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Constructions of Nontautological Classes on Moduli Spaces of Curves

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0. Introduction

The *tautological rings* $R^*(\bar{M}_{g,n})$ are natural subrings of the Chow rings of the Deligne–Mumford moduli spaces of pointed curves:

$$R^*(\bar{M}_{g,n}) \subset A^*(\bar{M}_{g,n}) \tag{1}$$

(the Chow rings are taken with $\mathbb{Q}$-coefficients). The system of tautological subrings (1) is defined to be the set of smallest $\mathbb{Q}$-subalgebras satisfying the following three properties [FP].

(i) $R^*(\bar{M}_{g,n})$ contains the cotangent line classes

$$\psi_1, \ldots, \psi_n \in A^1(\bar{M}_{g,n}).$$

(ii) The system is closed under push-forward via all maps forgetting markings:

$$\pi_*\colon R^*(\bar{M}_{g,n}) \to R^*(\bar{M}_{g,n-1}).$$

(iii) The system is closed under push-forward via all gluing maps:

$$\pi_*\colon R^*(\bar{M}_{g_1,n_1\cup\{*\}}) \otimes_{\mathbb{Q}} R^*(\bar{M}_{g_2,n_2\cup\{\bullet\}}) \to R^*(\bar{M}_{g_1+g_2,n_1+n_2}),$$
$$\pi_*\colon R^*(\bar{M}_{g_1,n_1\cup\{*,\bullet\}}) \to R^*(\bar{M}_{g_1+1,n_1}).$$

The tautological rings possess remarkable algebraic and combinatorial structures with basic connections to topological gravity. A discussion of these properties together with a conjectural framework for the study of $R^*(\bar{M}_{g,n})$ can be found in [F; FP].

In genus 0, the equality

$$R^*(\bar{M}_{0,n}) = A^*(\bar{M}_{0,n})$$

for $n \geq 3$ is well known from Keel's study [K].

Denote the image of $R^*(\bar{M}_{g,n})$ under the canonical map to the ring of *even* cohomology classes by

$$RH^*(\bar{M}_{g,n}) \subset H^{2*}(\bar{M}_{g,n}).$$

In genus 1, Getzler has claimed the isomorphisms

$$R^*(\bar{M}_{1,n}) \cong RH^*(\bar{M}_{1,n})$$

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and

$$RH^*(\bar{M}_{1,n}) = H^{2*}(\bar{M}_{1,n})$$

for $n \geq 1$; see [G1].

For $g > 1$, complete results are known only in codimension 1. The equality

$$R^1(\bar{M}_{g,n}) = A^*(\bar{M}_{g,n})$$

for $2g - 2 + n > 0$ is a consequence of Harer's cohomological calculations [Ha].

It is natural to ask whether all algebraic cycle classes on $\bar{M}_{g,n}$ are tautological. The existence of nontautological cycles defined over $\mathbb{C}$ may be deduced from the odd cohomology of $\bar{M}_{1,11}$. There are two arguments which may be used.

(i) By a theorem of Jannsen, since the map to cohomology

$$A^*(\bar{M}_{1,11}) \to H^*(\bar{M}_{1,11})$$

is not surjective, the map is not injective. We may then deduce the existence of a nontautological Chow class in $A^*(\bar{M}_{1,11})$ from Getzler's claims (see [B]).

(ii) More precisely, the existence of a holomorphic 11-form and a theorem of Srinivas together imply that $A_0(\bar{M}_{1,11})$ is an infinite-dimensional vector space, whereas $R_0(\bar{M}_{1,11}) \cong \mathbb{Q}$; see [GrV].

These arguments do not produce an explicit algebraic cycle that is not tautological. Several further questions are also left open. Are there nontautological cycles defined over $\mathbb{Q}$? Are there algebraic cycles with cohomological image not contained in $RH^*(\bar{M}_{g,n})$? Are there nontautological classes on the noncompact spaces $M_{g,n}$?

We answer all these questions in the affirmative by explicit construction of integrally defined algebraic cycles. Our basic criterion for detecting nontautological cycles is the following proposition.

PROPOSITION 1. *Let $\iota\colon \bar{M}_{g_1,n_1\cup\{*\}} \times \bar{M}_{g_2,n_2\cup\{\cdot\}} \to \bar{M}_{g,n_1+n_2}$ be the gluing map to a boundary divisor. If $\gamma \in RH^*(\bar{M}_{g_1+g_2,n_1+n_2})$, then $\iota^*(\gamma)$ has a tautological Künneth decomposition*:

$$\iota^*(\gamma) \in RH^*(\bar{M}_{g_1,n_1\cup\{*\}}) \otimes RH^*(\bar{M}_{g_2,n_2\cup\{\cdot\}}).$$

This result is well known to experts, but we know of no adequate reference and so give a proof in the Appendix.

Our strategy for finding nontautological classes combines Proposition 1 with the existence of odd cohomology on the moduli spaces of curves. We find loci in moduli space that restrict to diagonal loci of symmetric boundary divisors. By the existence of odd cohomology in certain cases, the Künneth decomposition of the diagonal is not tautological.

Let h be an *odd* integer and set $g = 2h$. Let $Y \subset \bar{M}_g$ denote the closure of the set of nonsingular curves of genus g that admit a degree-2 map to a nonsingular curve of genus h. Intersecting Y with the boundary map from $\bar{M}_{h,1} \times \bar{M}_{h,1}$ yields the diagonal. Pikaart has proven, for sufficiently large h, that $\bar{M}_{h,1}$ has odd cohomology [Pi]. Hence, we can conclude that Y is not a tautological class, even in homology.

THEOREM 1. *For all sufficiently large odd h,*

$$[Y] \notin RH^*(\bar{M}_{2h}).$$

Our other examples are loci in the moduli space of pointed genus-2 curves. We will use the odd cohomology of $\bar{M}_{1,11}$ to find nontautological Künneth decompositions.

Let σ in $\mathbb{S}_{20}$ be a product of ten disjoint 2-cycles,

$$\sigma = (1, 11)(2, 12) \cdots (10, 20),$$

inducing an involution on $\bar{M}_{2,20}$. Let Z denote the component of the fixed locus of the involution corresponding generically to a 20-pointed, nonsingular, bi-elliptic curve of genus 2 with the ten pairs of conjugate markings. Then Z is of codimension 11 in $\bar{M}_{2,20}$, and the intersection of Z with the boundary map

$$\iota\colon \bar{M}_{1,11} \times \bar{M}_{1,11} \to \bar{M}_{2,20} \tag{2}$$

yields the diagonal.

THEOREM 2. $[Z] \notin RH^*(\bar{M}_{2,20})$.

Although the methods used to prove Theorems 1 and 2 depend crucially on the structure of the boundary of the moduli space, in Section 3 we use Getzler's results on the cohomology of $\bar{M}_{1,n}$ to show that the class $[Z]$ is nontautological even on the interior.

THEOREM 3. $[Z] \notin R^*(M_{2,20})$.

Finally, although the diagonal loci were used in our deductions of the preceding results, we could not conclude that the diagonals were themselves nontautological. We show that a diagonal locus is nontautological in at least one case. Let ι denote the boundary inclusion

$$\iota\colon \bar{M}_{1,12} \times \bar{M}_{1,12} \to \bar{M}_{2,22},$$

and let Δ denote the class of the diagonal in $A^*(\bar{M}_{1,12} \times \bar{M}_{1,12})$.

THEOREM 4. *The push-forward $\iota_*[\Delta]$ is not a tautological class*:

$$\iota_*[\Delta] \notin RH^*(\bar{M}_{2,22}).$$

It seems likely the image of the diagonal by (2) in $\bar{M}_{2,20}$ is not tautological, but we do not have a proof.

To our knowledge there are still no (proven) examples of nontautological classes on M_g. Although the methods of our paper could perhaps be used to find such a class (in particular, the class of Theorem 1 may be nontautological when restricted to M_g), our techniques are unlikely to produce nontautological classes of low codimension. Because the tautological ring of M_g vanishes in codimension $g-1$ and higher, the question of nontautological classes on M_g of codimension less than $g-1$ is particularly interesting.

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1. Admissible Double Coverings

For the proofs of Theorems 1 and 2 we will require certain moduli spaces of double covers. Choose g and h with $g \geq 2h-1$. We let $M(g,h)$ denote the (open) space parameterizing *double* covers,

$$\pi : C_g \to C_h,$$

of curves of genus g and h, respectively, *together* with an ordering of the branch points of the morphism π. The space $M(g,h)$ is a finite étale cover of $M_{h,b}$, where $b = 2(g-2h+1)$ is the number of branch points of π. The map

$$\mu : M(g,h) \to M_{h,b}$$

is simply defined by

$$\mu([\pi]) = [C_h, p_1, \ldots, p_b],$$

where $p_1, \ldots, p_b$ are the ordered branch points.

There is a natural compactification by admissible double covers,

$$M(g,h) \subset \bar{M}(g,h),$$

over $\bar{M}_{h,b}$. An admissible double cover π of a stable curve is branched over the marked points and possibly the nodes. Over the nodes of the target, the map π is either étale or étale locally of the form

$$\pi : \operatorname{Spec}(\mathbb{C}[x,y]/(xy)) \to \operatorname{Spec}(\mathbb{C}[u,v]/(uv)), \quad u = x^2, \ v = y^2.$$

By construction, the space $\bar{M}(g,h)$ is equipped with maps to both $\bar{M}_{h,b}$ and $\bar{M}_g$. The latter map involves a stabilization process, since the source curve of an admissible covering need not be stable.

We will also require pointed moduli spaces of admissible covers, $\bar{M}_k(g,h)$. These pointed spaces are finite covers of $\bar{M}_{h,b+k}$, which parameterizes admissible double covers of a $(b+k)$-pointed nodal curve of genus h by a curve of genus g, with the ramification over the first b marked points and possibly the nodes of the target curve together with an ordering of the fibers of the last k marked points. The pointed spaces are equipped with natural morphisms to $\bar{M}_{h,b+k}$ and $\bar{M}_{g,2k}$. For the latter map, we adopt the ordering convention that the two points in the fiber over the $(b+i)$th marked point of the target curve have markings i and $k+i$ on the source.

Essentially, we require only one fact about the moduli spaces of admissible covers: $M(g,h) \subset \bar{M}(g,h)$ is dense (and similarly for the open subset $M_k(g,h) \subset \bar{M}_k(g,h)$). Over the complex numbers, the density is easily proven analytically:

one can locally smooth the double cover of a small neighborhood of the node and then glue the result together, with the restriction of the original cover away from the node. The local description of our double covers ensures that they can be smoothed locally. A treatment of the theory of admissible covers can be found in [HM].

2. Proof of Theorems 1 and 2

Let h be an odd positive integer, and let $g = 2h$. Consider the morphism

$$\phi\colon \bar{M}(g,h) \to \bar{M}_g.$$

The image cycle,

$$Y = \phi(\bar{M}(g,h)),$$

consists of those curves of genus g that are admissible double covers of a curve of genus h. Equivalently, Y is the closure of the set of nonsingular curves of genus g that admit a degree-2 map to a nonsingular curve of genus h. We want to apply Proposition 1 to conclude that $[Y]$ is not tautological. We will examine the pull-back of $[Y]$ under the gluing map

$$\iota\colon \bar{M}_{h,1} \times \bar{M}_{h,1} \to \bar{M}_g.$$

LEMMA 1. $\iota^*([Y]) = c[\Delta]$ *for some positive constant* c.

Proof. We first prove $\Delta \subset \iota^{-1}(Y)$. Let $[C, p] \in \bar{M}_{h,1}$. We will construct an admissible double cover with target C union a rational tail glued at p carrying the two branch markings. A double cover is given by two disjoint copies of C joined by a rational curve with a degree-2 mapping to the rational tail of the target branched over the two markings. Under stabilization, the domain is mapped to the diagonal point

$$\iota([C, p] \times [C, p]).$$

An easy count shows Δ to be an irreducible component of $\iota^{-1}(Y)$ of expected dimension.

To prove the lemma we need only show that $\Delta = \iota^{-1}(Y)$. Suppose there were another irreducible component I. Let

$$\pi : C_g \to C_h$$

be an admissible double cover corresponding to a general point of I. Then C_g may be expressed as a union of two curves of arithmetic genus h joined at a single node. The chosen node of C_g must map to a node of C_h. Since the space of admissible coverings is a finite cover of $\bar{M}_{h,2}$, the preimage of the locus of curves with two or more nodes is not a divisor. Hence, we conclude that C_h has *exactly* one node.

The node of C_h must be disconnecting because there are no reducible admissible double covers of an irreducible curve with branch points. We write $C_h = T_1 \cup T_2$. Since h is odd, we may assume (without loss of generality) that T_1 has genus greater than $h/2$.

Since C_h has one node, C_g must have either one or two nodes. Since any cover of T_1 by a curve of genus h must be unramified, C_h cannot have exactly one node.

The domain C_h must therefore have two nodes lying over the node of C_g. If the induced cover of T_1 were connected then neither node of C_g could be disconnecting. Hence, the cover of T_1 must be disconnected.

Therefore, each component of the cover of T_1 must map isomorphically to T_1. The cover of T_2 must be connected and of genus 0 in order for the assumed decomposition of C_g into curves of arithmetic genus h to exist. Therefore, we find that we are in the component Δ of $\iota^{-1}(Y)$. □

Pikaart [Pi] has shown that, for all sufficiently large values of h,

$$H^{33}(\bar{M}_{h,1}) \neq 0.$$

Hence, the diagonal in $\bar{M}_{h,1} \times \bar{M}_{h,1}$ does not have tautological Künneth decomposition. By Proposition 1, the proof of Theorem 1 is complete.

The argument for the nontautological cycle on $\bar{M}_{2,20}$ is similar. Let Z be the image of $\bar{M}_{10}(2,1)$ in $\bar{M}_{2,20}$. Consider the boundary stratum,

$$\iota\colon \bar{M}_{1,11} \times \bar{M}_{1,11} \to \bar{M}_{2,20},$$

obtained by (a) attaching at the last point on each marked curve and (b) numbering the markings of the glued curve in order, with the first ten markings from the first factor and the last ten from the second factor.

LEMMA 2. *$\iota^*([Z]) = c[\Delta]$ for some positive constant c.*

The proof of this lemma is essentially identical to the proof of Lemma 1. Theorem 2 is then a consequence of Proposition 1 and the existence of odd cohomology on $\bar{M}_{1,11}$.

3. Proof of Theorem 3

To deduce Theorem 3 from Theorem 2, we will need the following results announced by Getzler:

$$RH^*(\bar{M}_{1,n}) = H^{2*}(\bar{M}_{1,n}) \tag{3}$$

and, for all odd $k < 11$,

$$H^k(\bar{M}_{1,n}) = 0. \tag{4}$$

The statement (3) is equivalent to the generation of even cohomology by the classes of boundary strata for $\bar{M}_{1,n}$. Actually, we require the following consequences of Getzler's results.

LEMMA 3. *$\bar{M}_{1,n}$ exhibits the following three properties.*

(i) *Every algebraic cycle on $\bar{M}_{1,11} \times \bar{M}_{1,11}$ of complex codimension < 11 is homologous to a tautological class.*
(ii) *Every algebraic cycle on $\bar{M}_{1,m} \times \bar{M}_{1,n} \times \prod_i \bar{M}_{0,l_i}$ is homologous to a tautological class for $m < 11$.*
(iii) *Every algebraic cycle on $\bar{M}_{1,n} \times \prod_i \bar{M}_{0,l_i}$ is homologous to a tautological class.*

Proof. Let V be an algebraic cycle on $\bar{M}_{1,11} \times \bar{M}_{1,11}$ of complex codimension < 11. Consider the Künneth decomposition of $[V]$. There can be no odd terms by (4). Thus, by (3) we can write $[V]$ as a sum of products of tautological classes, proving (i). By (4) and Poincaré duality, all the cohomology of $\bar{M}_{1,m}$ is tautological when $m < 11$. By Keel's results, all the cohomology of $\bar{M}_{0,l_i}$ is tautological. Hence, in the Künneth decomposition of our cycle in parts (ii) and (iii), none of the odd cohomology of $\bar{M}_{1,n}$ can appear. □

Consider the class $[Z]$ on $\bar{M}_{2,20}$ constructed in Theorem 2. We claim that the image of $[Z]$ in $A^*(M_{2,20})$ is not tautological. The argument is by contradiction.

Suppose the image is tautological. There must exist a collection of cycles Z_i of codimension 11 in $\bar{M}_{2,20}$ that are supported on boundary strata for which $Z + \sum Z_i$ is tautological. Hence $\sum Z_i$ is not homologous to a tautological class when intersected with $\bar{M}_{1,11} \times \bar{M}_{1,11}$.

By Lemma 3(i), if any cycle Z_i is supported on the image stratum of $\bar{M}_{1,11} \times \bar{M}_{1,11}$ then Z_i is homologous to a tautological class (since the codimension of Z_i is less than 11 in the divisor). We discard all Z_i contained in the image of $\bar{M}_{1,11} \times \bar{M}_{1,11}$.

Let X be the union of boundary divisors supporting the remaining Z_i. The sum of the remaining Z_i is homologically nontautological when pushed into $\bar{M}_{2,20}$ and restricted to $\bar{M}_{1,11} \times \bar{M}_{1,11}$. However, it is clear that the push–pull will produce an algebraic cycle class supported on

$$X \cap \bar{M}_{1,11} \times \bar{M}_{1,11}. \tag{5}$$

Since X does not contain the image of $\bar{M}_{1,11} \times \bar{M}_{1,11}$, the locus (5) is contained in boundary strata that either have a genus-1 factor with fewer than 11 points or have fewer than two genus-1 factors. Parts (ii) and (iii) of Lemma 3 show that there are no homologically nontautological classes supported on these loci. This contradiction completes the proof of Theorem 3.

4. Proof of Theorem 4

4.1. Odd Cohomology of $\bar{M}_{1,n}$

We will require several properties of the odd cohomology of the moduli spaces $\bar{M}_{1,n}$ for the proof of Theorem 4. The first is a well-known specialization of (4).

PROPOSITION 2. *The odd cohomology groups of $\bar{M}_{1,n}$ vanish in case $1 \leq n < 10$.*

Observe that cusp forms of weight n may be used to construct cohomology classes in $H^{n-1,0}(\bar{M}_{1,n-1}, \mathbb{C})$. The discriminant form Δ, the unique cusp form of weight 12, yields a canonical nonzero element $s \in H^{11,0}(\bar{M}_{1,11}, \mathbb{C})$.

PROPOSITION 3. *The odd cohomology of $\bar{M}_{1,11}$ is concentrated in*

$$H^{11,0}(\bar{M}_{1,11}, \mathbb{C}) \cong \mathbb{C},$$

$$H^{0,11}(\bar{M}_{1,11}, \mathbb{C}) \cong \mathbb{C}.$$

Moreover, the $\mathbb{S}_{11}$-module in both cases is the alternating representation.

By the $\mathbb{S}_{11}$-module identification, the class s is *not* $\mathbb{S}_{11}$-invariant. Now let $t \in H^{0,11}(\bar{M}_{1,11}, \mathbb{C})$ denote the uniquely defined Poincaré dual class to s:

$$\int_{\bar{M}_{1,11}} s \cup t = 1.$$

Propositions 2 and 3 are both well known. Proofs can be found, for example, in [G2], where the $\mathbb{S}_n$-equivariant Hodge polynomials of $\bar{M}_{1,n}$ are calculated for all n. We will need a dimension calculation in the $n = 12$-pointed case [G2] as follows.

PROPOSITION 4. *The dimension of* $H^{11,0}(\bar{M}_{1,12}, \mathbb{C})$ *is* 11.

In fact, the odd cohomology of $\bar{M}_{1,12}$ is concentrated in $H^{11,0}$, $H^{0,11}$, $H^{12,1}$, and $H^{1,12}$ (all of which are 11-dimensional).

4.2. A Basis for $H^{11,0}(\bar{M}_{1,12}, \mathbb{C})$

Let $S = \{1, 2, 3, \ldots, 11, p\}$. For each index $1 \leq i \leq 11$, let

$$\pi_i \colon \bar{M}_{1,S} \to \bar{M}_{1,S-i} \cong \bar{M}_{1,11}$$

denote the forgetful map. Since we consider

$$S - i = \{1, 2, 3, \ldots, \hat{i}, \ldots, 11, p\}$$

as an *ordered* set, the last isomorphism above is canonical. Define the classes a_i and b_i by:

$$a_i = \pi_i^*(s) \in H^{11,0}(\bar{M}_{1,S}, \mathbb{C});$$

$$b_i = \pi_i^*(t) \in H^{0,11}(\bar{M}_{1,S}, \mathbb{C}).$$

For each index $1 \leq i \leq 11$, let ε_i be the map defined by the inclusion

$$\varepsilon_i \colon \bar{M}_{1,11} \cong \bar{M}_{1,S-i} \to \bar{M}_{1,S}.$$

Here, an S-pointed curve is obtained from an $(S-i)$-pointed curve by attaching a rational tail containing the markings i and p to the point p of the latter curve. The map ε_i is simply the inclusion of the boundary divisor D_{ip} with genus splitting $1 + 0$ and point splitting

$$\{1, \ldots, \hat{i}, \ldots, 11\} \cup \{i, p\}.$$

Define the classes c_i and d_i by:

$$c_i = \varepsilon_{i*}(s) \in H^{12,1}(\bar{M}_{1,S}, \mathbb{C});$$

$$d_i = \varepsilon_{i*}(t) \in H^{1,12}(\bar{M}_{1,S}, \mathbb{C}).$$

Here, the cohomological push-forward is defined by the equivalent equalities

$$\int_{\bar{M}_{1,S}} \varepsilon_{i*}(x) \cup y = \int_{\bar{M}_{1,11}} x \cup \varepsilon_i^*(y),$$

$$\int_{\bar{M}_{1,S}} y \cup \varepsilon_{i*}(x) = \int_{\bar{M}_{1,11}} \varepsilon_i^*(y) \cup x.$$

PROPOSITION 5. *The sets* $\{a_1, \ldots, a_{11}\}$ *and* $\{d_1, \ldots, d_{11}\}$ *form a pair of Poincaré dual bases of* $H^{11,0}(\bar{M}_{1,S}, \mathbb{C})$ *and* $H^{1,12}(\bar{M}_{1,S}, \mathbb{C})$.

Proof. By the dimension result of Proposition 4, it suffices to prove

$$\int_{\bar{M}_{1,S}} a_i \cup d_j = \delta_{ij}. \tag{6}$$

By definition of the cohomological push-forward,

$$\int_{\bar{M}_{1,S}} \pi_i^*(s) \cup \varepsilon_{i*}(t) = \int_{\bar{M}_{1,11}} s \cup t = 1. \tag{7}$$

The first equality in (7) is true exactly (not up to sign) by the precise ordering conventions used.

The vanishing of (6) when $i \neq j$ is a direct consequence of Proposition 2. We find

$$\int_{\bar{M}_{1,S}} \pi_i^*(s) \cup \varepsilon_{j*}(t) = \int_{\bar{M}_{1,11}} \varepsilon_j^* \pi_i^*(s) \cup t. \tag{8}$$

The composition $\pi_i \circ \varepsilon_j$ has image isomorphic to $\bar{M}_{1,10}$. Since the image supports no odd cohomology, the integral (8) vanishes. □

An identical argument proves the duality result for the classes c_i and b_i.

PROPOSITION 6. *The sets* $\{c_1, \ldots, c_{11}\}$ *and* $\{b_1, \ldots, b_{11}\}$ *form a pair of Poincaré dual bases of* $H^{12,1}(\bar{M}_{1,S}, \mathbb{C})$ *and* $H^{0,11}(\bar{M}_{1,S}, \mathbb{C})$. *The intersection form is*

$$\int_{\bar{M}_{1,S}} c_i \cup b_j = \delta_{ij}.$$

4.3. The Action of ψ_p

Let $\psi_p \in H^{1,1}(\bar{M}_{1,S}, \mathbb{C})$ denote the cotangent line class at the point p. Multiplication by ψ_p defines linear maps:

$$\Psi\colon H^{11,0}(\bar{M}_{1,S}, \mathbb{C}) \to H^{12,1}(\bar{M}_{1,S}, \mathbb{C});$$
$$\Psi\colon H^{0,11}(\bar{M}_{1,S}, \mathbb{C}) \to H^{1,12}(\bar{M}_{1,S}, \mathbb{C}).$$

These maps are completely determined by the following result.

PROPOSITION 7. *For all* $1 \leq i \leq 11$, $\Psi(a_i) = c_i$ *and* $\Psi(b_i) = d_i$.

Proof. Consider the morphism $\pi_i\colon \bar{M}_{1,S} \to \bar{M}_{1,S-i}$. A standard comparison result governing the cotangent line class is

$$\psi_p = \pi_i^*(\psi_p) + [D_{ip}],$$

where $\pi_i^*(\psi_p)$ denotes the pull-back of the cotangent class on $\bar{M}_{1,S-i}$. We then find that

$$\psi_p \cup \pi_i^*(s) = \pi_i^*(\psi_p \cup s) + [D_{ip}] \cup \pi_i^*(s). \tag{9}$$

Since $\bar{M}_{1,11}$ has odd cohomology only in degree 11, the first summand of (9) vanishes. The second summand is exactly equal to $\varepsilon_{i*}(s)$ (using the ordering conventions). We conclude that $\Psi(a_i) = c_i$. The derivation of $\Psi(b_i) = d_i$ is identical. □

4.4. *Proof of Theorem 4*

Set $\tilde{S} = \{\tilde{1}, \tilde{2}, \ldots, \widetilde{11}, \tilde{p}\}$. Consider the boundary map

$$\iota\colon \bar{M}_{1,S} \times \bar{M}_{1,\tilde{S}} \to \bar{M}_{2,22}$$

defined by attaching p to $\tilde{p}$ (and ordering the markings arbitrarily). Define γ by

$$\gamma = \iota_*[\Delta] \in H^{26}(\bar{M}_{2,22}),$$

where Δ is the diagonal subvariety of $\bar{M}_{1,S} \times \bar{M}_{1,\tilde{S}}$ (under the canonical isomorphism $\bar{M}_{1,S} \cong \bar{M}_{1,\tilde{S}}$).

Here, ι is easily seen to define an *embedding.* The normal bundle to ι in $\bar{M}_{2,22}$ has top Chern class $-\psi_p - \psi_{\tilde{p}}$. By the self-intersection formula,

$$\iota^*\iota_*[\Delta] = [\Delta] \cup (-\psi_p - \psi_{\tilde{p}}).$$

Let $X_1, \ldots, X_m$ be a basis of $H^*(\bar{M}_{1,S})$. Let $\tilde{X}_1, \ldots, \tilde{X}_m$ denote the corresponding basis of $H^*(\bar{M}_{1,\tilde{S}})$. The Künneth decomposition of $[\Delta]$ is determined by

$$[\Delta] = \sum_{i,j} g^{ij} X_i \otimes \tilde{X}_j \in H^*(\bar{M}_{1,S}) \times H^*(\bar{M}_{1,\tilde{S}}),$$

where

$$g_{ij} = \int_{\bar{M}_{1,S}} X_i \cup X_j.$$

In particular, if $X_1, \ldots, X_m$ is a self-dual basis, then

$$[\Delta] = \sum_i (-1)^{\nu_i \nu_i^\vee} X_i \otimes X_i^\vee,$$

where ν_i and $\nu_i^\vee$ are the degrees of X_i and $X_i^\vee$, respectively.

We are interested in the Künneth components of $[\Delta]$ of odd type—that is, Künneth components lying in

$$H^{\text{odd}}(\bar{M}_{1,S}) \otimes H^{\text{odd}}(\bar{M}_{1,\tilde{S}}).$$

By Propositions 5 and 6, the odd type summands of $[\Delta]$ are

$$\sum_{i=i}^{11} -a_i \otimes \widetilde{d_i} + b_i \otimes \widetilde{c_i} - c_i \otimes \widetilde{b_i} + d_i \otimes \widetilde{a_i}.$$

Hence, the odd summands of $\iota^*\iota_*[\Delta]$ are

$$\sum_{i=1}^{11} \Psi(a_i) \otimes \widetilde{d_i} - \Psi(b_i) \otimes \widetilde{c_i} + c_i \otimes \tilde{\Psi}(\widetilde{b_i}) - d_i \otimes \tilde{\Psi}(\widetilde{a_i}).$$

By Proposition 7, we find that the odd summands of $\iota^*\iota_*[\Delta]$ equal

$$\sum_{i=1}^{11} 2c_i \otimes \widetilde{d}_i - 2d_i \otimes \widetilde{c}_i.$$

Since the odd summands (10) do not vanish,

$$\iota_*[\Delta] \notin RH^*(\bar{M}_{2,22})$$

by Proposition 1. The proof of Theorem 4 is complete. □

Appendix A. Pull-backs in the Tautological Ring

A.1. Stable Graphs

The boundary strata of the moduli space of curves correspond to *stable graphs*

$$A = (V, H, L,\ g\colon V \to \mathbb{Z}_{\geq 0},\ a\colon H \to V,\ i\colon H \to H)$$

satisfying the following properties.

(i) V is a vertex set with a genus function g.
(ii) H is a half-edge set equipped with a vertex assignment a and fixed point–free involution i.
(iii) E, the edge set, is defined by the orbits of i in H (self-edges at vertices are permitted).
(iv) (V, E) define a *connected* graph.
(v) L is a set of numbered legs attached to the vertices.
(vi) For each vertex v, the stability condition holds:

$$2g(v) - 2 + n(v) > 0,$$

where $n(v)$ is the valence of A at v including both half-edges and legs.

The genus of A is defined by

$$g(A) = \sum_{v \in V} g(v) + h^1(A).$$

Let $v(A)$, $e(A)$, and $n(A)$ denote the cardinalities of V, E, and L, respectively. A boundary stratum of $\bar{M}_{g,n}$ naturally determines a stable graph of genus g with n legs by considering the dual graph of a generic pointed curve parameterized by the stratum.

Let A be a stable graph. Define the moduli space $\bar{M}_A$ by the product

$$\bar{M}_A = \prod_{v \in V(A)} \bar{M}_{g(v),n(v)}.$$

Let π_v denote the projection from $\bar{M}_A$ to $\bar{M}_{g(v),n(v)}$ associated to the vertex v. There is a canonical morphism $\xi_A\colon \bar{M}_A \to \bar{M}_{g,n}$ with image equal to the boundary stratum associated to the graph A. To construct ξ_A, a family of stable pointed curves over $\bar{M}_A$ is required. Such a family is easily defined by attaching the pull-backs of the universal families over each of the $\bar{M}_{g(v),n(v)}$ along the sections corresponding to half-edges.

A.2. Specialization

Our main goal in the Appendix is to understand the fiber product

$$\begin{array}{ccc} F_{A,B} & \longrightarrow & \bar{M}_B \\ \downarrow & & \downarrow{\scriptstyle \xi_B} \\ \bar{M}_A & \xrightarrow{\xi_A} & \bar{M}_{g,n}\,. \end{array}$$

Toward this end, we will require additional terminology. A stable graph C is a *specialization* of a stable graph A if C is obtained from A by replacing each vertex v of A with a stable graph of genus $g(v)$ with $n(v)$ legs. Specialization of graphs corresponds to specialization of stable curves.

There is a subtlety involved in the notion of specialization: A given graph C may arise as a specialization of A in more than one way. An *A-graph structure* on a stable graph C is a choice of subgraphs of C in bijective correspondence with $V(A)$ such that C can be constructed by replacing each vertex of A by the corresponding subgraph. If C has an A-structure, then (a) every half-edge of A corresponds to a particular half-edge of C and (b) every vertex of C is associated to a particular vertex of A.

A point of $\bar{M}_A$ is given by a stable curve together with a choice of A-structure on its dual graph. In fact, we can naturally identify the stack $\bar{M}_A$ with a stack defined in terms of A-structures. This identification will be useful for analyzing the fiber products of strata.

Define a stable A-curve over a connected base S,

$$\pi : \mathcal{C} \to S,$$

to be a stable $n(A)$-pointed curve of genus $g(A)$ over S together with:

(i) $e(A)$ sections $\sigma_1, \ldots, \sigma_{e(A)}$ of π with image in the singular locus of $\mathcal{C}$;
(ii) $2e(A)$ sections of the normalization of $\mathcal{C}$ along the sections $\{\sigma_i\}$ corresponding to the nodal separations;
(iii) $v(A)$ disjoint π-relative components of $\mathcal{C} \setminus \{\sigma_i\}$ whose union is $\mathcal{C} \setminus \{\sigma_i\}$; and
(iv) an isomorphism between A and the canonical stable graph defined by the dual graph of the $v(A)$ π-relative components and $2e(A)$ sections of the normalization (corresponding to half-edges).

Here, a π-relative component is a connected component of $\mathcal{C} \setminus \{\sigma_i\}$ that remains connected upon pull-back under an arbitrary morphism of connected schemes $h : T \to S$.

The data of a stable A-curve can be pulled back under any morphism of base schemes. After pull-back to a geometric point, an A-curve is exactly an A-structure on the dual graph of the corresponding curve.

A stack $\bar{M}'_A$ of curves with A-structure morphisms and respecting the A-structure may be defined. However, we find the following result.

PROPOSITION 8. *There is a natural isomorphism between $\bar{M}_A$ and $\bar{M}'_A$.*

Proof. A natural morphism from $\bar{M}_A$ to $\bar{M}'_A$ is obtained by assigning the canonical A-structure to the universal curve over $\bar{M}_A$. In the other direction, given an S-valued point of $\bar{M}'_A$, we naturally obtain a collection of $v(A)$ stable curves by analyzing the π-relative components of C normalized at the $e(A)$ nodes. Since we have a bijection between these curves and $v(A)$ as well as a bijection between the new markings and the $2e(A)$ sections, we obtain an S-valued point of $\bar{M}_A$. This correspondence induces a bijection on the space of morphisms between corresponding objects. □

A.3. Fiber Products

By definition, an S-valued point of $F_{A,B}$ is an S-valued point of $\bar{M}_A$, an S-valued point of $\bar{M}_B$, and a choice of isomorphism between the two pull-backs of the universal curve over $\bar{M}_{g,n}$ under the boundary inclusions. If S is $\mathrm{Spec}(\mathbb{C})$, we find that the dual graph C of the curve over S defined by the map to $\bar{M}_{g,n}$ is naturally equipped with both an A-structure and a B-structure. Conversely, given a curve C together with two such structures on the dual graph, we naturally obtain a point of $F_{A,B}$. A graph C equipped with both A- and B-structures will be called an (A, B)-graph.

An (A, B)-graph C is *generic* if every half-edge of C corresponds to a half-edge of A or a half-edge of B. The irreducible components of $F_{A,B}$ will correspond to generic (A, B)-graphs. A graph with an (A, B)-structure is canonically a specialization of a unique generic (A, B)-graph: the generic graph is obtained by contracting all those edges that do not correspond to edges of A or B.

Associated to an (A, B)-graph C, we obtain a moduli space $\bar{M}_C$ that naturally maps to $F_{A,B}$. The moduli space may be described either as $\prod_{v\in v(C)} \bar{M}_{g(v),n(v)}$ or in stack terms analogous to the preceding definition of $\bar{M}'_A$ (the stack does not depend on the (A, B)-structure on C, although the map to $F_{A,B}$ does). We find the following result.

PROPOSITION 9. *There is a canonical isomorphism between $F_{A,B}$ and the disjoint union of $\bar{M}_C$ over all generic (A, B)-graphs C.*

Proof. It will suffice to identify the categories involved over connected base schemes S. We will give the morphisms in both directions.

If C is an (A, B)-graph, then we clearly have a morphism from $\bar{M}_C$ to both $\bar{M}_A$ and $\bar{M}_B$, as well as a choice of isomorphism between the induced maps to $\bar{M}_{g,n}$.

In the other direction, suppose data corresponding to $F_{A,B}$ is given over S. In particular, we have a stable curve over S:

$$\pi : \mathcal{C} \to S.$$

Consider the π-fiber over a geometric point of S. The π-fiber has a dual graph equipped with an (A, B)-structure by virtue of the maps to $\bar{M}_A$ and $\bar{M}_B$. Let C be the unique generic (A, B)-graph that specializes to the (A, B)-structure found at the geometric point.

There is a canonical C-structure on $\mathcal{C}$. The half-edges of C are already naturally identified with half-edges of the graphs A or B; since $\mathcal{C}$ has A- and B-structures, it follows that π is equipped with sections associated to all of the half-edges. The C-structure is constant on $\mathcal{C}$ because of the connectedness of S.

The morphisms in the two categories are the same by a straightforward check. However, it is important to note that an automorphism of an object of $F_{A,B}$ must induce a trivial automorphism of the graph C, because each half-edge corresponds to an edge of either A or B. □

A.4. Pull-backs of Strata

The pull-backs of tautological classes to the boundary may now be explicitly determined. The basic calculation is the pull-back of the fundamental class of one boundary stratum to another. In terms of the diagram of Section A.2, we want to compute $\xi_A^*(\xi_{B*}[\bar{M}_B])$. Because we have identified $F_{A,B}$ explicitly as a smooth stack, the pull-back will be straightforward to compute. The intersection product is a sum of contributions of each component of $F_{A,B}$, and each contribution is the Euler class of an excess bundle on the component.

The components of $F_{A,B}$ have been identified in Proposition 9. Let C be a generic (A, B)-graph, and let $\bar{M}_C$ be the corresponding component of $F_{A,B}$. The excess bundle is easily identified on $\bar{M}_C$. First, we observe that the normal bundle to ξ_A naturally splits as a copy of $e(A)$ line bundles. Let the edge e be the join of the distinct half-edges h, h' incident to the vertices v, v' (which may coincide). The line bundle associated to e is

$$T_h \otimes T_{h'},$$

where T_h and $T_{h'}$ are the tangent lines at h and h' of the factors $\bar{M}_{g(v),n(v)}$ and $\bar{M}_{g(v),n(v')}$, respectively. The normal bundle to $\bar{M}_C$ in $\bar{M}_A$ is a sum of the analogous line bundles for those edges of C that do not correspond to edges of A. Precisely the same situation holds with respect to B. We can conclude that the excess normal bundle of $\bar{M}_C$, viewed as a component of $F_{A,B}$, is exactly the sum of the line bundles corresponding to those edges of C that correspond to edges of both A and B.

We have deduced the following formula:

$$\xi_A^*(\xi_{B*}([\bar{M}_B]) = \sum_C \xi_{C,A*}\Bigg(\prod_{e=h+h'} -\pi_v^*(\psi_h) - \pi_{v'}^*(\psi_{h'})\Bigg). \tag{11}$$

The sum is over all generic (A, B)-graphs C. The product is over all edges e of C that come from both an edge of A and an edge of B, and v, v' are the vertices joined by e. The morphism $\xi_{C,A}$ denotes the natural map from $\bar{M}_C$ to $\bar{M}_A$.

Formula (11) yields an explicit tautological Künneth decomposition of the pull-back class, since the morphism $\xi_{C,A}$ is simply the product of various boundary strata maps over the factors of $\bar{M}_A$.

We will compute a simple example to illustrate the formula. Consider the boundary divisor Δ_0 in $\bar{M}_g$ corresponding to the morphism

$$i\colon \bar{M}_{g-1,2} \to \bar{M}_g.$$

The graph A of Δ_0 has one vertex of genus $g-1$ and one self-edge. We will compute the self-intersection of the stratum: $\xi_A^*(\xi_{A*}[\bar{M}_A])$.

We first write down all generic (A, A)-graphs. They are A itself with the obvious (A, A)-structure, and then one graph for each integer from 0 to $\lfloor (g-1)/2 \rfloor$. Then C_0 is the graph with one vertex of genus $g-2$ and two loops. Observe that C_0 has two distinct isomorphism classes of (A, A)-structures but only one of them is generic: the (A, A)-structure where the edge contracted for the first A-structure is different from the edge contracted for the second A-structure. Similarly, C_i is the graph with a vertex of genus i and another vertex of genus $g-i-1$ connected to each other by two edges. The unique generic (A, A)-structure is obtained by contracting a different edge for the two A-structures. Applying formula (11), we find that

$$\xi_A^*(\xi_{A*}([\bar{M}_A])) = -\psi_1 - \psi_2 + \xi_{0*}([\bar{M}_{g-2,4}]) + \sum_{i=1}^{\lfloor (g-1)/2 \rfloor} \xi_{i*}([\bar{M}_{i,2} \times \bar{M}_{g-i-1,2}])$$

with hopefully evident notation.

Notice that the boundary strata corresponding to $\bar{M}_{i,3} \times \bar{M}_{g-i-1,1}$ do not appear in this formula because the corresponding dual graphs do not admit a generic (A, A)-structure. In more geometric terms, these strata do not contribute an extra term because they have only one nondisconnecting node.

A.5. Pull-backs of Tautological Classes

We observe that our calculations easily generalize to computing pull-backs of arbitrary tautological classes to boundary strata.

Define the tautological κ classes by

$$\pi_*(\psi_{n+1}^{l+1}) = \kappa_l \in R^*(\bar{M}_{g,n}),$$

where π is the map forgetting the last marking $n+1$. The first observation is the following result concerning the push-forwards of the ψ and κ classes.

PROPOSITION 10. *Let $\pi\colon \bar{M}_{g,n+m} \to \bar{M}_{g,n}$ be the map forgetting the last m points. The π push-forward of any element of the subring of $R^*(\bar{M}_{g,n+m})$ generated by*

$$\psi_1, \ldots, \psi_m, \{\kappa_i\}_{i \in \mathbb{Z}_{\geq 0}}$$

lies in the subring of $R^*(\bar{M}_{g,n})$ generated by

$$\psi_1, \ldots, \psi_n, \{\kappa_i\}_{i \in \mathbb{Z}_{\geq 0}}.$$

A proof can be found in [AC].

We can now describe a set of additive generators for $R^*(\bar{M}_{g,n})$. Let B be a stable graph of genus g with n legs. For each vertex v of B, let

$$\theta_v \in R^*(\bar{M}_{g(v),n(v)})$$

be an arbitrary monomial in the cotangent line and κ classes of the vertex moduli space.

PROPOSITION 11. *$R^*(\bar{M}_{g,n})$ is generated additively by classes of the form*

$$\xi_{B*}\Big(\prod_{v\in V(B)} \theta_v\Big).$$

Proof. By the definition of $R^*(\bar{M}_{g,n})$, the claimed generators lie in the tautological ring.

We first show that the span of the generators is closed under the intersection product. The closure follows from:

(i) the pull-back formula (11) for strata classes;
(ii) the trivial pull-back formula for cotangent lines under boundary maps; and
(iii) the pull-back formula for κ classes under boundary maps,

$$\xi_B^*(\kappa_i) = \sum_{v\in V(B)} \kappa_i$$

(see [AC]).

To prove that the claimed generators span $R^*(\bar{M}_{g,n})$, we must prove that the system defined by the generators is closed under push-forward by the forgetting maps and the gluing maps. Closure under push-forward by the forgetting maps is a consequence of Proposition 10. Closure under push-forward by the gluing maps is a trivial condition. □

COROLLARY 1. *$R^*(\bar{M}_{g,n})$ is a finite-dimensional $\mathbb{Q}$-vector space.*

Proof. The set of stable graphs B for fixed g and n is finite, and there are only finitely many nonvanishing monomials θ_v for each vertex v. □

PROPOSITION 12. *Let $\gamma \in R^*(\bar{M}_{g,n})$. Let A be a stable graph, and let*

$$\xi_A\colon \bar{M}_A \to \bar{M}_{g,n}.$$

Then $\xi_A^*(\gamma)$ has a tautological Künneth decomposition with respect to the product structure of $\bar{M}_A$.

Proof. This follows from Proposition 11 together with the pull-back formulas. The pull-back formulas for the three types of classes all yield tautological Künneth decompositions. □

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