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On the Geography of Stein Fillings of Certain 3-Manifolds

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1. Introduction

Let (Y, ξ) be a given (closed) contact 3-manifold. (For basic definitions regarding contact structures the reader is advised to consult e.g. [Ae; Et2].) A 4-manifold X is called a *Stein filling* of (Y, ξ) if X is a sublevel set of a plurisubharmonic function on a Stein surface and if (Y, ξ) is contactomorphic to ∂X (the contact structure induced by the complex tangencies). For a more detailed account regarding Stein fillings, see [LiMa].

Inspired by the geography problem of minimal surfaces of general type—that is, the determination of pairs (signature, Euler characteristic) $= (\sigma, \chi)$ of such 4-manifolds—we are led to the following.

Problem 1.1. Describe characteristic numbers of Stein fillings of a given contact 3-manifold (Y, ξ).

This problem has been solved for particular 3-manifolds—such as the 3-sphere S^3, the Poincaré sphere $\Sigma(2,3,5)$ (with both orientations), and lens spaces with specific contact structures—in a much stronger sense: for these examples also, the diffeomorphism classification of Stein fillings has been achieved (see [E1; Mc; OO]). For related results concerning $-\Sigma(2,3,11)$ and the 3-torus T^3, see [St1]. These examples led us to the following conjecture.

Conjecture 1.2. *The set*

$$\mathcal{CF}_{(Y,\xi)} = \{b_1(W), \sigma(W), \chi(W) \mid W \text{ is a Stein filling of } (Y,\xi)\}$$

of the characteristic numbers of Stein fillings of a given 3*-manifold* (Y, ξ) *is a finite set*.

Remark 1.3. A related conjecture could be formulated by examining finiteness properties of the set of diffeomorphism types of Stein fillings of a given contact 3-manifold (Y, ξ). In the light of a recent observation of I. Smith, this conjecture is too ambitious in general.

Our main result in this paper makes a minor step for verifying Conjecture 1.2 in general—and, in fact, proves the conjecture in some particular cases; see Corollaries 1.5 and 1.7.

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THEOREM 1.4. *For a given contact* 3*-manifold* (Y, ξ) *there exists a constant* $K_{(Y,\xi)}$ *such that, if* W *is a Stein filling of* (Y, ξ), *then* $3\sigma(W)+2\chi(W) \geq K_{(Y,\xi)}$. *In other words, the number* $c(W) = 3\sigma(W) + 2\chi(W)$ *for a Stein filling* W *of* (Y, ξ)*—a number that resembles the* c_1^2*-invariant of a closed complex surface—is bounded from below.*

COROLLARY 1.5. *If every Stein filling of* (Y, ξ) *has vanishing* b_2^+*-invariant, then* $\mathcal{CF}_{(Y,\xi)}$ *is finite.*

Proof. It is a standard fact that $b_1(W)$ of a Stein filling W is bounded above by $b_1(\partial W) = b_1(Y)$ (since W admits a handle decomposition with only 0-, 1-, and 2-handles; see also [St1]). By assumption, $b_2^+(W) = 0$ and hence the estimate of $\sigma(W)$ and $\chi(W)$ reduces to estimating $b_2^-(W)$. (Notice that $\chi(W) = 1 - b_1(W) + b_2^+(W) + b_2^-(W) + b_2^0(W)$ and, by the long exact sequence of $(W, \partial W)$, the term $b_2^0(W)$ is bounded by $b_1(\partial W) = b_1(Y)$.) Now the inequality of Theorem 1.4 can be rewritten as $5b_2^+(W) - b_2^-(W) + 2 - 2b_1(W) + 2b_2^0(W) \geq K_{(Y,\xi)}$. This implies

$$5b_2^+(W) + 2 - K_{(Y,\xi)} + 2b_1(Y) \geq b_2^-(W)$$

and, since $b_2^+(W) = 0$ by assumption, we obtain the desired upper bound for $b_2^-(W)$ in terms of invariants of (Y, ξ). □

REMARK 1.6. Recall that the intersection form Q_W of the 4-manifold W with boundary is *not* nondegenerate in general. In the preceding proof, the term $b_2^0(W)$ denoted the dimension of a maximal subspace of $H_2(W; \mathbb{R})$ on which Q_W vanishes. It is not hard to see that $b_2^0(W) \leq b_1(\partial W)$; there is no such trivial bound for the invariants $b_2^{\pm}(W)$, though.

COROLLARY 1.7. *If* Y *is a circle bundle over the Riemann surface* Σ *(of genus* $g(\Sigma)$*) with Euler number* n *satisfying* $|n| > 2g(\Sigma) - 2$, *then Conjecture 1.2 holds for* (Y, ξ) *with any contact structure* ξ.

Proof. For a Stein filling W of (Y, ξ) there exists a minimal surface X of general type and a Kähler embedding $f : W \to X$ such that $b_2^+(X - \operatorname{int} W) > 0$ (see [LiMa]; also cf. Theorem 2.1). By a result of Ozsváth and Szabó [OSz1], however, Y cannot be embedded in a surface of general type such that both components of its complement have nonvanishing b_2^+-invariant. Therefore $b_2^+(W) = 0$ and so Corollary 1.5 applies and provides the result. □

REMARKS 1.8. (a) That $b_2^+(W) = 0$ for all Stein fillings of any (Y, ξ) examined in Corollary 1.7 was first noticed by Akbulut and Ozbagci [AO2].

(b) Informally, the main idea in the proof of the result of [OSz1] is the realization that certain Floer homology groups (called $\mathrm{HF}^{\mathrm{SW}}_{\mathrm{red}}$) vanish for circle bundles of the type just described. Now the standard generalization of Donaldson's famous indecomposability theorem shows that Y cannot cut certain 4-manifolds into pieces with nonvanishing b_2^+. (Some care is needed, since for Y not a homology sphere the standard argument gives only a relation between invariants rather than their vanishing—as happens in the case of splitting along S^3.) Therefore, similar results to Corollary 1.7 hold for 3-manifolds with vanishing $\mathrm{HF}^{\mathrm{SW}}_{\mathrm{red}}$ invariants. More precisely, if $\mathfrak{s} \in \mathrm{Spin}^c(Y)$ is the spin^c structure generated by the contact structure ξ (as the corresponding 2-plane field), then the vanishing of $\mathrm{HF}^{\mathrm{SW}}_{\mathrm{red}}(Y, \mathfrak{s})$ is enough to show that all Stein fillings of (Y, ξ) have vanishing b_2^+-invariant. For more such examples, see [OSz2].

As the proof of Corollary 1.5 shows, once an upper bound on the b_2^+-invariants of the Stein fillings of (Y, ξ) is achieved, the finiteness of $\mathcal{CF}_{(Y,\xi)}$ follows easily from Theorem 1.4. An upper bound for b_2^+ seems to be hard to get in general, but for some particular 3-manifolds we can prove such a bound.

THEOREM 1.9. *If Y is the Seifert fibered 3-manifold $\Sigma(2, 3, 11)$ then a Stein filling W of (Y, ξ) (with any contact structure ξ) has $b_2^+(W) \le 2$; in particular, $\mathcal{CF}_{(\Sigma(2,3,11),\xi)}$ is finite. In addition, $\Sigma(2, 3, 11)$ does admit a Stein filling with nonvanishing b_2^+.*

Conjecture 1.2 (and so our partial solution to it) can be viewed from a different perspective. Recall that a contact structure on Y can be given as an open-book decomposition of the 3-manifold Y (this decomposition is unique up to positive stabilization/destabilization [Gi]), and a Stein filling corresponds to the factorization of the monodromy of (some corresponding) open-book decomposition into the product of right-handed Dehn twists [AO1; LP]. Now the finiteness in Conjecture 1.2 asserts that the number of right-handed Dehn twists in such a decomposition of a fixed element in the mapping class group of a surface with nonempty boundary is bounded from above. This observation follows from the fact that if $h \in \Gamma_F$ decomposes as $t_{C_1} \cdots t_{C_m}$ then the Euler characteristic of the corresponding Stein filling is $\chi(F) + m$, where F stands for the fiber of the open book. (Here Γ_F denotes the mapping class group of the surface-with-boundary F; for relations between Stein domains, Lefschetz fibrations, contact structures, and open books, see [AO1; Gi; LP].) Notice that such a bound obviously does not exist in the mapping class group of a closed surface: by fiber summing Lefschetz fibrations, we find arbitrary long decompositions of the unit element 1. (Interesting to note: If $\partial F \neq \emptyset$ then 1 cannot be (nontrivially) decomposed as a product of right-handed Dehn twists; see [St2].)

After recalling some background material in Section 2, we prove Theorem 1.4 in Section 3. The proof of Theorem 1.9 is given in Section 4.

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2. Background

One of the main ingredients in our proof of Theorem 1.4 is the following theorem.

THEOREM 2.1 [LiMa]. *Let W be a Stein filling. Then there exist a minimal complex surface X of general type and a $f : W \to X$ that is a Kähler embedding. In* *addition, we can assume that* $b_2^+(X - \operatorname{int} W) > 1$. *(With a slight abuse of notation we will always conflate the Stein filling* W *and its embedded image* $f(W) \subset X$, *denoting both simply by* W.*)*

The next ingredient we need consists of (generalized) surgeries, usually called cut-and-paste operations. It turns out that, under favorable circumstances, these operations can be performed symplectically. For a more general setup the reader may refer to [Et1]; the version we present here can be found in [LiMa].

THEOREM 2.2. *Suppose that* W_1, W_2 *are two Stein fillings of the given contact* 3*-manifold* (Y, ξ) *and that* $f: W_1 \to X_1$ *is the Kähler embedding of* W_1 *into the minimal complex surface* X_1 *of general type guaranteed by Theorem 2.1. Then the* 4*-manifold* $(X_1 - \operatorname{int} W_1) \cup_Y W_2$ *admits a symplectic structure.*

In understanding basic topological properties of Stein fillings, we will use gauge-theoretic arguments—in particular, we will carry out (partial) computations of Seiberg–Witten invariants of certain 4-manifolds. For a detailed discussion of the Seiberg–Witten equations and invariants we advise the reader to consult [F; M; W]; here we restrict ourselves to highlighting those properties and facts concerning these invariants that we will use in our subsequent discussions.

The Seiberg–Witten function $SW_X : \mathrm{Spin}^c(X) \to \mathbb{Z}$ is a diffeomorphism invariant of the smooth, closed, oriented 4-manifold X with $b_2^+(X) > 1$. It is defined as a suitable count of the solutions (up to symmetry) of a system of partial differential equations defined using a metric and a spin^c structure on X. A cohomology class $K \in H^2(X; \mathbb{Z})$ is called a *basic class* if there is a spin^c structure $\mathfrak{s} \in \mathrm{Spin}^c(X)$ such that $c_1(\mathfrak{s}) = K$ and $SW_X(\mathfrak{s}) \neq 0$. If X is a symplectic 4-manifold then it admits a canonical spin^c structure $\mathfrak{s}_0$ (with $c_1(\mathfrak{s}_0) = -c_1(X)$), and it has been shown by Taubes that SW_X takes values ± 1 both on $\mathfrak{s}_0$ and on its conjugate $\bar{\mathfrak{s}}_0$ (with $c_1(\bar{\mathfrak{s}}_0) = c_1(X)$). Furthermore, if X is a minimal surface of general type then these are the only spin^c structures with nonzero invariants. Using deep analytic arguments, Taubes managed to show a relation between Seiberg–Witten and Gromov–Witten invariants of symplectic 4-manifolds, and as a consequence of this theory he proved that $c_1^2(X) = 3\sigma(X) + 2\chi(X) \in \mathbb{Z}$ of a minimal symplectic 4-manifold X with $b_2^+(X) > 1$ is always nonnegative.

By studying translation-invariant solutions of the aforementioned equations on the product $N \times \mathbb{R}$ for a closed, oriented 3-manifold N, a related theory for 3-manifolds has been developed. In [F] it was shown (among other things) that if N is an integral homology sphere then, for a generic metric g and exact perturbation μ, the Seiberg–Witten equations admit (up to gauge equivalence) finitely many irreducible solutions $\{\gamma_1, \dots, \gamma_k\}$. (Here *irreducible* means that the spinor field of the solution is not identically zero.) Now the standard pull-apart argument together with the gluing construction shows that if $X = X_1 \cup_N X_2$ is a decomposition of X along the integral homology sphere N with $b_2^+(X_i) > 0$ $(i = 1, 2)$ then, for appropriate extended metric $\tilde{g}$, perturbation $\tilde{\mu}$, and spin^c structure $\mathfrak{s} \in \mathrm{Spin}^c(X)$, the moduli space $\mathcal{M}_{X,\tilde{g},\tilde{\mu}}(\mathfrak{s})$ of solutions of the Seiberg–Witten equations (modulo symmetry) is diffeomorphic to the union $\bigcup_{j=1}^k \mathcal{M}_{X_1}(\mathfrak{s}|_{X_1})[\gamma_j] \times \mathcal{M}_{X_2}(\mathfrak{s}|_{X_2})[\gamma_j]$.

(Here $\mathcal{M}_{X_i}(\mathfrak{s}|_{X_i})[\gamma_j]$ denotes the moduli space of those solutions of the Seiberg–Witten equations on X_i that converge to γ_j. To set this theory up, we consider a metric with cylindrical end on X_i—for more details see [F], and for a related theory consult [MMrR].) Notice that, since N is assumed to be an integral homology sphere, the spin^c structure $\mathfrak{s} \in \mathrm{Spin}^c(X)$ is determined by its restrictions $\mathfrak{s}|_{X_1}$ and $\mathfrak{s}|_{X_2}$. This product formula enables us to relate Seiberg–Witten invariants of various 4-manifolds constructed by cut-and-paste techniques.

Remark 2.3. The vectors $\Sigma_{i=j}^{k} \#(\mathcal{M}_{X_i}(\mathfrak{s}|_{X_i})[\gamma_j]) \cdot \gamma_i$ are expected to give rise to "relative invariants", in some metric independent Floer homology group of the boundary N, that satisfy a product formula analogous to the one just described. However, this theory is not yet fully developed.

3. The Proof

Recall that, since the cobordism group Ω_3 vanishes, any oriented 3-manifold is cobordant to S^3. Moreover, by surgering out the possible 1- and 3-handles of such cobordisms, for any (closed, oriented) 3-manifold Y we can find a cobordism V from Y to S^3 built on $Y \times [0, 1]$ by adding only 2-handles. A theorem of Eliashberg [E2] (see also [G; GSt]) shows that cobordisms built on contact 3-manifolds by attaching 2-handles along a Legendrian link with appropriate framings eventually support Stein structures. (The framing should be $\mathrm{tb}(K_i) - 1$ for each component K_i of the Legendrian link L.) Such cobordisms are usually called *Stein cobordisms*; for additional discussion, see [EtH].

We begin the proof of Theorem 1.4 with a lemma.

Lemma 3.1. *Let (Y, ξ) be a given contact 3-manifold. Then there exists a Stein cobordism W from (Y, ξ) to some (N, ξ') such that $b_2^+(W) > 1$ and $H_1(N; \mathbb{Z}) = H_1(S^3; \mathbb{Z}) = 0$ (i.e., N is an integral homology sphere).*

Proof. Let us take a cobordism V between Y and S^3, and suppose it is built on $Y \times [0, 1]$ by adding 2-handles only. Put the attaching circles of these two handles into Legendrian position with respect to the given contact structure ξ on Y. If the resulting Thurston–Bennequin invariant $\mathrm{tb}(K)$ of a component K is larger than the framing coefficient $\mathrm{fr}(K)$ of the knot K in V, then by adding zigzags to the Legendrian knot K we obtain K_{new} with $\mathrm{tb}(K_{\mathrm{new}}) - 1 = \mathrm{fr}(K)$, and without changing the diffeomorphism type of V. In case $\mathrm{fr}(K) \geq \mathrm{tb}(K)$, change the knot in a small Darboux neighborhood of a point $p \in K$ (disjoint from all other attaching circles) by connecting it to a Legendrian knot with high enough Thurston–Bennequin invariant. (This can be done because there are Legendrian knots with arbitrarily high Thurston–Bennequin invariants.) Notice that in this way we change the diffeomorphism type of both the cobordism V and its positive end S^3, but we leave their homologies unchanged: the chain complex computing the CW-homology of V reads only homological properties of the attaching circles (like their position in Y, their linkings and framings), but since our change is homologically invisible we never change either the homology of V or the homologies of its boundary components. At the end of this process we are left with a Stein cobordism from (Y, ξ) to a contact 3-manifold (N_1, ξ_1') with the homology of the 3-sphere S^3.

By adding some extra 2-handles attached to knots lying in a Darboux chart (disjoint from the other attaching circles) with Thurston–Bennequin numbers 2 (as described in [LiMa, Thm. 3.2]), we can easily arrange $b_2^+(W) > 1$. □

Next we collect the necessary constructions for the proof of Theorem 1.4. For the given contact 3-manifold (Y, ξ), fix a cobordism W provided by Lemma 3.1 and consider the set of irreducible solutions $\{\gamma_1, \dots, \gamma_k\}$ of the Seiberg–Witten equations on N for a generic metric g and a generic (exact) perturbation μ. (Notice that, since N is an integral homology sphere, it admits a unique spin^c structure.)

Suppose that $W_1, \dots, W_n, \dots$ are the diffeomorphism types of the Stein fillings of (Y, ξ). (This list might be empty, in which case the proof of the theorem is trivial.) Consider the Stein fillings $S_i = W_i \cup W$ of $\partial S_i = (N, \xi')$, and fix minimal surfaces X_i of general type with Kähler embeddings $S_i \to X_i$ such that $b_2^+(X_i - \operatorname{int} S_i) > 1$. (Such complex surfaces exist by Theorem 2.1.)

Our aim is to control the topology of S_i, and the main idea is as follows. Fix a Stein filling S_1 and consider $T_1 = X_1 - \operatorname{int} S_1$. Now for any other Stein filling S of (N, ξ') we can form $Z = T_1 \cup S$ and, according to Theorem 2.2, this is a symplectic 4-manifold with $b_2^+(Z) > 1$. Hence minimality of Z would imply $c_1^2(Z) \geq 0$, giving the desired lower bound for $3\sigma(S) + 2\chi(S)$ in terms of invariants of the fixed 4-manifold T_1. However, minimality of Z is hard to prove—although it seems to be true—so instead we use a larger (though still finite) set of test manifolds $X_i - \operatorname{int} S_i$ with which to compare the Stein filling S. Also, we may relax the minimality requirement by trying to prove that the number of blow-ups contained by the symplectic 4-manifold Z is bounded by some number depending only on (Y, ξ). This is exactly the line of reasoning we will follow in our proof of Theorem 1.4. To set the stage, we need a few definitions.

Let $\mathbb{A}_k$ stand for the set of subsets of $\{1, \dots, k\}$ with multiplicity ≤ 3, that is,

$$\mathbb{A}_k = \{(A_i)^j \mid A_i \subset \{1, \dots, k\} \text{ and } j \leq 3\}.$$

So a specific subset $A \subset \{1, \dots, k\}$ appears in $\mathbb{A}_k$ as A, then with multiplicity 2 as A^2, and finally as A^3. For example, $\mathbb{A}_1 = \{\emptyset, \emptyset^2, \emptyset^3, \{1\}, \{1\}^2, \{1\}^3\}$ and

$$\mathbb{A}_2 = \{\emptyset, \emptyset^2, \emptyset^3, \{1\}, \{1\}^2, \{1\}^3, \{2\}, \{2\}^2, \{2\}^3, \{1,2\}, \{1,2\}^2, \{1,2\}^3\}.$$

Notice that all $\mathbb{A}_k$ are finite sets. A Stein filling S_i determines a subset of $\mathbb{A}_k$ as follows. By extending the metric g and perturbation μ to $T_i = X_i - \operatorname{int} S_i$, for each spin^c structure $\mathfrak{s} \in \operatorname{Spin}^c(T_i)$ we obtain a vector $\mathrm{SW}_{T_i}(\mathfrak{s}) = (a_1, \dots, a_k)$, where $a_i = \#\mathcal{M}_{T_i, \tilde{g}, \tilde{\mu}}(\mathfrak{s})[\gamma_j]$—the number of Seiberg–Witten solutions for the extended (cylindrical) metric $\tilde{g}$, perturbation $\tilde{\mu}$, boundary value γ_j, and spin^c structure $\mathfrak{s}$. Now we assign $A \subset \{1, \dots, k\}$ to $\mathfrak{s} \in \operatorname{Spin}^c(T_i)$ with the property that $m \in A$ iff $a_m \equiv 1 \pmod 2$; that is, we take the mod 2 reduction of $(a_1, \dots, a_k)$ and identify the sequence of zeros and ones with the corresponding subset of $\{1, \dots, k\}$. If $\mathfrak{s}$ runs through $\operatorname{Spin}^c(T_i)$ then this assignment produces subsets of $\{1, \dots, k\}$, and if

$A \subset \{1, \ldots, k\}$ appears more than three times then we keep only three copies of it (and record the multiplicities as exponents). Consequently, each Stein filling S_i determines an element in the power set $\mathbb{P}(\mathbb{A}_k)$ of $\mathbb{A}_k$. (Note that, since $\mathbb{A}_k$ is finite, so is its power set.) Since the element associated in the manner just described to S_i uses Seiberg–Witten moduli spaces on T_i, we denote the resulting element as $\mathcal{SW}_{T_i} \in \mathbb{P}(\mathbb{A}_k)$. These elements are by no means (diffeomorphism) invariants of either S_i or T_i. The Stein filling S_i specifies T_i through the choice of X_i, and then $\mathcal{SW}_{T_i}$ is given by a count of certain Seiberg–Witten solutions over T_i using specific metrics and perturbations. The resulting element might depend on these choices, but we do not intend to deal with such dependencies because we are using $\mathcal{SW}_{T_i}$ only to learn something about the topology of S_i (in a quite roundabout way).

Now partition the Stein fillings S_i by saying that S_i is equivalent to S_j iff $\mathcal{SW}_{T_i} = \mathcal{SW}_{T_j}$ in $\mathbb{P}(\mathbb{A}_k)$, and choose representatives $S_{i_1}, \ldots, S_{i_N}$ from each equivalence class. (For choosing S_{i_j} first we must fix the X into which S embeds and determine $\mathcal{SW}_{X-\operatorname{int} S} = \mathcal{SW}_T$.)

Remark 3.2. We have just showed that there are finitely many Stein fillings that model all possible gauge-theoretic behavior—at least those of interest from our present point of view. That is, we count solutions to the Seiberg–Witten equations only mod 2 and with multiplicities at most 3. The reason for these simplifications is to have finitely many choices for $\mathcal{SW}_{T_i}$. That the information contained by the $\mathcal{SW}_{T_i}$ so defined is enough for our purposes will be clear when discussing minimality of glued-up manifolds.

Now, for a Stein filling S of (N, ξ'), consider S_{i_j} (for some $j \in \{1, \ldots, N\}$) from the chosen representatives equivalent to it.

Proposition 3.3. *The symplectic* 4*-manifold* $Z = T_{i_j} \cup S$ *satisfies* $c_1^2(Z) \geq -1$.

The proof of this proposition rests on the following simple observation.

Lemma 3.4. *If a closed symplectic* 4*-manifold* V *with* $b_2^+(V) > 1$ *has at most two* spin^c *structures with odd Seiberg–Witten invariants, then* $c_1^2(V) \geq -1$.

Proof. Suppose that $c_1^2(V) = -k < -1$. According to celebrated results of Taubes, a minimal symplectic 4-manifold (with $b_2^+ > 1$) has nonnegative c_1^2-invariant. Therefore V is at least a k-fold blow-up of another symplectic 4-manifold; in particular, it can be written as $V_1 \# 2\overline{\mathbb{CP}}^2$. Now the blow-up formula asserts that $\mathrm{SW}_V(\pm c_1(V_1) \pm E_1 \pm E_2) = \pm \mathrm{SW}_{V_1}(\pm c_1(V_1))$, where E_1 and E_2 denote the exceptional divisors of the blow-ups. Since V_1 is symplectic, another result of Taubes implies that $\mathrm{SW}_{V_1}(\pm c_1(V_1)) = \pm 1$. Hence, from the assumption $k < -1$, we found at most four basic classes $\pm c_1(V_1) \pm E_1 \pm E_2$ with odd Seiberg–Witten invariants on V. (Actually this argument produces eight basic classes with odd Seiberg–Witten invariants unless $c_1(V_1) = 0$, in which case we found four: $\pm E_1 \pm E_2$.) This contradiction proves the stated inequality. □

Proof of Proposition 3.3. In view of Lemma 3.4, the proposition follows from an appropriate estimate on the number of basic classes of Z with odd Seiberg–Witten invariant. Suppose there are three such classes corresponding to spin^c structures $\mathfrak{s}_1$, $\mathfrak{s}_2$, and $\mathfrak{s}_3$. Consider the previous spin^c structures restricted to T_{i_j}, and denote $\mathfrak{s}_m|_{T_{i_j}}$ by $\mathfrak{t}_m$ $(m = 1, 2, 3)$. Suppose first that all three $\mathfrak{t}_m$ are different. The corresponding vectors $\mathrm{SW}_{T_{i_j}}(\mathfrak{t}_m)$ then give odd dot products with the vectors $\mathrm{SW}_S(\mathfrak{s}_m|_S)$, since the result of this dot product is the (mod 2) Seiberg–Witten value of the corresponding spin^c structure $\mathfrak{s} \in \mathrm{Spin}^c(X)$. On the other hand, by the definition of our equivalence relation we know that there is a surface X of general type such that $X - \mathrm{int}\, S$ and T_{i_j} have identical $\mathcal{SW}$-invariants in $\mathbb{P}(\mathbb{A}_k)$. This, however, shows that there are three spin^c structures on X with odd Seiberg–Witten values, which contradicts that $\mathrm{SW}_X(K) = 0$ unless $K = \pm c_1(X)$. In the case $\mathfrak{t}_1 = \mathfrak{t}_2$, the same conclusion holds for the spin^c structures $\mathfrak{t}'\#(\mathfrak{s}_m|_S)$ $(m = 1, 2)$ and $\mathfrak{t}'_3\#(\mathfrak{s}_3|_S)$ on X. (Here $\mathfrak{t}'_m$ stands for a spin^c structure on $X - \mathrm{int}\, S$ with $\mathrm{SW}_{X-\mathrm{int}\, S}(\mathfrak{t}'_m) = \mathrm{SW}_{T_{i_j}}(\mathfrak{t}_m)$.) Finally, if all three restricted spin^c structures over T_{i_j} are equal to some $\mathfrak{t}$, then the three structures $\mathfrak{t}'\#(\mathfrak{s}_m|_S)$ $(m = 1, 2, 3)$ will provide the same contradiction. Therefore Z has only two spin^c structures with odd Seiberg–Witten invariants, hence Lemma 3.4 shows that $c_1^2(Z) \geq -1$. (Notice that, since N is an integral homology sphere, spin^c structures on the components of its complement simply add up (denoted by #) to give all the spin^c structures of the closed 4-manifold.) □

Proof of Theorem 1.4. The foregoing arguments show that, for any Stein filling S of (N, ξ'), there is some $j \in \{1, \ldots, N\}$ such that $c_1^2(T_{i_j} \cup S) \geq -1$. Since $\chi(T_{i_j} \cup S) = \chi(T_{i_j}) + \chi(S)$ and (by Novikov additivity) $\sigma(T_{i_j} \cup S) = \sigma(T_{i_j}) + \sigma(S)$, we get that $3\sigma(S) + 2\chi(S) \geq -1 - 3\sigma(T_{i_j}) - 2\chi(T_{i_j})$. Therefore, by setting

$$C_{(N,\xi')} = \max\{3\sigma(T_{i_j}) + 2\chi(T_{i_j}) + 1 \mid j = 1, \ldots, N\},$$

we obtain $3\sigma(S) + 2\chi(S) \geq -C_{(N,\xi')}$. Recall that initially we wanted to find an estimate on Stein fillings of a contact 3-manifold (Y, ξ). Now there is a fixed Stein cobordism W from (Y, ξ) to (N, ξ') and thus, for a Stein filling W_i of (Y, ξ), it follows that $3\sigma(W_i \cup W) + 2\chi(W_i \cup W) \geq -C_{(N,\xi')}$. Since W is fixed, if we set

$$K_{(Y,\xi)} = -C_{(N,\xi')} - (3\sigma(W) + 2\chi(W))$$

then the preceding inequality verifies $c(W_i) = 3\sigma(W_i) + 2\chi(W_i) \geq K_{(Y,\xi)}$, so the proof of Theorem 1.4 is complete. □

Notice that we made heavy use of the actual contact structure on Y in the proof of Lemma 3.1, since the Legendrian position (and hence the Thurston–Bennequin framing) of a knot depends on the chosen contact structure. Therefore, both W and N might be different for some other choice of ξ on Y. It seems to be quite difficult to get a hold on an actual value of $K_{(Y,\xi)}$, because it seems rather complicated to see anything from the topology of the surface of general type in which a Stein filling is embedded through Theorem 2.1.

4. On Fillings of $\Sigma(2, 3, 11)$

We conclude this paper with the proof of Theorem 1.9. It turns out that, by using the methods of [St1], we can determine the intersection form of all Stein fillings of $\Sigma(2, 3, 11)$ with positive b_2^+-invariant.

PROPOSITION 4.1. *If W is a Stein filling of $\Sigma(2, 3, 11)$ and if $b_2^+(W) > 0$, then the intersection form Q_W of W is equal to $2E_8 \oplus 2H$. In particular, $\chi(W) = 19$, $\sigma(W) = -16$, and W is spin.*

Proof. By embedding W into a minimal surface of general type, we conclude that the relative invariant $\mathcal{SW}(W)$ (defined in [StSz] and used in [St1]) is equal to ± 1. (This follows once we assume that the embedding of W into X has the property that $X - \operatorname{int} W$ is nonspin and has $b_2^+(X - \operatorname{int} W) > 0$, which is easy to achieve.) Notice that the product formula of [StSz] (as common in gauge theory) works only if both sides in the decomposition have positive b_2^+-invariants. We can arrange this to hold for $X - \operatorname{int} W$ but can only assume it for W. Now applying the product formula of [StSz] again, we see that $\mathcal{SW}(W \cup N_2) = \mathcal{SW}(W) \cdot \mathcal{SW}(N_2) = \pm 1$, where N_2 stands for the nucleus in the elliptic surface $E(2)$ (which is the K3-surface). This implies that $\mathrm{SW}_{W \cup N_2}(0) = \pm 1$; now a theorem of Morgan and Szabó [MSz] (cf. [St1, Thm. 2.9]) gives that $W \cup N_2$ is homeomorphic to the K3-surface. Since $Q_{\mathrm{K3}} = 2E_8 \oplus 3H$ and $Q_{N_2} = H$, the proposition follows. □

Proof of Theorem 1.9. As the previous proof shows, for Stein fillings of $\Sigma(2, 3, 11)$ with $b_2^+ \neq 0$ we have $b_2^+ = 2$. Hence the theorem and consequently the finiteness of $\mathcal{CF}_{(\Sigma(2,3,11),\xi)}$ (for any contact structure ξ) is proved.

Notice that, by deleting a nucleus N_2 (the neighborhood of a cusp fiber and a section) from an elliptic K3 surface, we are left with a Lefschetz fibration X over the disk D^2 with $T^2 - D^2$ as regular fiber. Such 4-manifolds, however, admit Stein structures [AO1], and it is not hard to see that $\partial X = \Sigma(2, 3, 11)$ (since X is just the corresponding compactified Milnor fiber). This shows that $\mathrm{K3} - \operatorname{int} N_2$ admits a Stein structure, providing a Stein filling of $\Sigma(2, 3, 11)$ (with some contact structure) with $b_2^+ = 2$. □

REMARK 4.2. Our construction of a Stein filling of $\Sigma(2, 3, 11)$ corresponds to the deformation of the singularity $\{x^2 + y^3 + z^{11} = 0\}$ in $\mathbb{C}^3$. By resolving this singularity we get a complex surface with $\Sigma(2, 3, 11)$ as its boundary. A detailed analysis of the resolution (together with Eliashberg's construction [E2]) shows that the resolution carries a Stein structure with $b_2^+ = 0$. (For the Kirby diagram of this resolution see [GhS]. In fact, the 3-manifold $\Sigma(2, 3, 11)$ carries only two (nonisotopic) tight contact structures [GhS], and both can be filled by some Stein structure on the 4-manifold given by Figure 12 of [GhS].) It seems natural to conjecture that any Stein filling of $\Sigma(2, 3, 11)$ is diffeomorphic either to this resolution or to the Milnor fiber described above.

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