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Rectification of Circles and Quaternions

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Introduction

Throughout this paper, the word "circle" means a circle or a straight line. We are always assuming that the space $\mathbb{R}^n$ is equipped with a fixed "standard" Euclidean inner product.

A collection of curves in $\mathbb{R}^n$ passing through 0 is said to be a *simple bundle of curves* if no two of them are tangent at 0. A simple bundle of curves is called *rectifiable* if there exists a germ of diffeomorphism in a neighborhood of the origin that sends all curves from this bundle to straight lines. Rectifiable bundles of curves appear, for example, in Riemannian geometry—the set of geodesics passing through a given point is rectifiable.

A. G. Khovanskii proved in [2] that a rectifiable simple bundle of more than six circles on plane necessarily passes through some point different from the origin. F. A. Izadi [1] generalized Khovanskii's arguments to dimension 3. A rectifiable simple bundle of circles in $\mathbb{R}^3$ containing sufficiently many circles in general position must pass through some other common point.

In dimension 4, this is not true. The simplest counterexample is a family of circles that are obtained from straight lines by some complex projective transformation (with respect to some identification $\mathbb{R}^4 = \mathbb{C}^2$ such that the multiplication by i is an orthogonal operator).

It turns out that, in dimension 4, there is a large family of transformations that round lines (i.e., take them to circles). To construct such a family, fix a quaternionic structure on $\mathbb{R}^4$ compatible with the Euclidean structure. If A and B are some affine maps, then the map $x \mapsto A(x)^{-1}B(x)$ rounds lines (the multiplication and the inverse are in the sense of quaternions). Such transformations will be called (left) *quaternionic fractional transformations*. Right quaternionic fractional transformations AB^{-1} also round lines. Any real projective, complex projective, or quaternionic projective transformation is quaternionic fractional.

In this paper, we will prove that a rectifiable simple bundle of circles containing sufficiently many circles in general position is the image of a bundle of straight lines under some left or right quaternionic fractional transformation.

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In arbitrary dimension, we have a purely algebraic description of rectifiable simple bundles of circles. So the analytic problem of classification of such bundles is reduced to an algebraic problem.

The paper is organized as follows. In Section 1, for a simple rectifiable bundle of circles we establish an algebraic condition on the second derivative of a rectifying map. This condition is formulated on the asymptotic cone $\{(x, x) = 0\} \subseteq \mathbb{C}^2$, where $(\cdot, \cdot)$ is the complexification of the usual inner product. This provides a simple proof of Izadi's theorem [1]. In Section 2, we show that this algebraic condition is not only necessary but also sufficient in a sense. Thus we obtain an algebraic description of rectifiable simple bundles of circles. In Section 3, we review some important properties of complex and quaternionic structures and relate them to the geometry of the asymptotic cone. In Section 4, we define quaternionic fractional transformations and list some of their properties. Section 5 contains the main rectification result and some of its geometrical consequences.

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1. Rectifiable Collections of Circles

The following result is true in dimensions 2 [2] and 3 [1].

THEOREM 1.1. *Consider a simple bundle of circles in $\mathbb{R}^2$ or $\mathbb{R}^3$ containing sufficiently many circles in general position. If this bundle is rectifiable, then all its circles pass through a common point different from the origin.*

On a plane, it is enough to take seven circles. Theorem 1.1 means, in particular, that if a generic family of circles can be rectified at all, then it can be rectified by means of some inversion. However, as we will see, this does not hold in dimension 4.

We need the following very simple lemma.

LEMMA 1.2. *Consider a polynomial map $F: \mathbb{R}^n \to \mathbb{R}^n$ such that $F(x)$ is everywhere proportional to x. Then $F(x) = G(x)x$ for some polynomial function $G: \mathbb{R}^n \to \mathbb{R}$. If F is homogeneous, then so is G.*

Proof. Introduce a coordinate system $(x_0, \dots, x_{n-1})$. Denote by F_i the ith component of F. Then the proportionality condition reads as $x_i F_0 - x_0 F_i = 0$. In particular, F_0 is divisible by x_0. Denote the quotient by G. Then from our equation we see that $F_i = Gx_i$. The last statement of the lemma is obvious. $\square$

Extend the standard inner product $(\cdot, \cdot)$ from $\mathbb{R}^n$ to $\mathbb{C}^n$ by complex bilinearity. The locus $(x, x) = 0$ is called the *asymptotic cone.* Denote this cone by C. The asymptotic cone describes the behavior of circles at infinity. Namely, any nondegenerate circle (not a line) is asymptotic to C.

Let $\Phi: (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be the germ of a diffeomorphism at 0 that sends several lines passing through the origin to circles. Suppose that the number of lines is big enough and that they are in general position; denote this set of lines by $\mathcal{L}$. We can assume without loss of generality that $d_0\Phi = \mathrm{id}$. To arrange this, it is enough to compose Φ with some linear transformation (namely, the inverse of $d_0\Phi$) that certainly takes lines to lines. Let $\Phi = x + \Phi_2(x) + \cdots$ near 0, where Φ_2 denotes the second-order terms.

PROPOSITION 1.3. *The quadratic map Φ_2 satisfies the following relations on the asymptotic cone*:

$$(\Phi_2(x), \Phi_2(x)) = 0, \qquad (\Phi_2(x), x) = 0.$$

This proposition means that Φ_2 preserves the asymptotic cone and takes each vector $x \in C$ to a vector $y \in C$ such that x and y span a subspace lying entirely in C. For an informal explanation of this result, let us assume the following.

(1) The diffeomorphism Φ takes germs of *all* lines passing through 0 to germs of circles.
(2) Our diffeomorphism can be extended to a neighborhood of the origin in $\mathbb{C}^n$ as a local holomorphic map.

Then Φ takes germs of complex lines to germs of some planar second-degree curves that approach the asymptotic cone at infinity.

Take a complex line L from C. We know that $\Phi(L)$ is tangent to L at 0 and asymptotic to C at infinity. Denote by M the plane where $\Phi(L)$ lies. Then either M is contained in C or $M \cap C$ is a pair of intersecting lines in M (that may be coincident). In the latter case $\Phi(L)$ must coincide with one of these lines. Indeed, $\Phi(L)$ intersects both lines at the origin and is asymptotic to one of them. But a plane curve of degree 2 cannot intersect its own asymptotic line. Note that L is clearly in $M \cap C$, so $\Phi(L) = L$.

In any case, L and $\Phi(L)$ span a vector subspace lying entirely in C. Hence $\Phi_2(L)$ lies in this subspace. From this the proposition follows.

The preceding argument can be extended to a rigorous proof but, in order to give a shorter proof, we will use another idea.

Proof of Proposition 1.3. Make the inversion I with respect to the origin and consider the composition $I \circ \Phi$. The diffeomorphism Φ takes a line from $\mathcal{L}$ to a tangent circle (owing to the condition $d_0\Phi = \mathrm{id}$), and I sends circles or lines tangent at 0 to parallel lines. Therefore, $I \circ \Phi$ maps each line from $\mathcal{L}$ to a parallel line.

Consider the Taylor series for Φ at the origin,

$$\Phi(x) = x + \Phi_2(x) + \Phi_3(x) + \cdots,$$

where $\Phi_k(x)$ denotes the order-k terms. Fix some nonzero vector x that spans a line from $\mathcal{L}$. This line can be parameterized as $\{xt\}$ where t is a parameter. Hence $I \circ \Phi(xt)$ runs over some line parallel to x as t runs over real numbers. This means that, in the expansion of $I \circ \Phi(xt)$, all terms with nonzero powers of t are proportional (parallel) to x. We will write down some initial terms of this expansion, dropping the terms with zero power of t and those obviously parallel to x:

$$\begin{aligned} I \circ \Phi(xt) = &\left(\frac{\Phi_3}{(x,x)} - \frac{2(\Phi_2, x)\Phi_2}{(x,x)^2} \right) t \\ &+ \left(\frac{\Phi_4}{(x,x)} - \frac{2(\Phi_2, x)\Phi_3}{(x,x)^2} \right. \\ &\qquad \left. - \frac{(\Phi_2, \Phi_2)\Phi_2 + 2(\Phi_3, x)\Phi_2}{(x,x)^2} + \frac{4\Phi_2 (x, \Phi_2)^2}{(x,x)^3} \right) t^2 + \cdots . \end{aligned}$$

The terms with t and t^2 must be proportional to x. The proportionality conditions are polynomial relations in x. If they hold for sufficiently many x in general position, then they hold everywhere.

The coefficient with t is equal to

$$\frac{\Phi_3}{(x,x)} - \frac{2(\Phi_2, x)\Phi_2}{(x,x)^2}.$$

Therefore, the map $\Phi_3 (x,x) - 2(\Phi_2, x)\Phi_2$ is everywhere proportional to x. In particular, the inner product of this map with x is identically zero on the asymptotic cone $\{(x,x) = 0\}$. This implies that $(\Phi_2, x) = 0$ on C. Hence (Φ_2, x) is divisible by (x,x), and so the map

$$\Phi_3 - \frac{2(\Phi_2, x)\Phi_2}{(x,x)}$$

is a polynomial proportional to x. By Lemma 1.2 this polynomial is divisible by x in the class of polynomials. Therefore, Φ_3 is a linear combination with polynomial coefficients of Φ_2 and x. Thus it always lies in the linear span of Φ_2 and x. In particular, $(\Phi_3, x) = 0$ on C.

The term with t^2 is

$$\frac{\Phi_4}{(x,x)} - \frac{2(\Phi_2, x)\Phi_3}{(x,x)^2} - \frac{(\Phi_2, \Phi_2)\Phi_2 + 2(\Phi_3, x)\Phi_2}{(x,x)^2} + \frac{4\Phi_2 (x, \Phi_2)^2}{(x,x)^3}.$$

Multiply this expression by $(x,x)^2$ and restrict it to the asymptotic cone. We obtain that $\Phi_2(\Phi_2, \Phi_2)$ is parallel to x on C (note that all other terms are zero on the asymptotic cone). This means that either Φ_2 is parallel to x on C or the coefficient is zero. In both cases we have $(\Phi_2, \Phi_2) = 0$ on C. □

EXAMPLE. Let us construct an example of transformation that takes all lines to circles and has the identical differential at 0. Pick up a point $a \in \mathbb{R}^n$ and compose the mirror reflection

$$x \mapsto x - 2\frac{(a,x)a}{(a,a)}$$

with respect to the orthogonal complement to a with the inversion

$$x \mapsto a + \frac{(a,a)(x-a)}{(x-a, x-a)}$$

with center a and radius $|a|$ (so that 0 is fixed). Denote the resulting local diffeomorphism by T^a. We have

$$T^a(x) = \frac{(a,a)x + (x,x)a}{(a,a) + 2(a,x) + (x,x)} = x + \frac{(x,x)a - 2(a,x)x}{(a,a)} + \cdots .$$

In particular, the quadratic term of T^a has the form

$$T_2^a(x) = \frac{(x,x)a - 2(a,x)x}{(a,a)},$$

which is obviously parallel to x on the asymptotic cone.

Now let us return to the general situation: we have a local diffeomorphism Φ that rounds a sufficiently big and sufficiently general collection $\mathcal{L}$ of lines passing through 0. Denote by $\mathcal{S}$ the corresponding set of circles.

PROPOSITION 1.4. *Suppose that Φ_2 is parallel to x on the asymptotic cone. Then all the circles from $\mathcal{S}$ pass through another common point different from the origin.*

In order to prove this, we need two very simple algebraic lemmas.

LEMMA 1.5. *Assume that a linear map $\Lambda\colon \mathbb{R}^n \to \Lambda^2\mathbb{R}^n$ satisfies the condition $\Lambda(x) \wedge x = 0$ everywhere. Then there is a vector $b \in \mathbb{R}^n$ such that $\Lambda(x) = b \wedge x$.*

Proof. Introduce a coordinate system $(x_0, \dots, x_{n-1})$ in $\mathbb{R}^n$. Let $\Lambda_{ij}(x)$ be the coordinates of $\Lambda(x)$ in the standard basis of $\Lambda^2\mathbb{R}^n$. These are linear functions in x. The condition $\Lambda \wedge x = 0$ can be written in coordinates as follows:

$$\Lambda_{ij}x_k + \Lambda_{jk}x_i + \Lambda_{ki}x_j = 0. \tag{$*$}$$

Formula $(*)$ implies that Λ_{ij} vanishes on the subspace $x_i = x_j = 0$. Hence $\Lambda_{ij} = b_{ij}x_j - c_{ij}x_i$, where b_{ij} and c_{ij} are some numbers. Substituting this equality into $(*)$ yields

$$(b_{ij}x_j - c_{ij}x_i)x_k + (b_{jk}x_k - c_{jk}x_j)x_i + (b_{ki}x_i - c_{ki}x_k)x_j = 0.$$

Equating the coefficient with x_ix_j to zero, we obtain $b_{ki} = c_{jk}$. This implies that:

(i) the coefficient b_{ki} is independent of i (denote it by b_k);
(ii) the coefficient c_{jk} is independent of j (denote it by c_k); and
(iii) $b_k = c_k$.

Now we have $\Lambda_{ij} = b_ix_j - b_jx_i$. This means that $\Lambda(x) = b \wedge x$, where b is the vector with coordinates $(b_0, \dots, b_{n-1})$. $\square$

Recall that a map $\Gamma\colon \mathbb{C}^n \to \mathbb{C}^n$ is *defined over reals* if it takes $\mathbb{R}^n \subset \mathbb{C}^n$ to $\mathbb{R}^n$.

LEMMA 1.6. *Let $\Gamma\colon \mathbb{C}^n \to \mathbb{C}^n$ be a vector-valued quadratic form* (*i.e., a homogeneous polynomial map of second degree*) *defined over reals and such that $\Gamma(x)$ is everywhere parallel to x on C. Then Γ has the form $\Gamma(x) = b(x,x) + \lambda(x)x$, where $b \in \mathbb{R}^n$ and λ is a linear functional.*

Proof. Since Γ is everywhere parallel to x on the cone C, we have $\Gamma(x) \wedge x = 0$ there. Consequently, $\Gamma \wedge x$ is divisible by (x,x). Denote the quotient by Λ; it is a linear map from $\mathbb{R}^n$ to $\Lambda^2\mathbb{R}^n$. Moreover, we have $\Lambda \wedge x = 0$ because $(\Gamma \wedge x) \wedge x = 0$. By Lemma 1.5 it follows that $\Lambda = b \wedge x$ and hence $(\Gamma - b(x, x)) \wedge x$ vanishes everywhere. This means that the polynomial map $\Gamma - b(x, x)$ is proportional to x. By Lemma 1.2 we have $\Gamma - b(x, x) = \lambda(x)x$, where λ is some linear function. □

Proof of Proposition 1.4. By Lemma 1.6 we know that the second-order part Φ_2 of a rectifying diffeomorphism Φ has the form $\Phi_2(x) = b(x, x) + \lambda(x)x$, where b is some vector from $\mathbb{R}^n$ and λ is a linear functional.

Consider a circle from $\mathcal{S}$ with the tangent vector x at 0. The acceleration with respect to the natural parameter is

$$2\frac{\Phi_2 - \frac{(\Phi_2, x)x}{(x,x)}}{(x, x)} = 2\frac{\Phi_2 - \lambda(x)x - (b, x)x}{(x, x)} = 2\left(b - \frac{(b, x)x}{(x, x)}\right),$$

which is the same as for the circle passing through $b/(b, b)$. But the circle is determined by its velocity $x/|x|$ and acceleration (both with respect to the natural parameter). It follows that all the circles from $\mathcal{S}$ pass through $b/(b, b)$. □

Now we can give a simple proof of Theorem 1.1.

Proof of Theorem 1.1. In dimensions 2 and 3, the asymptotic cone does not contain any plane. Therefore, Φ_2 must be parallel to x everywhere on the cone. Now Proposition 1.4 is applicable. □

EXAMPLE. In dimension 4, the statement of Theorem 1.1 does not hold. To construct a counterexample, introduce a complex structure on $\mathbb{R}^4$ and identify $\mathbb{R}^4$ with $\mathbb{C}^2$ by means of this complex structure. Consider any complex projective transformation Φ preserving the origin: it takes complex lines to complex lines, and on each line it induces a projective transformation. On the other hand, a complex projective transformation of a complex line takes real lines to circles. Hence Φ takes real lines to circles (note that each real line belongs to exactly one complex line). Thus we get a rectifiable family of circles (through 0). But these circles do not pass through a common point different from the origin, because different complex lines meet only at the origin.

Theorem 1.1 fails in dimension 4 for the following simple reason. The asymptotic cone now contains many planes, so there is no longer any reason for $\Phi_2(x)$ to be everywhere parallel to x on C.

2. Algebraic Criteria for Rectification

We shall now prove that the conditions on Φ_2 stated in Proposition 1.3 are not only necessary but also sufficient in a sense.

PROPOSITION 2.1. *If a vector-valued quadratic form* $\Gamma\colon \mathbb{C}^n \to \mathbb{C}^n$ *defined over reals satisfies the conditions* $(x, \Gamma(x)) = (\Gamma(x), \Gamma(x)) = 0$ *on the asymptotic cone, then there exists a germ of diffeomorphism* $\Phi\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ *that* *rounds lines passing through the origin and such that* $d_0\Phi = \mathrm{id}$ *and* $\Phi_2 = \Gamma$*; that is,* $\Phi = x + \Gamma$ *up to third-order terms.*

Proof. Let us introduce the following notation:

$$\lambda = \frac{(\Gamma, x)}{(x, x)}, \qquad \mu = \frac{(\Gamma, \Gamma)}{(x, x)}.$$

We know that λ and μ are polynomials in x (λ is a linear functional and μ is a quadratic form).

First assume that $\lambda = 0$ (i.e., $(\Gamma, x) = 0$ everywhere). Let us look for a diffeomorphism Φ of the form $\Phi(x) = x + \Gamma(x) f(x)$, where f is some smooth function that is equal to 1 at 0. We want Φ to take all lines (passing through 0) to circles. Denote by I the inversion with center at 0 and radius 1. Then the germ of diffeomorphism

$$I \circ \Phi = \frac{x + \Gamma f}{(x + \Gamma f, x + \Gamma f)} = \frac{1}{(x, x)} \frac{x + \Gamma f}{1 + \mu f^2}$$

sends a neighborhood of 0 to a neighborhood of ∞ and is supposed to take each line (passing through 0) to a parallel line. For that it suffices to require that $f/(1 + \mu f^2) = 1$. Indeed, under the latter requirement we have

$$I \circ \Phi(xt) = t^{-1} \frac{x}{(x, x)(1 + \mu f(xt)^2)} + \frac{\Gamma}{(x, x)},$$

and the right-hand side has the form "something parallel to x plus a term independent of t", which means that $I \circ \Phi(xt)$ runs over a line parallel to x as t runs over reals. Solving the corresponding quadratic equation on f, we obtain

$$f = \frac{1 - \sqrt{1 - 4\mu}}{2\mu}.$$

We see that f is a smooth analytic function near 0 such that $f(0) = 1$, as desired.

Now suppose that $\lambda \neq 0$. Let us look for a diffeomorphism Φ of the form $\Phi = T^a \circ \Psi$, where Ψ is some other local diffeomorphism at 0. If Ψ takes all lines passing through 0 to circles, then the same is true for Φ. We will try to kill λ by choosing an appropriate center a. For the second-order terms we have $\Phi_2 = \Psi_2 + T_2^a$. It therefore suffices to take a such that $\lambda(x) = -(a, x)/(a, a)$. Now $(\Psi_2, x) = 0$ everywhere, so our problem is reduced to the previous case ($\lambda = 0$), which we have already shown. $\square$

Consider a simple bundle $\mathcal{S}$ of circles passing through 0 such that in each direction there emanates a unique circle from $\mathcal{S}$. Such a bundle is called *complete.* Now we can give a description of complete rectifiable bundles of circles in pure algebraic terms.

THEOREM 2.2. *Complete rectifiable bundles of circles in* $\mathbb{R}^n$ *are in one-to-one correspondence with quadratic homogeneous maps* $\Gamma \colon \mathbb{C}^n \to \mathbb{C}^n$ *defined over reals and satisfying the conditions* $(x, \Gamma(x)) = (\Gamma(x), \Gamma(x)) = 0$ *on the asymptotic cone, modulo maps of the form* $x \mapsto \lambda(x)x$ (*where* λ *are linear functionals*).

Proof. To each complete rectifiable bundle $\mathcal{S}$ of circles assign the quadratic part Φ_2 of any rectifying diffeomorphism Φ. We know that any quadratic homogeneous map Φ_2 defined over reals and satisfying Proposition 1.3 can be obtained in this way. Let us see to what extent the quadratic map Φ_2 is unique. We saw already that, for each circle from $\mathcal{S}$, it is enough to know the acceleration at 0 with respect to the natural parameter. The acceleration of the circle with the tangent vector x is equal to

$$w(\Phi_2) = 2\frac{\Phi_2 - \frac{(\Phi_2, x)x}{(x,x)}}{(x,x)}.$$

However, this expression does not determine Φ_2. It is easy to see that, if Φ_2 and Φ_2' differ by $\lambda(x)x$ (where λ is a linear functional), then $w(\Phi_2) = w(\Phi_2')$ and so the corresponding families are the same. Indeed, this follows from the observation that $\Phi_2 - (\Phi_2, x)x/(x,x)$ is just the projection of Φ_2 to the orthogonal complement of x. Conversely, if $w(\Phi_2) = w(\Phi_2')$ then $\Phi_2 - \Phi_2'$ is everywhere parallel to x (since the projections to the orthogonal complement coincide). Hence $\Phi_2 - \Phi_2' = \lambda(x)x$, where λ is a linear functional. □

EXAMPLE. In dimension 4, the condition $(x, \Gamma(x)) = (\Gamma(x), \Gamma(x)) = 0$ on C can be interpreted in terms of algebraic geometry as follows. Denote by Q the projectivization of the asymptotic cone. This is a nondegenerate quadratic surface in $\mathbb{CP}^3$. Each point of Q belongs to two straight lines lying entirely in Q.

To describe all lines in Q, it is convenient to identify Q with the image of the Segre embedding

$$\mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^3,$$
$$([u_0 : u_1], [v_0 : v_1]) \mapsto [u_0 v_0 : u_0 v_1 : v_0 u_1 : u_1 v_1]$$

(recall that any nondegenerate quadratic surface in $\mathbb{CP}^3$ can be mapped to any other by a complex projective transformation). Under this embedding, all horizontal lines $\mathbb{CP}^1 \times \{p\}$ and all vertical lines $\{p\} \times \mathbb{CP}^1$ are mapped to straight lines. Hence we have two families of lines in Q such that every point of Q belongs to a unique line from each family. These families of lines are called *generating families of lines.* For each generating family of lines in Q there is the corresponding *generating family of planes* in C. So the cone C is covered by two generating families of planes, and every line in C belongs to exactly one plane from each generating family.

The conditions $(x, \Gamma(x)) = (\Gamma(x), \Gamma(x)) = 0$ on the asymptotic cone are equivalent to the following statement: The subspace spanned by x and $\Gamma(x)$ lies entirely in C. This means that Γ takes x to another point of some line or plane containing x and lying entirely in C. The map Γ is homogeneous, and thus it gives rise to a rational map $\gamma\colon \mathbb{CP}^3 \to \mathbb{CP}^3$ preserving the projectivization Q of the asymptotic cone C. We know that for each point $q \in Q$ there is a line lying entirely in Q and containing both q and its image $\gamma(q)$. We will deduce from this that Γ preserves at least one of the generating families of lines in Q (perhaps both)—in other words, that Γ takes each line from some generating family to itself. Indeed, being a rational map, γ cannot "switch" from one generating family to the other. We now provide a formal proof of this statement.

LEMMA 2.3. *The map γ preserves at least one generating family of lines in Q.*

Proof. The surface Q is isomorphic to $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$ via the Segre map. Hence γ can be given by the two rational maps

$$X\colon (x, y) \in \mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1 \mapsto X(x, y) \in \mathbb{C}\mathrm{P}^1,$$

$$Y\colon (x, y) \in \mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1 \mapsto Y(x, y) \in \mathbb{C}\mathrm{P}^1.$$

We know that, for each point $(x, y) \in \mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$, we have $X(x, y) = x$ or $Y(x, y) = y$. Therefore, Q is the union of two algebraic subsets defined by the equations $X(x, y) = x$ and $Y(x, y) = y$. Since Q is irreducible, at least one of our equations is satisfied identically, which means that γ preserves at least one of the generating families of lines in Q. $\square$

Now we can deduce the following.

PROPOSITION 2.4. *Polynomial homogeneous maps $\Gamma\colon \mathbb{C}^4 \to \mathbb{C}^4$ satisfying the conditions $(x, \Gamma(x)) = (\Gamma(x), \Gamma(x)) = 0$ on the asymptotic cone preserve some generating family of planes in C.*

3. Complex and Quaternionic Structures

From now on we will work in 4-dimensional space $\mathbb{R}^4$. This section reviews not only well-known classical facts about complex and quaternionic structures but also their relation to the geometry of the asymptotic cone C.

Recall that a *complex structure* in $\mathbb{R}^4$ is a linear operator $I\colon \mathbb{R}^4 \to \mathbb{R}^4$ such that $I^2 = -1$. We will always assume that the complex structure I is compatible with the Euclidean structure (i.e., that I preserves the inner product). A complex structure clearly defines an action of $\mathbb{C}$ on $\mathbb{R}^4$ via linear conformal maps. From the definition it follows immediately that I must be skew-symmetric, that is, $(x, Iy) = -(Ix, y)$ for all $x, y \in \mathbb{R}^4$. In particular, $(Ix, x) = 0$. Since the operator I is defined over reals and since $I^2 = -1$, it follows that I should have eigenvalues i and $-i$, both with multiplicity 2.

Note that, as an orthogonal operator, I preserves the asymptotic cone C. In particular, all eigenvectors of I belong to C. We know that $(Ix, x) = 0$ everywhere and in particular on C. From the conditions $(x, x) = (Ix, Ix) = (Ix, x) = 0$ on C it follows that the subspace spanned by x and Ix lies entirely in C. Hence I preserves one of the generating families of planes in C.

On the other hand, the complex structure I defines a canonical orientation on $\mathbb{R}^4$. Let us recall the definition. Take two vectors $x, y \in \mathbb{R}^4$ in general position. By definition, the canonical orientation is the orientation of the basis x, y, Ix, Iy. This orientation is well-defined (i.e., independent of the choice of x and y) because the set of degenerate pairs (x, y) (such that x, y, Ix, Iy are linearly dependent) has real codimension 2 in the space $\mathbb{R}^8$ of all pairs. Hence we can always avoid this set when going from any nondegenerate pair to any other. In fact, the degeneracy locus consists of all pairs x, y that are linearly dependent over $\mathbb{C}$, so it is a complex hypersurface.

PROPOSITION 3.1. *The space of all complex structures on $\mathbb{R}^4$ has two connected components. Complex structures from the same component preserve the same generating family of planes in C and provide the same canonical orientation.*

A connected component to which a complex structure I belongs will be called the *orientation* of I. Note that the orientation of I has nothing to do with $\det(I)$, which is always equal to 1—any complex structure preserves orientation of the ambient space.

Now let us pass to quaternionic structures. A *quaternionic structure* on $\mathbb{R}^4$ is a choice of three linear operators $I, J, K : \mathbb{R}^4 \to \mathbb{R}^4$ such that

$$I^2 = J^2 = K^2 = -1,$$
$$IJ = -JI = K, \quad JK = -KJ = I, \quad KI = -IK = J.$$

In particular, the operators I, J, K are complex structures. We will assume that they are compatible with the inner product. A quaternionic structure gives rise to an action of the skew-field $\mathbb{H}$ of quaternions on $\mathbb{R}^4$ via linear conformal maps. This action is called the *quaternionic multiplication*.

LEMMA 3.2. *Let (I, J, K) be any quaternionic structure on $\mathbb{R}^4$. Then all three complex structures I, J, K have the same orientation. Therefore, quaternionic multiplication preserves one of the generating families of planes in the asymptotic cone.*

Proof. Let us prove, for example, that I and J provide the same canonical orientation. Take any vector $e \in \mathbb{R}^4$. It is enough to show that the bases (e, Ke, Ie, IKe) and (e, Ke, Je, JKe) have the same orientation. But $IKe = -Je$ and $JKe = Ie$, so the statement becomes obvious. □

Let $a \in \mathbb{H}$ be a quaternion. It gives rise to the operator of multiplication $A : x \mapsto ax$. If $a = a_0 + a_1 i + a_2 j + a_3 k$, then the corresponding operator is $A = a_0 + a_1 I + a_2 J + a_3 K$. We know that the operator A satisfies the conditions $(x, Ax) = (Ax, Ax)$ on C. In particular, both forms (Ax, Ax) and (Ax, x) are divisible by (x, x). We can write down the quotients explicitly.

LEMMA 3.3. *If A is the operator of multiplication by a quaternion $a \in \mathbb{H}$ (with respect to some quaternionic structure on $\mathbb{R}^4$), then*

$$(Ax, Ax) = (a, a)(x, x), \qquad (Ax, x) = \mathrm{Re}(a)(x, x).$$

In particular, these forms are independent of the choice of quaternionic structure.

Proof. This is a very simple computation that is based on the fact that $(x, Ix) = (x, Jx) = (x, Kx) = 0$ for all $x \in \mathbb{R}^4$. $\square$

Let us summarize some properties of quaternionic structures that are of particular importance for us. These properties follow directly from what we have already seen.

PROPOSITION 3.4. *The set of all quaternionic structures in $\mathbb{R}^4$ has two connected components. Each component corresponds to a certain orientation of three complex structures involved. Quaternionic multiplications with respect to quaternionic structures from the same component preserve the same generating family of planes in C. Different components correspond to different families of generating planes.*

We will say that quaternionic structures from the same connected component have the same *orientation*. Note that the orientation of a quaternionic structure has nothing to do with determinants of quaternionic multiplications. Quaternionic multiplications (with respect to any quaternionic structure) always preserve the orientation of the ambient space.

EXAMPLE. Identify $\mathbb{R}^4$ with $\mathbb{H}$. Denote by I, J, K the operators of left multiplication by i, j, k respectively. The structure (I, J, K) is called the *left quaternionic structure* on $\mathbb{H}$. Taking right multiplication instead of left multiplication yields the *right quaternionic structure.* Left and right quaternionic structures on $\mathbb{H}$ have different orientations.

Let us introduce some notions. We say that a linear operator is *almost orthogonal* if it has the form $\mathrm{const} \cdot A$, where A is an orthogonal operator. Analogously, an operator is *almost skew-symmetric* if it has the form $\mathrm{const} + A$, where A is skew-symmetric.

PROPOSITION 3.5. *A linear operator $A\colon \mathbb{R}^4 \to \mathbb{R}^4$ is the multiplication by a quaternion* (*with respect to some quaternionic structure on $\mathbb{R}^4$*) *if and only if it is almost orthogonal and almost skew-symmetric. The property of being a quaternionic multiplication depends only on the orientation of a quaternionic structure, not on the structure itself.*

Proof. A quaternionic multiplication is clearly almost orthogonal and almost skew-symmetric; this follows from Lemma 3.3. Now consider an almost orthogonal and almost skew-symmetric operator A and present it by a matrix in some orthonormal basis. Denote by a_0, a_1, a_2, a_3 the entries of the first column of A. Since A is almost skew-symmetric, it has the form

$$\begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & \alpha & \beta \\ a_2 & -\alpha & a_0 & \gamma \\ a_3 & -\beta & -\gamma & a_0 \end{pmatrix}.$$

The columns must be orthogonal and have the same length. From the corresponding equations we obtain either that $\alpha = a_3$, $\beta = -a_2$, and $\gamma = a_1$ or that $\alpha = -a_3$, $\beta = a_2$, and $\gamma = -a_1$. The first case corresponds to the left multiplication by $a = a_0 + a_1 i + a_2 j + a_3 k$ with respect to the standard quaternionic structure (assigned to the given basis). The second case corresponds to the right multiplication by a. The equalities hold no matter what orthonormal basis we choose. Thus the second statement follows. □

4. Quaternionic Fractional Transformations

Let us identify $\mathbb{R}^4$ with the skew-field $\mathbb{H}$ of quaternions. Consider two affine maps $A, B: \mathbb{R}^4 \to \mathbb{R}^4$. The map $B^{-1}A$ (the multiplication and the inverse are in the sense of quaternions) is called a (left) *fractional quaternionic transformation* provided that it is defined and one-to-one in some open subset of $\mathbb{R}^4$. A *right quaternionic fractional transformation* is a local transformation of the form AB^{-1}, where A and B are some affine maps.

EXAMPLE 1. Any real projective transformation is quaternionic fractional. This corresponds to the case when B takes real values only.

EXAMPLE 2. Any complex projective transformation is quaternionic fractional. This happens if B takes complex values only and both A and B are complex linear (i.e., commute with the multiplication by i).

EXAMPLE 3. Consider a map of the form $x \mapsto (xa + b)^{-1}(xc + d)$, where a, b, c, d are quaternions. We are assuming that the denominator is not proportional to the numerator (in particular, the denominator is not identically zero). This map is called a (left) *quaternionic projective transformation*. Any quaternionic projective transformation is clearly quaternionic fractional. Note that each quaternionic projective transformation takes all lines to circles. Indeed, we have

$$(xa + b)^{-1}(xc + d) = (xa + b)^{-1}((xa + b)\alpha + \beta) = \alpha + (xa + b)^{-1}\beta,$$

where $\alpha = a^{-1}c$ and $\beta = d - b\alpha$. Hence a quaternionic projective transformation is a composition of a dilatation, reflected inversion, and a translation. This composition obviously rounds lines.

PROPOSITION 4.1. *Any quaternionic fractional transformation rounds lines* (*to be more precise*: *it takes germs of lines to germs of circles*).

Proof. Consider a line L in $\mathbb{R}^4$, and let t be a linear parameter on L. If A and B are some affine maps then their restrictions to L are $at + b$ and $ct + d$, respectively. On the line L the transformation $A^{-1}B$ therefore coincides with the quaternionic projective transformation $x \mapsto (ax + b)^{-1}(cx + d)$. But the latter rounds lines. □

REMARK. Note that a fractional quaternionic transformation can be described geometrically as follows. Consider an arbitrary embedding of $\mathbb{R}^4$ to $\mathbb{H}^2 = \mathbb{R}^8$ as a real affine subspace. A fractional quaternionic transformation is the composition of this embedding and a projection to some quaternionic line (from the origin). There are two types of projections, left and right. The left projection of a point $p \in \mathbb{H}^2$ to a left quaternionic line L is the intersection of L with the left quaternionic line passing through 0 and p (if L is parallel to this line, then the projection of p is not defined). Similarly, we can define the right projection to a right quaternionic line.

5. Rectification at a Point

In this section, we will prove the following theorem.

THEOREM 5.1. *Consider a simple bundle of circles in $\mathbb{R}^4$ containing sufficiently many circles in general position. If this bundle is rectifiable, then there exists a left or right quaternionic fractional transformation T such that T^{-1} sends all these circles to straight lines.*

Denote the given set of circles by $\mathcal{S}$. Let Φ be a local diffeomorphism such that $d_0\Phi = \mathrm{id}$ and Φ^{-1} rectifies all circles from $\mathcal{S}$. Then, by Proposition 1.3, the quadratic term Φ_2 satisfies the relations $(\Phi_2, x) = (\Phi_2, \Phi_2) = 0$ on the asymptotic cone. This means that Φ_2 preserves one of the generating families of planes in C.

LEMMA 5.2. *There exists a linear operator $A\colon \mathbb{R}^4 \to \mathbb{R}^4$ such that $\Phi_2(x) = A(x)x$ or $\Phi_2(x) = xA(x)$, where the product is in the sense of quaternions.*

Proof. Fix an identification $\mathbb{R}^4 = \mathbb{H}$. Extend the operators I, J, K of left multiplication by i, j, k (respectively) to $\mathbb{C}^4$ by complex linearity. Note that the operator I is quite different from the multiplication by $\sqrt{-1}$ in $\mathbb{C}^4$. By Proposition 3.4, the left quaternionic multiplication preserves one of the generating families of planes in C. Assume that Φ_2 preserves the same family. Otherwise we should consider the right multiplication instead of the left multiplication.

Recall that the *quaternionic conjugation* is the map

$$x = x_0 + x_1 i + x_2 j + x_3 k \mapsto \bar{x} = x_0 - x_1 i - x_2 j - x_3 k.$$

We can extend this map to $\mathbb{C}^4$ by complex linearity. Note that i is now a vector from $\mathbb{R}^4$, not a complex number. Let us multiply Φ_2 by $\bar{x}$ in the sense of quaternions. Note that

$$\Phi_2\bar{x} = (\Phi_2, x) + (\Phi_2, Ix)i + (\Phi_2, Jx)j + (\Phi_2, Kx)k.$$

But this expression is zero on the cone C since Φ_2, x, Ix, Jx, and Kx lie on the same plane belonging to C. Therefore, $\Phi_2\bar{x}$ is divisible by (x, x). The quotient is a linear map A. Since $\bar{x}/(x, x) = x^{-1}$ we have $\Phi_2 x^{-1} = A(x)$, that is, $\Phi_2(x) = A(x)x$. □

Now we can prove Theorem 5.1 and an even more precise statement as follows.

THEOREM 5.3. *Under the assumptions of Theorem 5.1, the family of circles can be obtained from the family of their tangent lines by one of the transformations*

$x \mapsto (1 - A(x))^{-1}x$ or $x \mapsto x(1 - A(x))^{-1}$, where A is some linear operator. This answer does not depend on the choice of a quaternionic structure.

Proof. Note that both transformations have the identical derivative at 0 and that their second-order terms are $A(x)x$ and $xA(x)$, respectively. These transformations are quaternionic fractional and so they round lines. The corresponding families of circles passing through 0 are determined by the second-order terms. But by Lemma 5.2, the quadratic maps $A(x)x$ and $xA(x)$ are the only possible second-order terms of transformations that round lines. $\square$

For a complete rectifiable bundle $\mathcal{S}$ of circles, there is a transformation of the form $x \mapsto (1-A(x))^{-1}x$ or $x \mapsto x(1-A(x))^{-1}$ that takes the family of all lines passing through 0 to $\mathcal{S}$. To fix the idea, assume that this is the left transformation $\Phi\colon x \mapsto (1 - A(x))^{-1}x$.

PROPOSITION 5.4. *The center of the circle from $\mathcal{S}$ with the tangent vector x at 0 is $-\frac{1}{2}(\operatorname{Im} A(x))^{-1}x$. This point can be infinite, which means that the corresponding circle is a straight line.*

Proof. We know that the acceleration with respect to the natural parameter is

$$w(x) = 2\frac{\Phi_2 - \frac{(\Phi_2,x)x}{(x,x)}}{(x,x)}.$$

Therefore, the center is located in the point

$$\frac{w}{(w,w)} = \frac{1}{2}\frac{\frac{\Phi_2}{(x,x)} - \frac{(\Phi_2,x)x}{(x,x)^2}}{\frac{(\Phi_2,\Phi_2)}{(x,x)^2} - \frac{(\Phi_2,x)^2}{(x,x)^3}}.$$

By Lemma 3.3 we have $(\Phi_2,\Phi_2) = (A,A)(x,x)$ and $(\Phi_2,x) = (\operatorname{Re} A)(x,x)$. Simplifying the expression just displayed yields the following formula for the center:

$$\frac{1}{2}\left(\frac{A - \operatorname{Re} A}{(A,A) - (\operatorname{Re} A)^2}\right)x = \frac{1}{2}\frac{\operatorname{Im}(A)}{(\operatorname{Im} A, \operatorname{Im} A)}x = -\frac{1}{2}(\operatorname{Im} A)^{-1}x. \qquad \square$$

Proposition 5.4 has the following geometric corollary.

COROLLARY 5.5. *The family $\mathcal{S}$ contains at least one line. The union of all straight lines from $\mathcal{S}$ is a vector subspace of $\mathbb{R}^4$.*

REMARK. We see that the set of all complete rectifiable families of circles passing through 0 is naturally identified with the union of two affine spaces of dimension 12 (= dimension of all possible $\operatorname{Im} A(x)$). The intersection of these components has dimension 4 and consists of all families rectifiable by means of inversions (i.e., families whose circles meet at a point different from 0). The two components can be distinguished by their “orientation”.

We can describe an affine structure on each component in geometric terms. Namely, take any two circles S_1 and S_2 tangent at 0. After an inversion, they become parallel lines. For two parallel lines L_1 and L_2 we can take their barycentric combination

$$L = \lambda L_1 + (1-\lambda)L_2 = \{\lambda x + (1-\lambda)y \mid x \in L_1,\ y \in L_2\}, \quad \lambda \in \mathbb{R}.$$

Make the inversion again. The line L goes to a circle S. Define

$$S = \lambda S_1 + (1-\lambda)S_2.$$

Now we can take barycentric combinations of complete bundles of circles. Namely, let the circle of the new bundle passing through 0 in direction x be $S = \lambda S_1 + (1-\lambda)S_2$, where S_1 and S_2 are circles from the old bundles going from 0 in direction x. It turns out that if two rectifiable bundles have the same "orientation" then their barycentric combinations are also rectifiable.

REMARK. We used Theorem 5.1 to classify all Kähler metrics in an open subset of $\mathbb{C}^2$ whose geodesics are circles. All such metrics are locally equivalent (by means of a complex projective transformation and multiplication by a constant factor) to Fubini metrics (i.e., to the Fubini–Study metric on $\mathbb{C}\mathrm{P}^2$ restricted to an affine chart, to the complex hyperbolic metric in the unit ball model, or to the Euclidean metric). A proof of this statement will appear in a separate paper.

OPEN QUESTION. How many complete rectifiable simple bundles of circles are there? We saw that in $\mathbb{R}^n$ the space of all complete rectifiable bundles of circles passing through 0 is finite-dimensional. What is its dimension (as a function of n)? Is there an explicit geometric description of such bundles in dimensions > 4?

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