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Total Masses of Mixed Monge–Ampère Currents

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1. Introduction

Our starting point is the classical problem on numeric characteristics for zero sets of polynomial mappings $P: \mathbf{C}^n \to \mathbf{C}^m$. If $m \geq n$ and P has discrete zeros then this is about the total number of zeros counted with multiplicities, and for $m < n$ the characteristics are the projective volumes of the corresponding holomorphic chains Z_P. When $m = 1$ the volume equals the degree of the polynomial P, but for $m > 1$ the situation becomes much more difficult. In particular, in the general case no exact formulas can be obtained in terms of the exponents and the problem reduces to finding appropriate upper bounds. An example of such a bound is given by Bezout's theorem: If $m = n$ and P has discrete zeros, then their number does not exceed the product of the degrees of the components of P. An alternative estimate is due to Kouchnirenko [11; 12]: The number of zeros is at most $n!$ times the volume of the *Newton polyhedron of P at infinity* (the convex hull of all exponents of P and the origin). A refined version of the latter result was obtained by Bernstein [3], who showed that the number of (discrete) zeros of a Laurent polynomial mapping P on $(\mathbf{C} \setminus \{0\})^n$ is not greater than $n!$ times the mixed volume of the *Newton polyhedra* (the convex hulls of the exponents) of the components of P.

Here we put this problem into a wider context of pluripotential theory. This can be done by considering plurisubharmonic functions $u = \log|P|$ and studying the Monge–Ampère operators $(dd^c u)^p$; we use the notation $d = \partial + \bar\partial$ and $d^c = (\partial - \bar\partial)/2\pi i$. The key relation is the King–Demailly formula, which implies that if the codimension of the zero set is at least p then $(dd^c u)^p \geq Z_P$ (with an equality if $p = m \leq n$). The problem of estimating total masses of the Monge–Ampère operators of plurisubharmonic functions u of logarithmic growth was studied in [22]. In particular, a relation was obtained in terms of the volume of a certain convex set generated by the function u, which in case $u = \log|P|$ is just the Newton polyhedron of P at infinity.

On the other hand, we know that the holomorphic chain Z_P with $m = p \leq n$ can be represented as the wedge product of the currents (divisors) $dd^c \log|P_k|$, $1 \leq k \leq m$, which leads to consideration of the *mixed* Monge–Ampère operators $dd^c u_1 \wedge \cdots \wedge dd^c u_m$ and estimating their total masses. Another motivation for this problem are generalized degrees $\int_{\mathbf{C}^n} T \wedge (dd^c \varphi)^p$ of positive closed currents T with respect to plurisubharmonic weights φ, due to Demailly [5].

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So, our main subject is mixed Monge–Ampère currents generated by arbitrary plurisubharmonic functions of logarithmic growth. Using the approach developed in [22], we obtain effective bounds for the masses of the currents. As a consequence, this gives us a plurisubharmonic version of Bernstein's theorem adapted, in particular, for polynomial mappings of $\mathbf{C}^n$. In addition, we get a representation for the generalized degrees of $(1,1)$-currents.

2. Preliminaries and Description of Results

We consider plurisubharmonic functions u of logarithmic growth in $\mathbf{C}^n$,

$$u(z) \le C_1 \log^+|z| + C_2$$

with some constants $C_j = C_j(u)$. The collection of all such functions will be denoted by $\mathcal{L}(\mathbf{C}^n)$ or simply by $\mathcal{L}$. Various results on such functions are presented, for example, in [1; 2; 15; 16; 23]. For general properties of plurisubharmonic functions and the complex Monge–Ampère operators, we refer the reader to [9; 10; 14; 17].

The (logarithmic) *type* of a function $u \in \mathcal{L}$ is defined as

$$\sigma(u) = \limsup_{|z|\to\infty} \frac{u(z)}{\log|z|},$$

which can be viewed as the Lelong number of u at infinity. The corresponding counterpart for the directional (refined) Lelong numbers are directional types

$$\sigma(u,a) = \limsup_{z\to\infty} \frac{u(z)}{\varphi_a(z)} \tag{2.1}$$

with

$$\varphi_a(z) = \sup_k a_k^{-1} \log|z_k|, \quad a = (a_1, \dots, a_n) \in \mathbf{R}_+^n; \tag{2.2}$$

see[22]. One can also consider the types $\sigma(u,\varphi)$ with respect to arbitrary plurisubharmonic exhaustive functions $\varphi \in \mathcal{L}$,

$$\sigma(u,\varphi) = \limsup_{z\to\infty} \frac{u(z)}{\varphi_(z)}. \tag{2.3}$$

One more characteristic is the *logarithmic multitype* $(\sigma_1(u), \dots, \sigma_n(u))$,

$$\sigma_1(u) = \sup\{\tilde\sigma_1(u; z') : z' \in \mathbf{C}^{n-1}\} \tag{2.4}$$

(see [16]), where $\tilde\sigma_1(u; z')$ is the logarithmic type of the function $u_{1,z'}(z_1) = u(z_1, z') \in \mathcal{L}(\mathbf{C})$ with $z' \in \mathbf{C}^{n-1}$ fixed, and similarly for $\sigma_2(u), \dots, \sigma_n(u)$. For example, if P is a polynomial of degree d_k in z_k, then $\sigma_k(\log|P|) = d_k$.

Another (and original) definition for the Lelong numbers is in terms of the currents $dd^c u$, which works for arbitrary positive closed currents. This leads to the notion of degree of a current. Let $\mathcal{D}_p^+(\Omega)$ be the collection of all closed positive currents of bidimension (p,p) on a domain $\Omega \subset \mathbf{C}^n$. We will consider currents $T \in \mathcal{D}_p^+(\mathbf{C}^n)$ with finite projective mass, or *degree*

$$\delta(T) = \int_{\mathbf{C}^n} T \wedge \left(\tfrac{1}{2} dd^c \log(1 + |z|^2)\right)^p;$$

the set of all such currents is denoted by $\mathcal{LD}_p^+$. The degree of $T \in \mathcal{LD}_p^+$ can also be represented as

$$\delta(T) = \int_{\mathbf{C}^n} T \wedge (dd^c \log|z|)^p$$

and as the density of the trace measure $\sigma_T = T \wedge \frac{1}{p!}\left(\frac{i}{2} \sum dz_j \wedge d\bar{z}_j\right)^p$ of the current T:

$$\delta(T) = \lim_{r\to\infty} \frac{\sigma_T(|z| < r)}{\mathrm{mes}_{2p}(|z| < r)}.$$

When $T = [A]$ is the current of integration over an algebraic set A of pure dimension p, the degree $\delta([A])$ coincides with the degree of the set A defined as the number of sheets in the ramified covering map $A \to L$ to a generic p-codimensional plane L. Note also that any current $T \in \mathcal{LD}_{n-1}^+$ has the form $T = dd^c u$ with $u \in \mathcal{L}$, and $\delta(dd^c u) = \sigma(u)$ (see [17]).

The *generalized degrees*

$$\delta(T, \varphi) := \int_{\mathbf{C}^n} T \wedge (dd^c \varphi)^p \tag{2.5}$$

with respect to plurisubharmonic weights φ were introduced in [5] as a powerful tool for studying polynomial mappings and algebraic sets.

We are concerned with the problem of evaluating

$$\mu(T, u_1, \ldots, u_p) := \int_{\mathbf{C}^n} T \wedge dd^c u_1 \wedge \cdots \wedge dd^c u_p$$

for currents $T \in \mathcal{LD}_p^+$ and functions $u_j \in \mathcal{L}$ in terms of the distribution of T and growth characteristics of u_k. The idea is to replace the functions u_k by certain plurisubharmonic functions v_k with simpler asymptotic properties. A relation between the corresponding total masses is provided by Theorem 3.1, which shows that the value of $\mu(T, u_1, \ldots, u_p)$ is a function of the asymptotic behavior of u_k at infinity. This comparison theorem is an extension of Taylor's theorem [24] on the total mass of $(dd^c u)^n$ of $u \in \mathcal{L} \cap L^\infty_{\mathrm{loc}}$. At the same time, it is an analogue for Demailly's second comparison theorem [9, Thm. 5.9] on generalized Lelong numbers.

Taking $v_k = \log|z|$ yields a bound in terms of the types of u_k and the degree of T (Corollary 3.1),

$$\mu(T, u_1, \ldots, u_p) \le \delta(T)\sigma(u_1) \ldots \sigma(u_p),$$

and the choice $v_k = \varphi_a$ leads to that in terms of the corresponding directional characteristics (Corollary 3.3).

Sharper bounds are obtained with $v_k = \Psi_{u_k,x}$, the *indicators* of u_k, introduced in [22] (see the definition and basic properties in Section 4). We have

$$\mu(T, u_1, \ldots, u_p) \le \mu(T, \Psi^+_{u_1,x}, \ldots, \Psi^+_{u_p,x})$$

(Proposition 4.1), and the problem reduces to evaluating the right-hand side. This can be done effectively in the case $p = n-1$, $T = dd^c u_n$. Namely, for $u \in \mathcal{L}$, the convex function $\psi^+_{u,x}(t) := \Psi^+_{u,x}(\exp t_1, \dots, \exp t_n)$, $t \in \mathbf{R}^n$, is the support function to the convex set

$$\Theta^u = \{a \in \mathbf{R}^n : \langle a, t\rangle \le \psi^+_{u,x}(t)\ \forall t \in \mathbf{R}^n\},$$

and

$$\mu(\Psi^+_{u_1,x}, \dots, \Psi^+_{u_n,x}) = n!\,\mathrm{Vol}(\Theta^{u_1}, \dots, \Theta^{u_n}), \tag{2.6}$$

Minkowski's mixed volume of the sets Θ^{u_k} (Theorem 4.1).

The foregoing considerations are applied in Section 5 to investigation of the generalized degrees $\delta(T, \varphi)$ defined in (2.5). By Proposition 4.1, we are reduced to the values $\delta(T, \Psi_{\varphi,y})$. When $T = dd^c u$, we prove the relation $\delta(dd^c u, \Psi_{\varphi,y}) = \delta(dd^c \Psi_{u,x}, \Psi_{\varphi,y})$ for all $x, y \in \mathbf{C}^n$. We study the "swept out" Monge–Ampère measures of indicator weights in Theorem 5.3. As a consequence, we derive a representation for $\delta(dd^c u, \Psi_{\varphi,y})$ in terms of the sets Θ^u and Θ^φ and a relation between $\sigma(u, \varphi)$ and $\delta(dd^c u, \varphi)$ in Corollary 5.2.

Finally, in Section 6 we specify our results for currents generated by polynomial mappings. In particular, we observe that (2.6) implies the following analogue for Bernstein's inequality (Corollary 6.1): the projective volume $\delta(Z_P)$ of the holomorphic chain Z_P generated by a polynomial mapping $P = (P_1, \dots, P_p)$ in general position, $1 \le p \le n$, has the bound

$$\delta(Z_P) \le n!\,\mathrm{Vol}(G_1^+, \dots, G_p^+, \Delta, \dots, \Delta),$$

where G_j^+ is the Newton polyhedron of the polynomial P_j at infinity and $\Delta = \{t \in \mathbf{R}^n_+ : \sum t_j \le 1\}$ is the standard simplex in $\mathbf{R}^n$. We also derive a number of other bounds (like Bezout's and Tsikh's theorems) as direct consequences of our general results on mixed Monge–Ampère operators.

3. Comparison Theorem for Mixed Operators

A q-tuple of plurisubharmonic functions $u_1, \dots, u_q$ will be said to be *properly intersected*, or *in general position*, with respect to a current $T \in \mathcal{D}_p^+$ ($p \ge q$) if their unboundedness loci $A_1, \dots, A_p$ satisfy the following condition: For all choices of indices $j_1 < \dots < j_k$ ($k \le q$), the $(2q-2k+1)$-dimensional Hausdorff measure of the set $A_{j_1} \cap \dots \cap A_{j_k} \cap \operatorname{supp} T$ equals zero. If this is the case, then the current $T \wedge dd^c u_1 \wedge \dots \wedge dd^c u_q$ is well-defined and has locally finite mass [9, Thm. 2.5].

We recall that a function u in $\mathbf{C}^n$ is called *semi-exhaustive* on a set A if $\{u < R\} \cap A \subset\subset \mathbf{C}^n$ for some real R, and *exhaustive* if this is valid for all R.

THEOREM 3.1 (Comparison Theorem). *Let $T \in \mathcal{L}\mathcal{D}_p^+$ and $u_1, \dots, u_p \in \mathcal{L}$ be properly intersected with respect to T, and let $v_1, \dots, v_p \in \mathcal{L}$ be semi-exhaustive on* $\operatorname{supp} T$. *If, for any $\eta > 0$,*

$$\limsup_{|z|\to\infty,\ z \in \operatorname{supp} T} \frac{u_j(z)}{v_j(z) + \eta \log|z|} \le l_j, \quad 1 \le j \le p,$$

then $\mu(T, u_1, \dots, u_p) \le l_1 \dots l_p \mu(T, v_1, \dots, v_p)$.

Proof. It suffices to show that the conditions

$$\limsup_{|z|\to\infty,\, z\in\operatorname{supp} T} \frac{u_j(z)}{v_j(z)+\eta\log|z|} < 1 \quad \forall \eta > 0, \quad 1 \le j \le p, \tag{3.1}$$

imply

$$\mu(T, u_1, \dots, u_p) \le \mu(T, v_1, \dots, v_p). \tag{3.2}$$

Without loss of generality, we can also take $\varphi > 0$ on $\mathbf{C}^n$.

For any $N > 0$, the functions $u_{j,N} := \max\{u_j, -N\}$ still satisfy (3.1). Then, for any $\eta > 0$ and $C > 0$, the set

$$E_j(C) = \{z \in \operatorname{supp} T : v_j(z) + \eta \log|z| - C < u_{j,N}(z)\}$$

is compactly supported in the ball B_{α_j} for some $\alpha_j = \alpha_j(C, \eta, u_{j,N}, v_j)$. Put $\alpha = \max_j \alpha_j$, $E(C) = \bigcup_j E_j(C)$, $F(C) = \bigcap_j E_j(C)$, and

$$w_{j,C} = \max\{v_j(z) + \eta \log|z| - C, u_{j,N}\}.$$

Since $w_{j,C} = v_j(z) + \eta\log|z| - C$ near $\partial B_\alpha \cap \operatorname{supp} T$, we have

$$\begin{aligned}\int_{B_\alpha} T \bigwedge_{1\le j\le p} dd^c w_{j,C} &= \int_{B_\alpha} T \bigwedge_{1\le j\le p} dd^c(v_j + \eta\log|z|)\\ &\le \int_{\mathbf{C}^n} T \bigwedge_{1\le j\le p} dd^c(v_j + \eta\log|z|).\end{aligned}$$

Note that, for any compact subset K of $\operatorname{supp} T$, one can find C_K such that $K \subset F(C)$ for all $C > C_K$; hence

$$\int_{B_R} T \bigwedge_{1\le j\le p} dd^c w_{j,C} \le \int_{\mathbf{C}^n} T \bigwedge_{1\le j\le p} dd^c(v_j + \eta\log|z|)$$

for any $R > 0$ and all $C > C_R$. In addition,

$$T \bigwedge_{1\le j\le p} dd^c w_{j,C} \to T \bigwedge_{1\le j\le p} dd^c u_{j,N}$$

as $C \to +\infty$ (the functions $w_{j,C}$ decrease to $u_{j,N}$) and therefore

$$\begin{aligned}\int_{B_R} T \bigwedge_{1\le j\le p} dd^c u_{j,N} &\le \limsup_{C\to\infty} \int_{B_R} T \bigwedge_{1\le j\le p} dd^c w_{j,C}\\ &\le \int_{\mathbf{C}^n} T \bigwedge_{1\le j\le p} dd^c(v_j + \eta\log|z|).\end{aligned}$$

Since η is arbitrary, we derive the inequality

$$\int_{B_R} T \bigwedge_{1\le j\le p} dd^c u_{j,N} \le \int_{\mathbf{C}^n} T \bigwedge_{1\le j\le p} dd^c v_j;$$

finally, letting $N \to \infty$, we have

$$\int_{B_R} T \bigwedge_{1\le j\le p} dd^c u_j \le \int_{\mathbf{C}^n} T \bigwedge_{1\le j\le p} dd^c v_j$$

for any $R > 0$, which gives us (3.2) and thus completes the proof. □

REMARK. As follows from the theorem,

$$\mu(T, u_1, \dots, u_p) \le \mu(T, \max\{u_1, \alpha_1\}, \dots, \max\{u_p, \alpha_p\})$$

for any $\alpha \in \mathbf{R}^p$, and the right-hand side is independent of α. The inequality here can be strict, which follows from the consideration of the function $u(z_1, z_2) = \log(|z_1|^2 + |z_1 z_2 + 1|^2)$. We have $\mu(u, u) = 0$ while $\mu(u^+, u^+) = 4$, the latter relation verified by comparing u^+ with the function $\max\{\log|z_1|, \log|z_1 z_2|, 0\}$, whose total Monge–Ampère mass can be calculated by Proposition 4.2. This shows that the condition on the functions v_j in Theorem 3.1 to be semi-exhaustive is essential.

An immediate application of Theorem 3.1 is the following bound for the total mass of the current $T \wedge dd^c u_1 \wedge \cdots \wedge dd^c u_p$ in terms of the degree $\delta(T)$ of T and logarithmic types $\sigma(u_j)$ of u_j.

COROLLARY 3.1. *If $T \in \mathcal{LD}_p^+$ and $u_1, \dots, u_p \in \mathcal{L}$ are properly intersected with respect to T, then*

$$\mu(T, u_1, \dots, u_p) \le \delta(T)\sigma(u_1) \dots \sigma(u_p).$$

In particular, $T \wedge dd^c u_1 \wedge \cdots \wedge dd^c u_{p-k} \in \mathcal{LD}_k^+$ for $0 \le k \le p$.

Moreover, we have the following refined bound via the generalized characteristics $\sigma(u, \varphi)$ and $\delta(T, \varphi)$ (see (2.3) and (2.5), respectively) with regard to plurisubharmonic weights φ.

COROLLARY 3.2. *Let $T, u_1, \dots, u_p$ satisfy the conditions of Corollary 3.1 and let $\varphi \in \mathcal{L}$ be an exhaustive weight. Then*

$$\mu(T, u_1, \dots, u_p) \le \delta(T, \varphi)\sigma(u_1, \varphi) \dots \sigma(u_p, \varphi).$$

For the specified case of $T = 1$, $p = n$, and $\varphi = \varphi_a$, this gives us the following bound in terms of the directional types $\sigma(u_j, a)$ of u_j (cf. (2.1)).

COROLLARY 3.3. *If the functions $u_1, \dots, u_n \in \mathcal{L}$ are properly intersected, then*

$$\mu(u_1, \dots, u_n) \le \inf_{a \in \mathbf{R}_+^n} \frac{\sigma(u_1, a) \dots \sigma(u_n, a)}{a_1 \dots a_n}.$$

4. Bounds in Terms of Indicators

More precise bounds can be obtained by means of indicators of functions from the class $\mathcal{L}$.

Developing the notion of local indicator introduced in [18], the (global) *indicator* of a function $u \in \mathcal{L}$ at $x \in \mathbf{C}^n$ was defined in [22] as

$$\Psi_{u,x}(y) = \lim_{R\to+\infty} R^{-1} \sup\{u(z) : |z_k - x_k| \le |y_k|^R,\ 1 \le k \le n\} \quad \text{for } y_1 \dots y_n \ne 0,$$

and it extends to a plurisubharmonic function of the class $\mathcal{L}$ depending only on $|z_1|, \dots, |z_n|$ and satisfying $\Psi_{u,x}(|z_1|^c, \dots, |z_n|^c) = c\Psi_{u,x}(|z_1|, \dots, |z_n|)$ for all $c > 0$. The indicator controls the behavior of u in the whole $\mathbf{C}^n$,

$$u(z) \le \Psi_{u,x}(z - x) + C_x \quad \forall z \in \mathbf{C}^n \tag{4.1}$$

[22, Thm. 1], with C_x equal the supremum of u on the unit polydisk centered at x. Besides, the indicator is a (unique) logarithmic tangent to u at x, that is, the weak limit in $L^1_{\text{loc}}(\mathbf{C}^n)$ of the functions

$$u_m(y) = m^{-1}u(x_1 + y_1^m, \dots, x_n + y_n^m) \tag{4.2}$$

as $m \to \infty$ [22, Thm. 2].

Note that the indicator of $\log|z|$ at x equals $\max_k \log|y_k|$ if $x = 0$ and equals $\max_k \log^+|y_k|$ for any other point x.

The asymptotic characteristics (types) of u can be easily expressed in terms either of its indicator or (more conveniently) of the convex *image*

$$\psi_{u,x}(t) = \Psi_{u,x}(e^{t_1}, \dots, e^{t_n}), \quad t \in \mathbf{R}^n,$$

of the indicator [22, Prop. 3]: For each $x \in \mathbf{C}^n$,

$$\sigma(u) = \sigma(u, (1, \dots, 1)) = \psi_{u,x}(1, \dots, 1),$$

$$\sigma(u, a) = \psi_{u,x}(a) \quad \forall a \in \mathbf{R}^n_+, \tag{4.3}$$

$$\sigma_k(u) = \psi_{u,x}(e_k), \quad 1 \le k \le n, \tag{4.4}$$

with $e_1, \dots, e_n$ the standard basis of $\mathbf{R}^n$. Note that the restriction of $\psi_{u,x}$ to $\mathbf{R}^n_+$ is independent of $x \in \mathbf{C}^n$ [22, Prop. 7].

By (4.1), Theorem 3.1 implies the following.

Proposition 4.1. *Let $T \in \mathcal{LD}^+_p$ and $u_1, \dots, u_p \in \mathcal{L}$ be properly intersected with respect to T, and let $x \in \mathbf{C}^n$. Then*

$$\mu(T, u_1, \dots, u_p) \le \mu(T, \Psi^+_{u_1,x}, \dots, \Psi^+_{u_p,x}).$$

Corollary 4.1. *Let $u_1, \dots, u_n \in \mathcal{L}$ be properly intersected and let $x^k \in \mathbf{C}^n$ $(1 \le k \le n)$. Then*

$$\mu(u_1, \dots, u_n) \le \mu(u_1, \dots, u_{n-1}, \Psi^+_{u_n,x^n}) \le \mu(u_1, \dots, \Psi^+_{u_{n-1},x^{n-1}}, \Psi^+_{u_n,x^n}) \dots$$
$$\le \mu(u_1, \Psi^+_{u_1,x^1}, \dots, \Psi^+_{u_n,x^n}) \le \mu(\Psi^+_{u_1,x^1}, \dots, \Psi^+_{u_n,x^n}).$$

Remark. The choice of $x \in \mathbf{C}^n$ can affect the value of the total Monge–Ampère mass of the indicators. For example, let $u(z_1, z_2) = \frac{1}{2}\log(1 + |z_1 z_2|^2)$; then $\Psi_{u,0}(y) = \log^+|y_1 y_2|$ has zero mass, while the mass of $\Psi_{u,(1,1)}(y) = \max\{\log^+|y_1|, \log^+|y_2|, \log^+|y_1 y_2|\}$ equals 2 (see Proposition 4.2).

To get an interpretation for the masses of indicators, we proceed as in [22]. Let Φ be an abstract indicator in $\mathbf{C}^n$, that is, a plurisubharmonic function depending only on $|z_1|, \dots, |z_n|$ and satisfying the homogeneity condition

$$\Phi(|z_1|^c, \dots, |z_n|^c) = c\Phi(|z_1|, \dots, |z_n|) \quad \forall c > 0. \tag{4.5}$$

The functions φ_a (see (2.2)) are particular examples of the indicators. It is clear that $\Phi \le 0$ in the unit polydisk $\mathbf{D}$ and is strictly positive on

$$\mathbf{D}^{-1} = \{z \in \mathbf{C}^n : |z_k| > 1,\ 1 \le k \le n\}$$

unless $\Phi \equiv 0$.

Let us assume $\Phi \ge 0$ on $\mathbf{C}^n$. In this case, $(dd^c\Phi)^n$ is supported by the distinguished boundary $\mathbf{T}$ of $\mathbf{D}$ [22, Thm. 6]. Denote

$$\varphi(t) = \Phi(e^{t_1}, \dots, e^{t_n}), \quad t \in \mathbf{R}^n, \tag{4.6}$$

the convex image of Φ in $\mathbf{R}^n$, and

$$\Theta^\Phi = \{a \in \mathbf{R}^n : \langle a, t\rangle \le \varphi(t)\ \forall t \in \mathbf{R}^n\}. \tag{4.7}$$

It is easy to see that Θ^Φ is a convex compact subset of $\overline{\mathbf{R}^n_+}$. By the construction, φ is the support function of Θ^Φ. The *real* Monge–Ampère operator applied to φ gives us the δ-function δ_0 with mass $\mathrm{Vol}(\Theta^\Phi)$. By comparing the real and complex Monge–Ampère operators we obtain

$$\mu(\Phi, \dots, \Phi) = n!\,\mathrm{Vol}(\Theta^\Phi) \tag{4.8}$$

(see the details in [22, Thm. 6]).

This can be extended to the mixed Monge–Ampère operators of indicators as follows.

PROPOSITION 4.2. *Let $\Phi_1, \dots, \Phi_n$ be nonnegative indicators. Then*

$$dd^c\Phi_1 \wedge \dots \wedge dd^c\Phi_n = \mu\, dm,$$

where dm is the normalized Lebesgue measure on $\mathbf{T}$, and

$$\mu = \mu(\Phi_1, \dots, \Phi_n) = n!\,\mathrm{Vol}(\Theta^{\Phi_1}, \dots, \Theta^{\Phi_n}).$$

Proof. By the polarization formula for the complex Monge–Ampère operator,

$$\bigwedge_k dd^c\Phi_k = \frac{(-1)^n}{n!}\sum_{j=1}^n (-1)^j \sum_{1\le i_1<\dots<i_j\le n}\Big(dd^c\sum_{k=1}^j \Phi_{j_k}\Big)^n. \tag{4.9}$$

Because the sum of indicators is itself an indicator, the support of $\bigwedge_k dd^c\Phi_k$ is a subset of $\mathbf{T}$. In view of the translation invariance of this measure, it has the form $\mu(2\pi)^{-n}\, d\theta_1 \dots d\theta_n$ with a nonnegative constant μ.

By (4.8), the right-hand side of (4.9) is the alternating sum of the corresponding volumes. Hence, the definition of the mixed volume gives the desired expression for μ and thus completes the proof. □

Now Corollary 4.1 and Proposition 4.2 easily give us the main result of the section.

THEOREM 4.1. *Let functions $u_1, \dots, u_n \in \mathcal{L}$ be properly intersected, and let $x^k \in \mathbf{C}^n$ for $1 \le k \le n$. Then*

$$\mu(u_1, \dots, u_n) \le n!\,\mathrm{Vol}(\Theta^{\Phi_1}, \dots, \Theta^{\Phi_n}),$$

where the sets Θ^{Φ_k} are defined by (4.6)–(4.7) *for $\Phi = \Phi_k = \Psi^+_{u_k, x^k}$.*

As a consequence, we can derive a bound for $\mu(u_1, \dots, u_n)$ in terms of the types $\sigma_k(u_j)$ of u_j with respect to z_k (cf. (2.4)).

COROLLARY 4.2. *If $u_1, \dots, u_n \in \mathcal{L}$ are properly intersected, then*

$$\mu(u_1, \dots, u_n) \le n!\,\mathrm{per}(\sigma_k(u_j))^n_{j,k=1},$$

where per A *denotes the permanent of the matrix A.*

Proof. As follows from (4.4), the set Θ^{Φ_j} is a subset of the rectangle $[0, \sigma_1(u_j)] \times \dots \times [0, \sigma_n(u_j)]$, and the mixed volume of the rectangles $[0, a_{1j}] \times \dots \times [0, a_{nj}]$ $(1 \le j \le n)$ equals $\mathrm{per}(a_{jk})^n_{j,k=1}$. □

5. Degrees with Respect to Plurisubharmonic Weights

Given a subset A of $\mathbf{C}^n$, we denote by $W(A)$ the collection of all functions (*weights*) $\varphi \in \mathcal{L}$ that are continuous as mappings to $[-\infty, +\infty)$ and are exhaustive on A.

Let $T \in \mathcal{LD}^+_p$; then the current (measure) $T \wedge (dd^c\varphi)^p \in \mathcal{LD}^+_0$ is well-defined for any $\varphi \in W(\mathrm{supp}\, T)$. Let

$$\delta(T, \varphi) = \int_{\mathbf{C}^n} T \wedge (dd^c\varphi)^p$$

be the (generalized) degree of T with respect to the weight φ (see [5]).

Observe that $\delta(T, \varphi) = \delta(T, \varphi^+)$ since φ is assumed to be exhaustive on supp T. Note also that $\delta(T, \varphi) = \delta(T)$ if $\varphi(z) = \log|z|$.

Generalized degrees of currents can be viewed as generalized Lelong numbers at infinity, and we start here with two semicontinuity properties parallel to those for the Lelong numbers (cf. [9]).

PROPOSITION 5.1. *Let $T_m, T \in \mathcal{LD}^+_p$ and $T_m \to T$. Then, for any weight φ from $W\big(\overline{\bigcup_m \mathrm{supp}\, T_m}\big)$,*

$$\delta(T, \varphi) \le \liminf_{m\to\infty} \delta(T_m, \varphi).$$

Proof. This follows immediately from [9, Prop. 3.12]. □

PROPOSITION 5.2. *Let weights $\varphi_k, \varphi \in W(\mathrm{supp}\, T)$ be such that, for some $t \in \mathbf{R}$, the functions $\max\{\varphi_k, t\}$ converge to $\max\{\varphi, t\}$ uniformly on compact subsets of $\mathbf{C}^n$. Then*

$$\delta(T, \varphi) \le \liminf_{m\to\infty} \delta(T, \varphi_k).$$

Proof. Since $\delta(T, \max\{\varphi, t\}) = \delta(T, \varphi)$ for any weight $\varphi \in W(\mathrm{supp}\, T)$, we can take $\varphi_k \to \varphi$ uniformly on compact subsets of $\mathbf{C}^n$.

For any $R > 0$, consider $\eta \in C^\infty(\mathbf{C}^n)$, $0 \le \eta \le 1$, such that $\operatorname{supp}\eta \subset B_R$ and $\eta \equiv 1$ on $B_{R/2}$. The relation

$$\lim_{k\to\infty} \int \eta T \wedge (dd^c\varphi_k)^p = \int \eta T \wedge (dd^c\varphi)^p$$

implies that

$$\liminf_{k\to\infty} \delta(T,\varphi_k) \ge \int \eta T \wedge (dd^c\varphi)^p,$$

and the assertion follows. □

Comparison Theorem 3.1 for the degrees reads as follows.

PROPOSITION 5.3. *If two weights $\varphi, \psi \in W(\operatorname{supp} T)$ for a current $T \in \mathcal{LD}_p^+$ and if*

$$\limsup_{|z|\to\infty,\, z\in\operatorname{supp} T} \frac{\varphi(z)}{\psi(z)} \le l,$$

then $\delta(T,\varphi) \le l^p \delta(T,\psi)$.

When applied to indicators, this gives us the following corollary.

COROLLARY 5.1. *For any current $T \in \mathcal{LD}_p^+$, any weight $\varphi \in W(\operatorname{supp} T)$, and $y \in \mathbf{C}^n$, we have $\delta(T,\varphi) \le \delta(T,\Psi_{\varphi,y}^+) = \delta(T,\Psi_{\varphi,y})$.*

More can be said if $T = dd^c u$ $(u \in \mathcal{L})$. In this case, the generalized degrees can be represented by means of the swept-out Monge–Ampère measures introduced by Demailly [7]. For $\varphi \in W(\mathbf{C}^n)$, let $B_r(\varphi) = \{z : \varphi(z) < r\}$, $S_r(\varphi) = \{z : \varphi(z) = r\}$, and $\varphi_r = \max\{\varphi, r\}$. The *swept-out Monge–Ampère measure* μ_r^φ is defined as

$$\mu_r^\varphi = (dd^c\varphi_r)^n - \chi_r (dd^c\varphi)^n,$$

where χ_r is the characteristic function of $\mathbf{C}^n \setminus B_r(\varphi)$. It is a positive measure on $S_r(\varphi)$ with the total mass $\mu_r^\varphi(S_r(\varphi)) = (dd^c\varphi)^n(B_r(\varphi))$. If $\operatorname{supp}(dd^c\varphi)^n \subset B_R(\varphi)$, then $\mu_r^\varphi = (dd^c\varphi_r)^n$ for all $r > R$.

For $\varphi(z) = \log|z-x|$, μ_r^φ is the normalized Lebesgue measure on the sphere $\{z : |z-x| = e^r\}$, and for $\varphi = \varphi_a$ it is supported by the set

$$\mathbf{T}_{ra} = \{z : z_k = \exp(ra_k + i\theta_k),\ 0 \le \theta_k \le 2\pi,\ 1 \le k \le n\} \tag{5.1}$$

and has the form $\mu_r^{\varphi_a} = (a_1 \dots a_n)^{-1}(2\pi)^{-n}\, d\theta_1 \dots d\theta_n$ (see [9]).

The role of the measures μ_r^φ is clarified by the Lelong–Jensen–Demailly formula [7; 9]: For any function u that is plurisubharmonic in $B_R(\varphi)$,

$$\mu_r^\varphi(u) - \int_{B_r(\varphi)} u (dd^c\varphi)^n = \int_{-\infty}^r \int_{B_t(\varphi)} dd^c u \wedge (dd^c\varphi)^{n-1}\, dt \quad \forall r < R.$$

THEOREM 5.1 (cf. [7; 9]). *Let $u \in \mathcal{L}$ and $\varphi \in W(\mathbf{C}^n)$. Then*

$$\limsup_{r\to\infty} \frac{\mu_r^\varphi(u)}{r} \le \delta(dd^c u, \varphi) \le \liminf_{r\to\infty} \frac{\mu_r^\varphi(u^+)}{r}. \tag{5.2}$$

If, in addition,

$$\operatorname{supp}(dd^c\varphi)^n \subset B_{r_0}(\varphi) \tag{5.3}$$

for some r_0, *then* $r \mapsto \mu_r^\varphi(u)$ *is a convex function of* $r \in (r_0, \infty)$ *and*

$$\delta(dd^c u, \varphi) = \lim_{r\to+\infty} \frac{\mu_r^\varphi(u)}{r}. \tag{5.4}$$

Proof. From the Lelong–Jensen–Demailly formula, for any $r > r_0$ we have

$$\mu_r^\varphi(u) = \int_{B_{r_0}(\varphi)} u(dd^c\varphi)^n + \int_{B_r(\varphi)\setminus B_{r_0}(\varphi)} u(dd^c\varphi)^n + \int_{-\infty}^{r} \delta(dd^c u, \varphi, t)\, dt \tag{5.5}$$

with

$$\delta(dd^c u, \varphi, t) = \int_{B_t(\varphi)} dd^c u \wedge (dd^c\varphi)^{n-1}.$$

If φ satisfies (5.3) then the right-hand side is a convex function of r, and (5.4) follows. When (5.3) is not assumed, take any $\varepsilon > 0$ and choose r_0 such that $(dd^c\varphi)^n(\mathbf{C}^n \setminus B_{r_0}(\varphi)) < \varepsilon$ and $u(z) \le (\sigma(u,\varphi) + \varepsilon)\varphi(z)$ for all $z \in \mathbf{C}^n \setminus B_{r_0}(\varphi)$. Then we have

$$\mu_r^\varphi(u) \le \text{Const} + (\sigma(u,\varphi) + \varepsilon) r\varepsilon + \int_{-\infty}^{r} \delta(dd^c u, \varphi, t)\, dt,$$

which gives us the first inequality in (5.2). To get the second inequality, consider the functions $u_N(z) = \max\{u(z), -N\}$ for $N > 0$. By Proposition 5.1, the number N can be chosen such that $\delta(dd^c u_N, \varphi) \ge \delta(u,\varphi) + \varepsilon$. Application of (5.5) to the function u_N gives us

$$\mu_r^\varphi(u^+) \ge \mu_r^\varphi(u_N) \ge \text{Const} - N\varepsilon + \int_{-\infty}^{r} \delta(dd^c u, \varphi, t)\, dt$$

and thus finishes the proof. □

As follows from definition of the generalized type (2.3) and inequality (4.1), $\sigma(u,\varphi) \ge \sigma(u, \Psi_{\varphi,0})$ for every $\varphi \in W(\mathbf{C}^n)$, and Corollary 5.1 shows that $\delta(dd^c u, \varphi) \le \delta(dd^c u, \Psi_{\varphi,y})$ with any $y \in \mathbf{C}^n$. This motivates consideration of *homogeneous* weights, or (abstract) indicators, Φ that depend only on $|z_1|, \ldots, |z_n|$ and satisfy the homogeneity condition (4.5). Note that a homogeneous weight Φ is exhaustive on $\mathbf{C}^n$ if and only if $\Phi > 0$ on $\mathbf{C}^n \setminus \mathbf{D}$.

It is easy to see that the type $\sigma(u,\Phi)$ with respect to a homogeneous exhaustive weight Φ can be computed as

$$\sigma(u,\Phi) = \max\{\Psi_{u,0}(z) : \Phi(z) = 1\}.$$

It is interesting that the degrees $\delta(dd^c u, \Phi)$ can also be represented in terms of the indicators.

THEOREM 5.2. *For any function* $u \in \mathcal{L}$, *any* $x \in \mathbf{C}^n$, *and any homogeneous weight* $\Phi \in W(\mathbf{C}^n)$, *the equality* $\delta(dd^c u, \Phi) = \delta(dd^c \Psi_{u,x}, \Phi)$ *holds. In particular,* $\delta(dd^c u, \varphi) \le \delta(dd^c u, \Psi_{\varphi,y}) = \delta(dd^c \Psi_{u,x}, \Psi_{\varphi,y})$ *for any weight* $\varphi \in W(\mathbf{C}^n)$ *and* $y \in \mathbf{C}^n$.

Proof. Consider the family of functions u_m defined by (4.2), $x \in \mathbf{C}^n$. As mentioned in Section 4, the u_m converge (in L^1_{loc}) to $\Psi_{u,x}$ as $m \to \infty$, so $dd^c u_m \to dd^c \Psi_{u,x}$. By Proposition 5.1,

$$\delta(dd^c \Psi_{u,x}, \Phi) \le \liminf_{m\to\infty} \delta(dd^c u_m, \Phi).$$

However, the homogeneity of Φ gives us $\delta(dd^c u_m, \Phi) = \delta(dd^c u, \Phi)$ for each m, so $\delta(dd^c u, \Phi) \ge \delta(dd^c \Psi_{u,x}, \Phi)$; the desired equality then follows from Corollary 5.1. The theorem is proved. □

REMARK. As mentioned in Section 2, $\sigma(u) = \delta(dd^c u)$. It is not hard to see that, more generally, the directional type $\sigma(u, a)$ as described in equation (2.1) is equal to $a_1 \dots a_n \delta(dd^c u, \varphi_a)$, where the weights φ_a are defined by (2.2); in other words, $\delta(dd^c u, \varphi_a) = \sigma(u, \varphi_a)\mu(\varphi_a, \dots, \varphi_a)$. As can be seen from Corollary 5.2, a relation between the type and the degree with respect to an arbitrary homogeneous exhaustive weight Φ is not so perfect: $\delta(dd^c u, \Phi) \le \sigma(u, \Phi)\mu(\Phi, \dots, \Phi)$, and an equality for all u implies that $\Phi^+ = c\varphi_a^+$ with some $c > 0$ and $a \in \mathbf{R}^n_+$.

The structure of the swept-out Monge–Ampère measures for homogeneous weights is given by our next theorem.

THEOREM 5.3. *Let $\Phi \in W(\mathbf{C}^n)$ be a homogeneous weight. For any function u that is plurisubharmonic in $B_R(\Phi)$ for $R > 0$, the swept-out Monge–Ampère measure μ_r^Φ on the set $S_r(\Phi)$, $0 < r < R$, is determined by the formula*

$$\mu_r^\Phi(u) = n! \int_{E^\Phi} \lambda(u, rt)\, d\gamma_1^\Phi(t),$$

where $\lambda(u, rt)$ is the mean value of u over the distinguished boundary of the polydisk $\{|z_k| < \exp\{rt_k\},\ 1 \le k \le n\}$ and where the measure γ_1^Φ on the set E^Φ of extreme points of the convex set $\{t \in \mathbf{R}^n : \Phi(e^{t_1}, \dots, e^{t_n}) \le 1\}$ is given by the relation $\gamma_1^\Phi(F) = \mathrm{Vol}\,\Theta_F^\Phi$ for compact subsets F of E^Φ, with the set Θ_F^Φ defined by relations (5.8), (5.9), *and* (5.6).

COROLLARY 5.2. *For any $\varphi \in W(\mathbf{C}^n)$, $u \in \mathcal{L}$, and $x, y \in \mathbf{C}^n$,*

$$\delta(dd^c u, \varphi) \le \delta(dd^c u, \Psi_{\varphi,y}) = n! \int_{E^\Phi} \psi_{u,x}(t)\, d\gamma_1^\Phi(t),$$

where $\Phi = \Psi_{\varphi,y}$ and $\psi_{u,x}$ is the convex image of the indicator $\Psi_{u,x}$. In particular, $\delta(dd^c u, \varphi) \le \sigma(u, \varphi)\mu(\Psi_{\varphi,0}, \dots, \Psi_{\varphi,0})$.

REMARK. A description for the swept-out Monge–Ampère measures for (negative) *local indicators* was given in [21]; the result was that the generalized Lelong numbers with arbitrary homogeneous weight can be recovered from those with respect to the weights φ_a (its directional Lelong numbers along $a \in \mathbf{R}^n_+$). As follows from Corollary 5.2, this is not always the case for the generalized degrees (the measure γ_1^Φ can charge $\mathbf{R}^n \setminus \mathbf{R}^n_+$).

Proof of Theorem 5.3. Since $(dd^c\Phi)^n = 0$ on $\{\Phi > 0\}$, we have $\mu_r^\Phi = (dd^c\Phi_r)^n$ for each $r > 0$. By the rotation invariance,

$$\mu_r^\Phi = (2\pi)^{-n} d\theta \otimes d\rho_r^\Phi$$

with some measure ρ_r^Φ supported by $S_r(\Phi) \cap \mathbf{R}^n$. Moreover, since μ_r^Φ has no masses on the pluripolar set $S_r(\Phi) \cap \{z : z_1 \ldots z_n = 0\}$, we can pass to the coordinates $z_k = \exp\{t_k + i\theta_k\}$ $(-\infty < t_k < \infty, 1 \le k \le n)$. The functions

$$\varphi(t) := \Phi(e^{t_1}, \ldots, e^{t_n}) \quad \text{and} \quad \varphi_r(t) = \Phi_r(e^{t_1}, \ldots, e^{t_n}) = \max\{\varphi(t), r\}$$

are convex in $\mathbf{R}^n$ and increasing in each t_k. Simple calculations show that, in these coordinates, ρ_r^Φ transforms into the measure

$$\gamma_r^\Phi = n!\, \mathcal{MA}[\varphi_r],$$

where $\mathcal{MA}$ is the *real* Monge–Ampère operator (see e.g. [20] for details). We recall that, for smooth functions v,

$$\mathcal{MA}[v] = \det\left(\frac{\partial^2 v}{\partial t_j \partial t_k}\right) dt,$$

and it can be extended as a positive measure to any convex function (see [19]). Thus we have

$$\mu_r^\Phi(u) = \int_{\mathbf{R}^n} (2\pi)^{-n} \int_{[0,2\pi]^n} u(z_1 e^{i\theta_1}, \ldots, z_n e^{i\theta_n})\, d\theta\, d\rho_r^\Phi(|z_1|, \ldots, |z_n|)$$
$$= n! \int_{\mathbf{R}^n} \lambda(u,t)\, d\gamma_r^\Phi(t) = n! \int_{\mathbf{R}^n} \lambda(u, rt)\, d\gamma_1^\Phi(t)$$

since $\varphi_r(t) = r\varphi_1(t/r)$, and we need only to find an explicit expression for the measure γ_1^Φ supported in the level set

$$L^\Phi = \{t \in \mathbf{R}^n : \varphi(t) = 1\}. \tag{5.6}$$

As follows from properties of the real Monge–Ampère operator,

$$\int_F \mathcal{MA}[\varphi_1] = \mathrm{Vol}(\omega(F, \varphi_1)) \quad \forall F \subset L^\Phi, \tag{5.7}$$

where

$$\omega(F, \varphi_1) = \bigcup_{t^0 \in F} \{a \in \mathbf{R}^n : \varphi_1(t) \ge 1 + \langle a, t - t^0\rangle \ \forall t \in \mathbf{R}^n\}$$

is the gradient image of the set F.

Given a subset F of L^Φ, we put

$$\Gamma_F^\Phi = \left\{a \in \mathbf{R}_+^n : \sup_{t \in F}\langle a, t\rangle = \sup_{t \in L^\Phi}\langle a, t\rangle = 1\right\} \tag{5.8}$$

and

$$\Theta_F^\Phi = \{\lambda a : 0 \le \lambda \le 1,\ a \in \Gamma_F^\Phi\}. \tag{5.9}$$

Note that $\Theta^{\Phi}_{L^{\Phi}}$ is a bounded convex subset of $\mathbf{R}^n_+$ and that φ is its support function. We claim that, *for any compact subset F of L^{Φ}, $\Theta^{\Phi}_F = \omega(F, \varphi_1)$.*

If $a \in \omega(F, \varphi_1)$ then, for some $t^0 \in F$,

$$\langle a, t^0\rangle \geq \langle a, t\rangle - \varphi_1(t) + 1 \quad \forall t \in \mathbf{R}^n. \tag{5.10}$$

In particular,

$$\langle a, t^0\rangle \geq \langle a, t\rangle \quad \forall t \in L^{\Phi}. \tag{5.11}$$

When $t = ct_0$ with $c > 1$, (5.10) implies $\langle a, t^0\rangle \leq 1$. In view of (5.11) it follows that $a \in \Theta^{\Phi}_F$ and thus $\omega(F, \varphi_1) \subset \Theta^{\Phi}_F$.

Let now $a \in \Theta^{\Phi}_F$, so $a = \lambda a^0$ for $a^0 \in \Gamma^{\Phi}_F$ and $0 \leq \lambda \leq 1$. Then there is a point $t^0 \in F$ such that

$$\langle a, t^0\rangle = \sup_{t\in F}\langle a, t\rangle = \sup_{t\in L^{\Phi}}\langle a, t\rangle = \lambda.$$

Take any $t \in \mathbf{R}^n$. If $\varphi(t) \leq 1$, then $\langle a, t^0\rangle \geq \langle a, t\rangle$ and thus $\varphi_1(t) \geq 1 + \langle a, t - t^0\rangle$. If $\varphi(t) = \alpha > 1$, then $t/\alpha \in L^{\Phi}$ and

$$\begin{aligned}\langle a, t\rangle - \varphi_1(t) + 1 = \alpha\langle a, t/\alpha\rangle + 1 - \alpha &\leq \alpha \sup_{s\in L^{\Phi}}\langle a, s\rangle + 1 - \alpha \\ &= \alpha\langle a, t^0\rangle + 1 - \alpha = \alpha\lambda + 1 - \alpha \leq \lambda = \langle a, t^0\rangle,\end{aligned}$$

so $a \in \omega(F, \varphi_1)$. The claim is proved.

Finally, let E^{Φ} be the set of extreme points of L^{Φ} (i.e., those not situated inside intervals on L^{Φ}). Since $L^{\Phi} \subset \{t \in \mathbf{R}^n \setminus \mathbf{R}^n_- : t_k \leq b_k, 1 \leq k \leq n\}$ for some $b \in \mathbf{R}^n_+$, we have

$$\sup_{t\in L^{\Phi}}\langle a, t\rangle = \sup_{t\in E^{\Phi}}\langle a, t\rangle \quad \forall a \in \mathbf{R}^n_+,$$

so that $\Theta^{\Phi}_{L^{\Phi}} = \Theta^{\Phi}_{E^{\Phi}}$. Hence $\gamma^{\Phi}_1(L^{\Phi}) = \gamma^{\Phi}_1(E^{\Phi})$ and then $\operatorname{supp}\gamma^{\Phi}_1 \subset E^{\Phi}$. The proof is complete. □

6. Algebraic Case: Newton Polyhedra

Here we test our results for the case of currents generated by polynomial mappings.

When $u = \log|P|$ for a polynomial P, it follows that $\sigma(u)$ is the degree of P, $\sigma_k(u)$ is its degree with respect to z_k, and $\psi_{u,x}(t) = \max\{\langle t, J\rangle : J \in \omega_x(P)\}$, where

$$\omega_x(P) = \left\{J \in \mathbf{Z}^n_+ : \frac{\partial^J P}{\partial z^J}(x) \neq 0\right\}$$

(see [22]). Note that the maximum is attained on the set of extreme points $E_x(P)$ of the set $\omega_x(P)$. In particular, $E_0(P)$ coincides with the set of extreme points of the convex hull of the exponents of the polynomial P. This means that the set Θ^{Φ} with $\Phi = \Psi_{u,0}$ equals

$$G^+(P) = \operatorname{conv}(E_0(P) \cup \{0\}), \tag{6.1}$$

the *Newton polyhedron of P at infinity* as defined in [11].

Let now $P = (P_1, \dots, P_p)$ be a polynomial mapping; if $u_j = \log|P_j|$ then $Z_j = dd^c u_j$ is the divisor of P_j. Let the zero sets A_j of P_j be properly intersected—that is, let $\operatorname{codim} A_{j_1} \cap \dots \cap A_{j_m} \ge m$ for all choices of indices $j_1, \dots, j_m$ $(m \le p)$. Then the holomorphic chain Z of the mapping P is the intersection of the divisors Z_j: $Z = Z_1 \wedge \dots \wedge Z_p$ [9, Prop. 2.12].

In this setting, Corollary 3.1 turns into the bound

$$\int_{\mathbf{C}^n} T \wedge Z \le \delta(T)\delta_1 \dots \delta_p$$

via the degrees δ_j of P_j. For $T = 1$ this gives Bezout's inequality for the projective volume of the chain Z. The specification of Corollary 3.3 with $a \in \mathbf{Z}^n_+$ gives a bound by means of the degrees of $P_j(z^a)$, a global counterpart for the Tsikh–Yuzhakov theorem on multiplicity of holomorphic mappings in terms of the quasi-homogeneous (or weighted homogeneous) initial polynomial terms [26] (see also [4, Thm. 10.3.2′]). And Corollary 4.2 becomes exactly a result of Tsikh [25].

Theorem 4.1 now takes the following form.

Corollary 6.1. *The degree* (*projective volume*) $\delta(Z)$ *of the holomorphic chain* Z *generated by a polynomial mapping* $P = (P_1, \dots, P_p)$, *where* $p \le n$ *and the zero sets of components* P_k *are properly intersected, has the bound*

$$\delta(Z) \le n! \operatorname{Vol}(G_1^+, \dots, G_p^+, \Delta, \dots, \Delta).$$

Here G_j^+ *is the Newton polyhedron of the polynomial* P_j *at infinity* (*defined by* (6.1)) *and* $\Delta = \{t \in \mathbf{R}^n_+ : \sum t_j \le 1\}$, *the standard simplex in* $\mathbf{R}^n$. *In particular, if* $p = n$ *then the number of zeros of* P *counted with their multiplicities does not exceed* $n! \operatorname{Vol}(G_1^+, \dots, G_n^+)$.

When $P_j(0) = 0$, the set G_j^+ is strictly greater than the convex hull $E_0(P_j)$ of the set $\omega_0(P_j)$ appearing in Bernstein's theorem. But in return we take care of all the zeros whereas Bernstein's theorem estimates only those in $(\mathbf{C} \setminus \{0\})^n$. Actually, no bound for the total number is possible in terms of just the convex hulls of the exponents (see e.g. $f(z) \equiv z$).

An algebraic specification of Theorem 5.3 and Corollary 5.2 is as follows. Let Φ be the indicator of $\log|P|$ for a polynomial mapping $P : \mathbf{C}^n \to \mathbf{C}^p$ $(p \ge n)$ with discrete zeros. Then the set $\Gamma^\Phi_{L^\Phi}$ is the Newton diagram for P and $\Theta^\Phi_{L^\Phi} = G^+(P)$ is the Newton polyhedron for P at infinity—that is, the convex hull of the sets $G^+(P_k)$, $1 \le k \le p$. In this case, the set $E^\Phi = \{t^1, \dots, t^N\}$ is finite; it consists simply of normals to the $(n-1)$-dimensional faces $\Gamma_j(P)$ of the polyhedron situated outside the coordinate planes, with the condition $\Phi(t^j) = 1$. The measure γ_1^Φ charges t^j with the volume of the convex hull $G_j^+(P)$ of the corresponding face $\Gamma_j(P)$ and 0, so

$$\delta(dd^c u, \log|P|) \le \delta(dd^c u, \Psi_{\log|P|,0}) = n! \sum_{1 \le j \le N} \psi_{u,0}(t^j) \operatorname{Vol}(G_j^+(P)).$$

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