means that there is a smooth $(2n-1)$-form σ on $Y \setminus P$ with $\sigma|S \setminus P = 0$ such that $d(\varphi - \sigma)$ extends smoothly to Y.

We have proved the following result.

THEOREM. *Let X be a domain in $\mathbb{C}^n$, $z \in X$, $f \in \mathcal{O}(X)$, and S a smooth hypersurface in Y such that P_z, Q, and S are in general position and S satisfies the* *topological condition. For Leray's third formula to hold for this data, it is necessary that*

$$\mathrm{Res}_{(P_z,S)}\, f\Psi_z = 0 \ \textit{ in } H^{2n-1}(P_z, S)$$

and sufficient that

$$\mathrm{Res}_{P_z\cap S}\, f\Phi_z = 0 \ \textit{ in } H^{2n-2}(P_z\cap S).$$

When X is contractible, this sufficient condition is also necessary.

In order to derive a characterization of the third formula, we do not actually need X to be contractible. The preceding argument shows that it suffices to have $H_{2n-2}(P\cap Q)$ and $H_{2n-2}(P) = H_{2n-2}(\mathbb{P}^{n-1}\times X)$ be 1-dimensional (each generated by β).

There is no shortage of other sufficient conditions for Leray's third formula to hold. One is that $\varphi|S\setminus P$ extend holomorphically to S: then, surely, its residue along $P\cap S$ vanishes. Examples B and C in Section 3 illustrate this. Another is for $H_{2n-1}(S\setminus P)\to H_{2n-1}(S)$ to be injective: then every class in $H^{2n-1}(S\setminus P)$ lies in the image of $H^{2n-1}(S)$. A third sufficient condition is that $\partial_Q^*\varphi$ and $d\varphi$ have the same residue along $(P, Q\cup S)$, as is evident from our discussion preceding the statement of the formula. This is equivalent to the existence of a smooth $(2n-1)$-form σ on $Y\setminus P$ with $\sigma|S\setminus P = 0$ and $\sigma=\varphi$ on $Q\setminus P$ such that $d(\varphi-\sigma)$ extends smoothly to Y.

We have pinched Leray's third formula between this sufficient condition and the necessary condition that such a form σ exist without requiring that $\sigma=\varphi$ on $Q\setminus P$. We have also pinched Leray's third formula between the sufficient condition that $\mathrm{Res}_{P\cap S}\,\varphi = 0$ and the (same) necessary condition that $\partial^*\,\mathrm{Res}_{P\cap S}\,\varphi = -\mathrm{Res}_{(P,S)}\,d\varphi = 0$.

We conclude this section by pointing out that if S contains a fiber of the projection $P\to X$ (which, by Example C of Section 3, is stronger than the topological condition), then the sufficient condition $\Phi|S\setminus P = 0$ actually implies our strongest and most obvious sufficient condition: that S be the preimage of a hypersurface in X. We may assume that $z=0$. First, in the affine coordinates $y_j = \xi_j/\xi_0$, we have

$$\Phi = y_1^n d\left(\frac{y_2}{y_1}\right)\wedge\cdots\wedge d\left(\frac{y_n}{y_1}\right)\wedge\omega(x);$$

so, changing to the coordinates $w_1 = y_1$ and $w_j = y_j/y_1$ for $j = 2,\ldots,n$, we have $\Phi = 0$ on the subset of $S\setminus P$ where $\xi_1\neq 0$ if and only if $dw_2\wedge\cdots\wedge dw_n\wedge dx_1\wedge\cdots\wedge dx_n = 0$ there. This is equivalent to $\partial s/\partial w_1 = 0$ for any local defining function s for S, that is, to S consisting of curves on which $w_2,\ldots,w_n,x_1,\ldots,x_n$ are constant. Hence, for every $x\in X$, $S\cap(\mathbb{C}^n\times\{x\})$ is a union of lines through the origin, or is the origin itself, or is empty. Adding the point at infinity to each of these affine lines yields the closure of $S\setminus P$ in Y, which equals S since $S\setminus P$ is dense in S (this clearly follows from P and S being in general position, but $P\not\subset S$ is enough). Hence, the assumption that S contains a P-fiber implies that S contains the whole fiber $\mathbb{P}^n\times\{x\}$ for some $x\in X$. Because it is a hypersurface, S cannot also contain $\{0\}\times X$, so its image by the projection $Y\to X$ (which is proper and hence takes a subvariety to a subvariety) must be a hypersurface in X with S as its preimage in Y.

3. Examples

In this section, we present the following five examples of smooth hypersurfaces S in $\mathbb{P}^n \times X$ such that P_z, Q, and S are in general position for $z = 0$ and S contains a fiber of the projection $P \to X$, so S satisfies the topological condition; that is, the basic assumptions in the third Cauchy–Fantappiè formula are in place.

A. Examples with $X = \mathbb{C}$, where the third formula holds (to show what it looks like in the simplest case) and where it fails.
B. A more involved example with $X = \mathbb{C}$, where $\Phi | S \setminus P$ extends holomorphically to S and so the third formula holds, although S is not the preimage of a subset of X.
C. Same as Example B, but with $X = \mathbb{C}^2$; also, a modification of this example that satisfies the topological condition even though it does not contain a P-fiber.
D. An example with $n = 2$, where the necessary condition $\operatorname{Res}_{(P,S)} f\Psi = 0$ (and hence the third formula) fails for $f = 1$; here, X is contractible but not convex.
E. An example with $X = \mathbb{C}^2$ where the necessary condition fails for $f = 1$; here, the computations are considerably more complicated than in Example D.

EXAMPLE A. We first consider the third Cauchy–Fantappiè formula in the simplest case, with $X = \mathbb{C}$ and $z = 0$. Letting $\eta = \xi_0/\xi_1$, we have $P = \{\eta = 0\} \cong \mathbb{C}$, $Q = \{\eta + x = 0\}$, and $\Psi = \eta^{-2} dx \wedge d\eta$. Take $S = \{a\eta + x = 1\}$ with $a \in \mathbb{C}$. Then P, Q, and S are in general position and S satisfies the topological condition. In fact, S contains the P-fiber above $x = 1$, and $\gamma \in H_1(P, \{0, 1\})$ is represented by any path $\langle 1, 0\rangle$ in $\mathbb{C}$ from 1 to 0. Let

$$\begin{aligned}\sigma &= \frac{a\eta + x - 1}{\eta}(d\eta + dx) - \frac{\eta + x}{\eta}(a\,d\eta + dx)\\ &= \left((1-a)\frac{x}{\eta} - \frac{1}{\eta}\right)d\eta + \left(a - 1 - \frac{1}{\eta}\right)dx.\end{aligned}$$

The first expression for σ shows that it vanishes on $Q \cup S$, and the second one shows that

$$d\sigma = (a-1)\frac{d\eta}{\eta} \wedge dx - \Psi,$$

so $\operatorname{Res}_{(P,Q\cup S)} \Psi$ is represented by the form $(a-1)\,dx$. If $f \in \mathcal{O}(\mathbb{C})$, then similarly $f\sigma$ vanishes on $Q \cup S$ and

$$f\Psi + d(f\sigma) = \frac{d\eta}{\eta} \wedge d(f(x)((a-1)x + 1)),$$

so $\operatorname{Res}_{(P,Q\cup S)} f\Psi$ is represented by the form $d(f(x)((a-1)x+1))$, and the third formula looks as follows:

$$f(0) = \int_{\langle 1,0\rangle} d(f(x)((a-1)x+1)) = f(0) - af(1).$$

Clearly, the formula holds for all $f \in \mathcal{O}(\mathbb{C})$ only when $a = 0$, but it fails for f with $f(1) \neq 0$ when $a \neq 0$.

EXAMPLE B. Again, we take $X = \mathbb{C}$ and $z = 0$, but now S is defined by the equation

$$s(\xi_0, \xi_1, x) = \xi_0^2 + \xi_1(\xi_0 + \xi_1)(x-1) = 0.$$

Clearly, S is not the preimage of a subset of $\mathbb{C}$, and $P \cap S$ is the point above 1 in $\mathbb{C}$. It is easily verified that S is smooth and that P, Q, and S are in general position.

We claim that $\Phi|S \setminus P$ (and hence $f\Phi|S \setminus P$ for every $f \in \mathcal{O}(X)$) extends holomorphically to S. Working near P, where $\xi_1 \neq 0$, and using the affine coordinate $\eta = \xi_0/\xi_1$, we have $\Phi = dx/\eta$. On S near P we have $x = 1 - \eta^2/(\eta+1)$, so $dx = -\eta(\eta+2)/(\eta+1)^2 d\eta$ and $\Phi = -(\eta+2)/(\eta+1)^2 d\eta$, which extends holomorphically across $\eta = 0$. Hence, the third formula holds. Let us check this by calculating the integrand $\mathrm{Res}_{(P,Q\cup S)} f\Psi$. We work on the neighborhood of P where $\xi_1 \neq 0$. Let q be the defining function $\eta + x$ for Q. Let

$$\sigma = \frac{f}{\eta}(sdq - qds) = -\frac{f(x)(x-1)^2}{\eta} d\eta - \frac{f}{\eta} dx + \theta,$$

where θ is smooth near P. Then σ vanishes on $Q \cup S$ and

$$f\Psi + d\sigma = \frac{d\eta}{\eta} \wedge d(f(x)(x-1)^2) + d\theta,$$

so the residue $\mathrm{Res}_{(P,Q\cup S)} f\Psi$ is represented by the form $d(f(x)(x-1)^2)$. Integrating this form along a path from 1 to 0 gives $f(0)$.

EXAMPLE C. Here, $X = \mathbb{C}^2$, $z = 0$, and S is defined by the equation

$$s(\xi_0, \xi_1, \xi_2, x_1, x_2) = \xi_0^3 + \xi_1^3(x_1 - 1) + \xi_2^3(x_2 - 2) = 0.$$

Note that S is not the preimage of a curve in X and that S contains the P-fiber above $(1, 2)$. Lengthy but routine computations show that S is smooth and that P, Q, and S are in general position. We will show that $\Phi|S \setminus P$ extends holomorphically to S, so the third formula holds. Let U_k be the subset of S where $\xi_k \neq 0$ $(k = 1, 2)$. We shall verify that $\Phi|S \setminus P$ extends across P on U_2; the case of U_1 is analogous. In the affine coordinates $y_0 = \xi_0/\xi_2$ and $y_1 = \xi_1/\xi_2$, we have $\Phi = -y_0^{-2} dy_1 \wedge dx_1 \wedge dx_2$. On $S \cap U_2$ we have $x_2 = 2 - y_0^3 - y_1^3(x_1 - 1)$, so there $\Phi = 3dy_0 \wedge dy_1 \wedge dx_1$, which clearly extends across P.

Now take

$$s(\xi_0, \xi_1, \xi_2, x_1, x_2) = \xi_0^3 + \xi_1^3(x_1 - 1) + \xi_2^3(x_2 - 2) + 2\xi_1^2\xi_2.$$

Just as before, S is smooth, P, Q, and S are in general position, and $\Phi|S \setminus P$ extends holomorphically to S. Also, S is not the preimage of a curve in X; in fact, it is easy to see that S does not even contain a P-fiber. All the same, S satisfies the topological condition, as we will now verify. Consider the projection $\pi : P \cap S \hookrightarrow P = \mathbb{P}^1 \times X \to \mathbb{P}^1$. Since $P \to \mathbb{P}^1$ is a homotopy equivalence, if we can show that π is a homotopy equivalence then so is the inclusion $P \cap S \hookrightarrow P$, and the topological condition follows.

We will show that π is a fibration with contractible fibers. Now $P \cap S$ is given by the equations $\xi_0 = 0$ and $\xi_1^3(x_1 - 1) + \xi_2^3(x_2 - 2) + 2\xi_1^2\xi_2 = 0$. The fibers of π are lines in $\mathbb{C}^2$ because the coefficients of x_1 and x_2 do not vanish simultaneously. We claim that π is surjective. Namely, if $\xi = [\xi_1, \xi_2] \in \mathbb{P}^1$ and $\xi_1 \neq 0$, we set $x_2 = 0$ and $x_1 = \xi_1^{-3}(\xi_1^3 + 2\xi_2^3 - 2\xi_1^2\xi_2)$. Then $(\xi, x) \in P \cap S$ and $\pi(\xi, x) = \xi$. If $\xi_2 \neq 0$, we take $x_1 = 0$ and $x_2 = \xi_2^{-3}(\xi_1^3 + 2\xi_2^3 - 2\xi_1^2\xi_2)$. Finally, π is locally trivial: over $U = \{\xi \in \mathbb{P}^1 : \xi_1 \neq 0\}$, the trivialization map is $\pi^{-1}(U) \to U \times \mathbb{C}$, $(\xi, x_1, x_2) \mapsto (\xi, x_2)$, and the inverse map is given by $x_1 = 1 - \xi_1^{-3}(\xi_2^3(x_2 - 2) + 2\xi_1^2\xi_2)$. Over $\{\xi_2 \neq 0\}$, the trivialization is given by $(\xi, x_1, x_2) \mapsto (\xi, x_1)$ and the inverse map is obtained in the same way.

EXAMPLE D. Here, $n = 2$, $z = 0$, and S is defined by the equation

$$s(\xi_0, \xi_1, \xi_2, x_1, x_2) = \xi_0^2 + \xi_1(\xi_1 + \xi_2)(x_1 - 1)x_2 + \xi_2^2(x_2^2 + 1) = 0$$

over a contractible domain X in $\mathbb{C}^2$ that will be specified later. Note that S contains the P-fibers over $(1, \pm i)$. Computations (whose details will not be reproduced here) show that:

(i) S is smooth outside P;
(ii) at each point of $P \cap S$, S is smooth and in general position with respect to P except over the point $(1, 0)$;
(iii) at each point of $(Q \cap S) \setminus P$, Q and S are in general position except over a finite set in $\mathbb{C}^2$ that does not contain $(1, \pm i)$ or the base point $(0, 0)$; and
(iv) at each point of $P \cap Q \cap S$, the three surfaces are in general position.

Let us show that the necessary condition fails—in other words, that $\operatorname{Res}_{(P,S)} \Psi \neq 0$. Let $U = \{\xi_2 \neq 0\}$. It suffices to prove that $\operatorname{Res}_{(P\cap U, S\cap U)} \Psi|U \neq 0$. In the following, we will work on U but omit it from the notation. In the affine coordinates $y_0 = \xi_0/\xi_2$ and $y_1 = \xi_1/\xi_2$,

$$\Psi = y_0^{-3} dy_0 \wedge dy_1 \wedge dx_1 \wedge dx_2$$

and

$$s = y_0^2 + y_1(y_1 + 1)(x_1 - 1)x_2 + x_2^2 + 1.$$

Let

$$\tau = \frac{s}{y_0^2} dy_1 \wedge dx_1 \wedge dx_2 + \frac{(x_1 - 1)dy_1 \wedge dx_2 - x_2 dy_1 \wedge dx_1}{2y_0^2} \wedge ds.$$

Then $\tau|S$ clearly vanishes and

$$d\tau = 2\left(\frac{1}{y_0} - \frac{1}{y_0^3}\right) dy_0 \wedge dy_1 \wedge dx_1 \wedge dx_2,$$

so

$$\Psi + \frac{1}{2}d\tau = \frac{dy_0}{y_0} \wedge dy_1 \wedge dx_1 \wedge dx_2$$

and $\operatorname{Res}_{(P,S)} \Psi$ is represented by the form $dy_1 \wedge dx_1 \wedge dx_2 = d((1-x_1)dy_1 \wedge dx_2)$. We need to show that this form is not null-cohomologous on P relative to S. By Leray's long exact sequence and contractibility of X, it suffices to show that $(1-x_1)dy_1 \wedge dx_2$ is not exact on $S \cap P$. Now on $S \cap P$ we have

$$1 - x_1 = \frac{x_2^2 + 1}{x_2 y_1 (y_1 + 1)}$$

(where the denominator does not vanish), and integrating the form $(1-x_1)dy_1 \wedge dx_2$ over the 2-cycle

$$y_1 = \varepsilon e^{i\theta}, \quad x_2 = \varepsilon e^{i\eta}, \quad x_1 = 1 - \frac{1 + \varepsilon^2 e^{2i\eta}}{\varepsilon^2 e^{i(\theta+\eta)}(1 + \varepsilon e^{i\theta})}, \quad \theta, \eta \in [0, 2\pi],$$

in $S \cap P$ with $0 < \varepsilon < 1$ (using the 1-dimensional residue formula) does *not* yield zero.

Therefore, X must satisfy the following conditions: X is contractible, X contains $(0,0)$ and $(1, \pm i)$, X avoids a certain finite set not containing these points, and X contains the set of points (x_1, x_2) as before with $\theta, \eta \in [0, 2\pi]$ for some $\varepsilon \in (0,1)$. This set is a 2-dimensional real submanifold of $\mathbb{C}^2$ that does not disconnect $\mathbb{C}^2$. We can, for instance, take X to be the complement of a broken half-line joining the points we must avoid and going out to infinity.

Example E. Here, $X = \mathbb{C}^2$, $z = 0$, and S is defined by the equation

$$s(\xi_0, \xi_1, \xi_2, x_1, x_2) = \xi_0^2 + (\xi_1^2 + 3\xi_1\xi_2 x_2 + 2\xi_2^2 x_2^2)(x_1 - 1) + \xi_2^2(x_2^3 + 1) = 0.$$

Clearly, S contains the three P-fibers above $\left(1, \sqrt[3]{-1}\right)$. Calculations—which are considerably more complicated than the corresponding ones in Example D (and for which one may want to use a computer algebra system)—show that S is smooth and that P, Q, and S are in general position.

Working on $U = \{\xi_2 \neq 0\}$ (which again will be omitted from the notation) with affine coordinates $y_0 = \xi_0/\xi_2$ and $y_1 = \xi_1/\xi_2$, we shall show that $\operatorname{Res}_{(P,S)} \Psi \neq 0$. Let

$$\tau = \frac{s}{y_0^2} dy_1 \wedge dx_1 \wedge dx_2 - \frac{y_1 dx_1 \wedge dx_2 - (x_1 - 1)dy_1 \wedge dx_2 + x_2 dy_1 \wedge dx_1}{3y_0^2} \wedge ds.$$

We see that $\tau | S = 0$ and compute that

$$d\tau = 2\left(\frac{1}{y_0} - \frac{1}{y_0^3}\right) dy_0 \wedge dy_1 \wedge dx_1 \wedge dx_2.$$

Hence, as before, $\operatorname{Res}_{(P,S)} \Psi$ is represented by the form $dy_1 \wedge dx_1 \wedge dx_2$, and we must show that $\theta = (1-x_1)dy_1 \wedge dx_2$ is not exact on $S \cap P$.

The projection $(y_1, x_1, x_2) \mapsto (y_1, x_2)$ restricts to an isomorphism of $S \cap P \setminus \{(y_1 + x_2)(y_1 + 2x_2) = 0\}$ onto $\mathbb{C}^2 \setminus \{(y_1 + x_2)(y_1 + 2x_2) = 0\}$ with inverse given by

$$1 - x_1 = \frac{x_2^3 + 1}{(y_1 + x_2)(y_1 + 2x_2)}.$$

Set $u = y_1 + x_2$ and $v = y_1 + 2x_2$, so now $x_2 = v - u$, $y_1 = 2u - v$, and $dy_1 \wedge dx_2 = du \wedge dv$. The map $(y_1, x_1, x_2) \mapsto (u, v)$ is an isomorphism of $S \cap P \setminus \{(y_1 + x_2)(y_1 + 2x_2) = 0\}$ onto $\mathbb{C}^2 \setminus \{uv = 0\}$, and the push-forward $\frac{(v-u)^3+1}{uv} du \wedge dv$ of θ is not exact on the image: just integrate it over the product of two small circles centered at the origin, using the 1-dimensional residue formula.

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