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Metric Definition of μ-Homeomorphisms

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Dedicated to Fred and Lois Gehring

1. Introduction

The analytic definition of quasiconformality declares that a homeomorphism f between domains Ω and Ω' in $\mathbf{R}^n$, $n \geq 2$, is quasiconformal if $f \in W^{1,n}_{\text{loc}}(\Omega, \Omega')$ and there exists a constant K such that

$$|Df(x)|^n \leq KJ_f(x) \ \text{ a.e. in } \Omega.$$

Because the Jacobian of any homeomorphism $f \in W^{1,1}_{\text{loc}}(\Omega, \Omega')$ is locally integrable, the regularity assumption on f in this definition can naturally be relaxed to $f \in W^{1,1}_{\text{loc}}(\Omega, \Omega')$. There has been considerable interest recently in so-called μ-homeomorphisms that form a natural generalization of the concept of a quasiconformal mapping in dimension 2. To be more precise, we consider homeomorphisms $f \in W^{1,1}_{\text{loc}}(\Omega, \Omega')$ such that

$$|Df(x)|^2 \leq K(x)J_f(x) \ \text{ a.e. in } \Omega \tag{1}$$

with $K(x) \geq 1$ and $\exp(\lambda K) \in L^1_{\text{loc}}(\Omega)$ for some $\lambda > 0$. A class of mappings equivalent to this was introduced by David in [2] and further studied in [17; 18]. David considered the Beltrami equation

$$\bar{\partial} f(z) = \mu(z)\partial f(z)$$

and essentially showed that a homeomorphic solution $f \in W^{1,1}_{\text{loc}}(\Omega, \Omega')$ exists (in the planar case) when $|\mu(z)| \leq 1$ almost everywhere and

$$\exp\left(C\frac{1+|\mu(z)|}{1-|\mu(z)|}\right) \in L^1_{\text{loc}}(\Omega)$$

for some $C > 0$; for this generality see [18]. These mappings in fact belong to $\bigcap_{p<2} W^{1,p}_{\text{loc}}(\Omega, \Omega')$; they are differentiable a.e. and preserve the null sets for the 2-dimensional Lebesgue measure. These conclusions hold with 2 replaced by n in any dimension for mappings with an exponentially integrable distortion in the sense of (1); see [13; 14].

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Quasiconformal mappings can alternatively be defined using metric quantities: Let Ω and Ω' be domains in $\mathbf{R}^n$ and let $f: \Omega \to \Omega'$ be a homeomorphism. Recall that f is then either sense-preserving or sense-reversing; throughout this paper, we will assume that all the homeomorphisms we deal with are sense-preserving. Then the two distortion functions of f that are of interest to us at a point $x \in \Omega$ are

$$H_f(x) = \limsup_{r \to 0} H_f(x, r) \tag{2}$$

and

$$h_f(x) = \liminf_{r \to 0} H_f(x, r), \tag{3}$$

where

$$H_f(x, r) = \frac{L_f(x, r)}{l_f(x, r)}$$

and

$$L_f(x, r) := \sup\{|f(x) - f(y)| : |x - y| \le r\},$$
$$l_f(x, r) := \inf\{|f(x) - f(y)| : |x - y| \ge r\}.$$

By $|x - y|$ we denote the Euclidean distance between x and y. Now f is quasiconformal if and only if the distortion H_f is uniformly bounded—that is, iff

$$H_f(x) \le H < \infty \quad \text{for all } x \in \Omega. \tag{4}$$

According to a result by Gehring [3, Thm. 8], the uniform boundedness of H_f can be relaxed to the requirement that $H_f(x) < \infty$ outside a set E of σ-finite $(n-1)$-dimensional measure and $H_f(x) \le H$ a.e. with respect to the Lebesgue measure. It has recently been observed that, first of all, H_f in (4) can be replaced with h_f. For this result, which quickly found applications in complex dynamics, see [7]. Second, we established in [11] a version of the result of Gehring's by showing that it suffices to assume that $h_f(x) \le H$ outside a set of σ-finite $(n-1)$-measure. This result was partially motivated by the need for tools of this type in complex dynamics (see [4]).

In the case of μ-homeomorphisms—or, more generally, homeomorphisms $f \in W^{1,1}_{\text{loc}}(\Omega, \Omega')$ that satisfy (1) with some suitably well-integrable K—there is no real hope of obtaining a metric definition that would characterize the class of mappings in question. Indeed, the size condition on the exceptional set in the metric definition cannot be relaxed even in the quasiconformal setting; moreover, under integrability conditions on K, we know that H_f can well be infinite in a set of dimension larger than $n-1$. Thus the best one can hope for is a sufficient metric condition. Again, the quest for such a condition is partially motivated by complex dynamics; μ-homeomorphisms appear naturally in conjugation problems (see [5; 6]). According to a result of Kallunki and Martio [12], in the planar case, a homeomorphism $f: \Omega \to \Omega'$ for which $H_f \in L^p_{\text{loc}}(\Omega)$ for some $p > 2$ and $H_f(x) < \infty$ outside a set of σ-finite length indeed belongs to $W^{1,1}_{\text{loc}}(\Omega, \Omega')$ and is differentiable a.e. See also [18] for a related result. For simplicity and for the relevance to complex dynamics, here and in the sequel we mostly concentrate on the planar case. In this paper we establish new results in terms of h_f. They rely on our first theorem, which gives control on the distortion of shapes by means of integrals of h_f.

THEOREM 1.1. *Let f be a homeomorphism between domains $\Omega, \Omega' \subset \mathbf{R}^2$ such that $h_f(x) < \infty$ outside a set E of σ-finite length and $h_f \in L^2_{\mathrm{loc}}(\Omega)$. Then*

$$L_f(x,r) \le l_f(x,r) \exp\left(C \fint_{B(x,2r)} h_f^2(y)\, dy \right) \tag{5}$$

for each $x \in \Omega$ and every $r > 0$ such that $B(x, 2r) \subset \Omega$. The constant C is an absolute constant. In particular, f is differentiable almost everywhere.

By the σ-finite length of a set we mean that the set has a countable cover by sets of finite Hausdorff 1-measure.

As an immediate consequence of this theorem and its higher-dimensional analog (given in Section 4), we obtain the following corollary, which gives a full extension of the result of Gehring's discussed previously; see Theorem 1.3. It is an improvement on our main result in [11].

COROLLARY 1.2. *Let $\Omega, \Omega' \subset \mathbf{R}^n$ be domains and suppose that $f : \Omega \to \Omega'$ is a homeomorphism. Suppose that there exist a set E of σ-finite $(n-1)$-measure and a constant H such that $h_f(x) < \infty$ outside E in Ω and*

$$h_f(x) \le H$$

almost everywhere in Ω. Then f is quasiconformal.

We close this introduction with a regularity result that gives a sufficient metric condition for a mapping to be a μ-homeomorphism.

THEOREM 1.3. *Let f be a homeomorphism between domains $\Omega, \Omega' \subset \mathbf{R}^2$ such that $h_f(x) < \infty$ outside a set E of σ-finite length. There is an absolute constant C' such that*

$$\exp(C' h_f^2) \in L^1_{\mathrm{loc}}(\Omega)$$

implies that $f \in W^{1,2}_{\mathrm{loc}}(\Omega, \Omega')$ and that (1) *holds with* $\exp(C' K^2) \in L^1_{\mathrm{loc}}(\Omega)$.

Notice that we obtain a stronger conclusion than simply the exponential integrability of the distortion and also that the asserted regularity of the mapping is stronger than one would expect. The regularity will be deduced from [8] and [9]. It would be interesting to know if the exponential integrability of h_f could already guarantee that the mapping belongs to $W^{1,1}_{\mathrm{loc}}(\Omega, \Omega')$.

The paper is organized as follows. In Section 2 we prove Theorem 1.1, and Section 3 is devoted to the proof of Theorem 1.3. The last section contains the formulation and the outline of the proof of Theorem 1.1 in $\mathbf{R}^n$.

2. The Local Quasisymmetry Condition

In this section we prove Theorem 1.1.

Proof of Theorem 1.1. If inequality (5) holds, then

$$H_f(x) = \limsup_{r \to 0} \frac{L_f(x,r)}{l_f(x,r)} < \infty \quad \text{for a.e. } x \in \Omega.$$

This guarantees the differentiability of f almost everywhere, owing to the Rademacher–Stepanov theorem (see e.g. [12]).

The proof of inequality (5) is somewhat technical. The argument is an improvement on the techniques in [7] and [11]; for the convenience of the reader we will repeat even the part of the original reasoning from [7] that need not be altered.

First fix $x_0 \in \Omega$ and $r > 0$ with $B(x_0, 2r) \subset \Omega$. We can assume that

$$L_f(x_0, r) > 3l_f(x_0, r).$$

Let $1 \le p < 2$ and $\varepsilon > 0$. Define

$$A = \bar{B}(f(x_0), L) \setminus B(f(x_0), l),$$

where $L = L_f(x_0, r)$ and $l = l_f(x_0, r)$. For each $k = 0, 1, 2, \dots,$ write

$$A_k = \{y \in f^{-1}(A) \cap B(x_0, 2r) : 2^k \le h_f(y) < 2^{k+1}\}.$$

The set A_k is a Borel set, $f^{-1}(A) \cap B(x_0, 2r) \setminus E = \bigcup_k A_k$, and for every k there exist open U_k such that $A_k \subset U_k$ and

$$|U_k| \le |A_k| + \frac{\varepsilon}{(2^{2p/(2-p)})^k}.$$

Here $|A|$ denotes the Lebesgue measure of a set A. Fix k. Now, for every $y \in A_k$, there is a $r_y > 0$ such that

(i) $0 < r_y < \frac{1}{10} \min\{d(f^{-1}(\bar{B}(f(x_0), l)), f^{-1}(\mathbf{R}^2 \setminus B(f(x_0), L))),$
$d(y, \partial B(x_0, 2r))\},$

(ii) $\operatorname{diam}(fB_y) < 2^{-j_0-3}L$,

(iii) $H_f(y, r_y) < 2^{k+1}$, and

(iv) $B_y \subset U_k$.

Here $B_y = B(y, r_y)$ and j_0 is the least positive integer with $2^{-j_0}L < l$.

We have obtained a family of balls B_y that satisfy conditions (i) and (ii) and such that, if $y \in A_k$, then B_y satisfies condition (iii) for k. By the Besicovitch covering theorem we may find balls $\bar{B}_1, \bar{B}_2, \dots$ from balls $\bar{B}(y, r_y)$, so that

$$f^{-1}(A) \cap B(x_0, 2r) \setminus E \subset \bigcup_j \bar{B}_j \subset B(x_0, 2r)$$

and $\sum_j \chi_{\bar{B}_j}(x) \le C(2)$ for every $x \in \mathbf{R}^2$. Here and in what follows, notation like $C(2)$ indicates that this constant will depend on the dimension when the argument is extended to cover the higher-dimensional setting. For these balls, we know that

$$|f\bar{B}_j| \le C(2) \operatorname{diam}(f\bar{B}_j)^2$$

and, when $y_j \in A_k$ (here y_j is the center of $\bar{B}_j$),

$$|f\bar{B}_j| \ge \frac{C(2) \operatorname{diam}(f\bar{B}_j)^2}{2^2(2^{k+1})^2}.$$

Let us define

$$\rho(x) = \left(\log \frac{L}{l}\right)^{-1} \sum_j \frac{\operatorname{diam}(f\bar{B}_j)}{d(fB_j, f(x_0))} \frac{1}{\operatorname{diam}(B_j)} \chi_{2B_j}(x).$$

The function ρ is measurable because it is a countable sum of simple functions.

The next step is to estimate the L^p-norm, $1 \le p < 2$, of ρ. In the planar case we can simply take $p = 1$, but in higher dimensions an exponent $n - 1 < p < n$ will be needed. We thus write this proof for general p so that the extension to higher dimensions becomes transparent (cf. Section 4).

By a general estimate on L^p-norms of weighted sums of characteristic functions, the L^p-norms ($1 \le p < 2$) of ρ are comparable to the corresponding norms of the function where the characteristic functions χ_{2B_j} are replaced with χ_{B_j} (cf. [1]). Thus, knowing that $\sum \chi_{B_j} \le C(2)$, we arrive at the estimate

$$\int_{\mathbf{R}^n} \rho(x)^p\,dx \le C(2,p)\left(\log\frac{L}{l}\right)^{-p}\sum_j\left(\frac{\operatorname{diam}(f\bar{B}_j)}{d(fB_j, f(x_0))}\frac{1}{\operatorname{diam}(B_j)}\right)^p |B_j|.$$

Using Hölder's inequality and the fact that $\operatorname{diam}(fB_j)^2 \le C(2)|fB_j|(2^{k+1})^2$ when $y_j \in A_k$, we thus obtain

$$\int_{\mathbf{R}^n} \rho(x)^p\,dx \le C(2,p)\left(\log\frac{L}{l}\right)^{-p}\left(\sum_j \frac{|fB_j|}{d(fB_j, f(x_0))^2}\right)^{p/2} \times\left(\sum_k\sum_{y_j\in A_k}(2^k)^{2p/(2-p)}\operatorname{diam}(B_j)^2\right)^{(2-p)/2}.$$

Regrouping the balls depending on their distance from $f(x_0)$ and then using the estimate $\sum \chi_{B_i} \le C(2)$, it is easy to see that

$$\sum_j \frac{|fB_j|}{d(fB_j, f(x_0))^2} \le C(2)\log\frac{L}{l}.$$

The approximation of the second term is a little bit trickier. First observe that $\operatorname{diam}(B_j)^2 \le C(2)|B_j| = C(2)(|B_j \cap A_k| + |B_j \setminus A_k|)$. The double sum over the $|B_j \cap A_k|$-terms can be estimated by the integral of $h_f^{2p/(2-p)}$ and the double sum over the $|B_j \setminus A_k|$-terms turns out to be no more than a constant times ε, because $\bigcup_{y_j\in A_k} B_j \subset U_k$ and $|U_k| \le |A_k| + \varepsilon/(2^{2p/(2-p)})^k$. Therefore,

$$\sum_k\sum_{y_j\in A_k}(2^k)^{2p/(2-p)}\operatorname{diam}(B_j)^2 \le C(2)\left(\int_{B(x_0,2r)} h_f(x)^{2p/(2-p)}\,dx + \varepsilon\right).$$

Because ε was arbitrary, we conclude that

$$\int_{\mathbf{R}^n} \rho(x)^p\,dx \le C(2,p)\left(\log\frac{L}{l}\right)^{-p/2}\left(\int_{B(x_0,2r)} h_f(x)^{2p/(2-p)}\,dx\right)^{(2-p)/2}.$$

In the following we will actually choose $p = 1$.

Our next goal is to find a lower bound on the integral of ρ. For this, define $F_1 = f^{-1}(\bar{B}(f(x_0), l))$ and $F_2 = f^{-1}(\mathbf{R}^2 \setminus B(f(x_0), L)) \cap B(x_0, 2r)$. Take a point $y \in F_1 \cap S(x_0, r)$. By applying an auxiliary rotation, we may assume that $y = x_0 + (r, 0)$. Consider the line segments J_t parallel to the imaginary axis through the points $x_0 + (t, 0)$, $0 \le t \le r$, that join two points of $S(x_0, 2r)$. Assume first that $\int_{J_t} \rho \geq \frac{1}{2000}$ for each t in a set $A \subset [0, r]$ with $m(A) > r/2$. Here m refers to the Lebesgue measure on the line. Then it follows from the Fubini theorem that

$$\int_{B(x_0, 2r)} \rho \geq \frac{r}{4000}.$$

Suppose then that $\int_{J_t} \rho \leq \frac{1}{2000}$ for each t in a set A with $m(A) > r/2$. Now $m(\{0 \leq t \leq r : E \cap J_t \text{ is uncountable}\}) = 0$, and $m(\{r \leq s \leq 2r : E \cap S(x_0, s)$ is uncountable$\}) = 0$ because E has σ-finite length (cf. [19, 30.16]).

Take a radius $r < s < 2r$ and a number $t \in A$ such that both $E \cap S(x_0, s)$ and $E \cap J_t$ are countable. Pick the balls $V_1, V_2, \dots$ from the balls $\bar{B}_1, \bar{B}_2, \dots$ for which $V_i \cap S(x_0, s) \neq \emptyset$ or $V_i \cap J_t \neq \emptyset$. Write $\gamma = J_t \cup S(x_0, s)$. Then the connected set γ intersects both F_1 and F_2 and thus $f(\gamma)$ is a connected set that intersects both $\bar{B}(f(x_0), l)$ and $\mathbf{R}^2 \setminus B(f(x_0), L)$. Moreover, the sets $f(V_i)$ cover $f(\gamma)$ up to a countable set.

Now

$$\int_\gamma \rho \geq \frac{1}{2}\left(\log \frac{L}{l}\right)^{-1} \sum_i \frac{\operatorname{diam}(fV_i)}{d(fV_i, f(x_0))}.$$

If $f(V_i)$ touches the annulus $A_j = B(f(x_0), 2^{j+1}l) \setminus B(f(x_0), 2^j l)$, $j = 0, \dots, j_0 - 1$, then $d(fV_i, f(x_0)) \leq 2^{j+1}l$, and because the connected sets $f(V_i)$ cover $f(\gamma)$ up to a countable set,

$$\int_\gamma \rho \geq \left(\log \frac{L}{l}\right)^{-1} \sum_{j=0}^{j_0-1} \frac{1}{4} \geq \frac{1}{1000}.$$

Because $t \in A$, we have

$$\int_{J_t} \rho \leq \frac{1}{2000}$$

and it follows that

$$\int_{S(x_0, s)} \rho \geq \frac{1}{2000}.$$

From this estimate and using the Fubini theorem, we obtain that

$$\int_{B(x_0, 2r)} \rho(x) \geq \int_r^{2r} \left(\int_{S(y, s)} \rho \right) ds \geq \frac{1}{2000} r.$$

Thus, in both cases we have the estimate

$$\int_{B(x_0, 2r)} \rho(x)\, dx \geq Cr.$$

Combining now the lower bound with the upper bound, we finally have (with the choice $p = 1$) that

$$C(2)r \leq C(2)\left(\log \frac{L}{l}\right)^{-1/2} \left(\int_{B(x_0, 2r)} h_f(x)^2\, dx \right)^{1/2}.$$

This gives the claim. $\square$

3. A Metric Condition for μ-Homeomorphisms

In this section we will prove Theorem 1.3. For the proof we need two lemmas, the first of which is a version (tailored for our setting) of the standard absolute continuity result for quasiconformal mappings.

LEMMA 3.1. *Let f be a homeomorphism between domains $\Omega, \Omega' \subset \mathbf{R}^2$ such that*

$$L_f(x,r) \le l_f(x,r)\varphi(x) \quad \textit{whenever} \;\; B(x,2r) \subset \Omega, \tag{6}$$

where $\varphi \in L^2_{\mathrm{loc}}(\Omega)$. Then f is absolutely continuous on almost all lines parallel to the coordinate axes.

Proof. Let $Q \subset\subset \Omega$ be an open 2-interval and suppose that $Q = I \times J$, where $I =]a,b[\in \mathbf{R}^1$ and $J =]c,d[\subset \mathbf{R}^1$. For each Borel set $E \subset I$ we set $\eta(E) = |f(E \times J)|$. Then η is a finite Borel measure in I and hence, by the Radon–Nikodym theorem, it has a finite derivative $\eta'(y)$ for almost every $y \in I$. Choose $y \in I$ such that (i) $\eta'(y)$ exists and (ii) $\varphi \in L^2(\{y\} \times J)$. The latter is possible because of the Fubini theorem. We will prove that f is absolutely continuous on the segment $\{y\} \times J$, which will prove the theorem.

Define $J' = \{y\} \times J$. Now let $F \subset J'$ be compact. We wish to estimate $\mathcal{H}^1(fF)$. Choose $0 < \varepsilon < \operatorname{dist}(F, \partial J)/4$ and $t > 0$. Let $0 < \delta_1 \le 1$ be the number given by [19, Lemma 31.1] for the set F. We will soon state what this lemma gives us. Choose δ_2 such that, if $0 < r < \delta_2$, then $|f(x) - f(z)| < t$ whenever $x, z \in Q$ and $|x - z| \le 2r$. Denote $\delta = \min\{\delta_1, \delta_2, \varepsilon\}$. Choose $0 < r < \delta$. Now [19, Lemma 31.1] gives a covering $\triangle_1, \dots, \triangle_p$ of F with intervals in J' such that (i) $\operatorname{diam}(\triangle_i) = r$ for $1 \le i \le p$, (ii) each point of J' belongs to at most two different $\triangle_i$, and (iii) each $\bar{\triangle}_i$ is contained in the ε-neighborhood of F in J'.

Now, because $\varphi \in L^2(J')$, there are points $x_i \in \bar{\triangle}_i$ such that

$$\varphi(x_i) \le 2 \inf_{x \in \bar{\triangle}_i} \varphi(x) < \infty. \tag{7}$$

Set $A_i = B^2(x_i, r)$. Now $\triangle_i \subset A_i$ and $A_i \subset \bar{B}^1(y,r) \times J$. Because $\operatorname{diam}(fA_i) < t$ we have that $\mathcal{H}^1_t(fF) \le \sum \operatorname{diam}(fA_i) \le 2\sum L_i$, where $L_i = L_f(x_i, r)$. Denote similarly $l_i = l_f(x_i, r)$. Using (6), we obtain the estimate

$$\begin{aligned} \mathcal{H}^1_t(fF)^2 &\le 2^2\Big(\sum_i L_i\Big)^2 \le 2^2\Big(\sum_i l_i\varphi(x_i)\Big)^2 \\ &= \frac{2^2}{r}\Big(x\sum_i l_i r^{1/2}\varphi(x_i)\Big)^2; \end{aligned}$$

notice that $B(x_i, 2r) \subset \Omega$. By Hölder's inequality we further conclude that

$$\mathcal{H}^1_t(fF)^2 \le \frac{2^2}{\Omega_2 r}\sum_i |f(A_i)| \sum_i r\varphi^2(x_i),$$

where $\Omega_2 = |B^2(0,1)|$. Because no point belongs to more than two of the sets A_i, it follows that $\sum |fA_i| \le 10\eta(\overline{B^1}(y,r))$. Since the points x_i satisfy (7), we arrive at

$$\mathcal{H}^1_t(fF)^2 \le \frac{10 \cdot 2^2}{\Omega_2} \frac{\eta(\overline{B^1}(y,r))}{r} \left(\int_{F+\varepsilon} \varphi^2(x)\,dx \right).$$

Here $F+\varepsilon$ is the ε-neighborhood of F in J'. Letting first $r \to 0$ and then $\delta_1 \to 0$ and finally $\varepsilon \to 0$ and $t \to 0$, we deduce that

$$\mathcal{H}^1(fF)^2 \le C(2)\eta'(y)\left(\int_F \varphi^2(x)\,dx \right).$$

The absolute continuity of f on J' follows from this estimate; see [19, Lemma 30.9]. □

LEMMA 3.2. *Let u be a nonnegative function such that*

$$\exp(C'u) \in L^1_{\mathrm{loc}}(\Omega), \tag{8}$$

and let $p > 1$. Then, for each compact set $F \subset \Omega$,

$$\exp(\varepsilon C' M(\chi_F u)) \in L^p_{\mathrm{loc}}(\Omega), \tag{9}$$

where ε depends only on p and where M is the usual Hardy–Littlewood maximal operator.

The proof of this lemma is a simple computation based on (a) the fact that the Hardy–Littlewood maximal operator is bounded from L^q to L^q when $1 < q < \infty$ and (b) the expansion of the exponential function as a power series. Indeed,

$$\int_{\mathbf{R}^2} (Mv)^q\,dx \le C \int_{\mathbf{R}^2} |v|^q\,dx$$

whenever $q \ge 2$, where $C = c2^q q/(q-1)$ for c an absolute constant; see [15]. Moreover,

$$\int_E Mv\,dx \le \left(\int_E (Mv)^2\,dx \right)^{1/2} |E|^{1/2}$$

by Hölder's inequality. Thus the claim follows from the power series expansion

$$\exp(p\varepsilon C' M(\chi_F u)) = \sum_k \frac{(p\varepsilon C' M(\chi_F u))^k}{k!}.$$

Proof of Theorem 1.3. From Theorem 1.1 we have that f is differentiable almost everywhere. Next we would like to show that f is absolutely continuous on lines. For this it is enough to show that f is absolutely continuous on lines in every square $Q \subset\subset \Omega$. Fix such a square Q. Now, if $r < \frac{1}{2}\operatorname{dist}(Q, \partial\Omega)$ and $x \in Q$ then inequality (5) gives that

$$L_f(x,r) \le l_f(x,r)\exp(CM(\chi_Q h_f^2)(x)).$$

Lemma 3.2 shows that $\exp(CM(\chi_Q h_f^2)(x)) \in L^2_{\text{loc}}(\Omega)$ when the constant C' is chosen correctly. By Lemma 3.1, we see that f is absolutely continuous on almost all lines parallel to the coordinate axes.

The next goal in our proof is to check that $f \in W^{1,1}_{\text{loc}}(\Omega, \Omega')$. Because f is absolutely continuous on almost all lines parallel to the coordinate axes, it suffices to show that $|Df| \in L^1_{\text{loc}}(\Omega)$. Here Df is the matrix obtained from the partial derivatives of the coordinate mappings, and it exists at a.e. point of Ω. Now, for a.e. $x \in \Omega$, it follows that f is differentiable, $Df(x)$ exists, and $h_f(x) < \infty$. Fix such a point x. Because f was assumed to be sense-preserving, the Jacobian determinant $J_f(x)$ must be nonnegative. If $J_f(x) > 0$, then clearly

$$|Df(x)|^2 \le h_f(x) J_f(x). \tag{10}$$

On the other hand, if $J_f(x) = 0$, then elementary linear algebra shows that $\min_{|e|=1}|Df(x)e| = 0$. From the assumption $h_f(x) < \infty$ it then follows that $Df(x)$ must be the zero matrix. Thus (10) holds for a.e. $x \in \Omega$. Because the Jacobian of each a.e. differentiable homeomorphism is locally integrable (cf. [16, p. 360]), we thus conclude from (10) and our integrability assumption on h_f that $|Df|$ is locally integrable—in fact, locally p-integrable for any $p < 2$.

We are left to show that $f \in W^{1,2}_{\text{loc}}(\Omega, \Omega')$. By the previous paragraph, $f \in W^{1,1}_{\text{loc}}(\Omega, \Omega')$, $J_f \in L^1_{\text{loc}}(\Omega)$, and

$$|Df(x)|^2 \le K(x) J_f(x)$$

a.e., where $K = h_f$ satisfies $\exp(C'K^2) \in L^1_{\text{loc}}(\Omega)$. Then $\exp(\lambda K) \in L^1_{\text{loc}}(\Omega)$ for every $\lambda > 0$ and so [8] (or the main theorem in [9]) gives us the desired regularity. □

4. The Higher-Dimensional Setting

The higher-dimensional setting is somewhat more technical. Corollary 1.2 is an immediate consequence of the following result.

THEOREM 4.1. *Let f be a homeomorphism between domains $\Omega, \Omega' \subset \mathbf{R}^n$ such that $h_f(x) < \infty$ outside a set E of σ-finite $(n-1)$-measure and $h_f \in L^{p^*}_{\text{loc}}(\Omega)$, with some $n-1 < p < n$. Here $p^* = pn/(n-p)$. Then*

$$L_f(x,r) \le l_f(x,r)\exp\left(\left(C(n,p) \fint_{B(x,2r)} h_f^{p*}(y)\,dy\right)^{\frac{1}{p^*}\frac{n}{n-1}}\right) \tag{11}$$

for every $x \in \Omega$ and each $r > 0$ such that $B(x,2r) \subset \Omega$. In particular, f is differentiable almost everywhere.

Proof. The first part of the proof is the same as the beginning of the proof of Theorem 1.1. We simply replace the number 2 there with n. We then end up with the estimate

$$\int_{\mathbf{R}^n} \rho^p(y)\,dy \le C(n,p)\left(\log\frac{L}{l}\right)^{(p/n)(1-n)}\left(\int_{B(x_0,2r)} h_f(x)^{np/(n-p)}\,dx\right)^{(n-p)/n}.$$

The claim follows from this inequality, provided we can find a suitable lower bound on the p-integral of ρ. Such an estimate is obtained by a simple modification of the reasoning in [11] (Lemma 2.1 there holds with n replaced by p when $p > n-1$, and in Lemma 2.3 there the set E_b is not needed). Explicitly, we have the lower bound

$$\int_{\mathbf{R}^n} \rho^p(y)\,dy \ge C(n,p)r^{n-p};$$

see [10, Lemma 2.6]. By combining these two facts, the claim follows. □

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