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Reinhardt Domains and Toric Models

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1. Introduction

This article is motivated by the construction of unitary representations of the torus, based on Kähler structures on strictly pseudoconvex Reinhardt domains. Let T be the compact n-torus. In the language of geometric quantization [7], the "classical" picture is a T-manifold X along with a T-invariant symplectic form ω, while the "quantum" picture is a unitary T-representation H. The process of transforming (X, ω) to $H = H_\omega$ is called *geometric quantization*. If H_ω contains every irreducible T-representation exactly once, it is called a *model*. This terminology is due originally to I. M. Gelfand and A. Zelevinski [4], who construct models of the classical groups. Since T is a torus, we also call it a *toric model*. A space where T acts naturally is the Reinhardt domain $X \subset \mathbf{C}^n$, since $(z_1, \dots, z_n) \in X$ implies that $(e^{i\theta_1}z_1, \dots, e^{i\theta_n}z_n) \in X$. Consider the setting (X, ω), where ω is a T-invariant Kähler form on the Reinhardt domain X. The central issue of this article is: *When does (X, ω) provide a toric model H_ω?*

We shall describe the Kähler structures ω, construct H_ω, and show that the conditions for H_ω to be a toric model are closely related to the convergence of the integrals

$$\int_{x \in \Omega} e^{-F(x)+\lambda x}\, dV, \quad \lambda \in \mathbf{Z}^n.$$

Here $\Omega \subset \mathbf{R}^n$ is a strictly convex domain, $F \in C^\infty(\Omega)$ a strictly convex function, and dV the Lebesgue measure. This integral will be our major concern. We now outline our projects in better detail.

We restrict our consideration to strictly pseudoconvex Reinhardt domains X with free T-action. Free T-action implies that if $(z_1, \dots, z_n) \in X$ then $z_i \neq 0$ for all i. By the exponential map and normalization $2\pi \sim 1$, it follows that T and X have the following convenient descriptions:

$$T = \mathbf{R}^n/\mathbf{Z}^n, \qquad X = \left\{x + \sqrt{-1}y : x \in \Omega,\ y \in T\right\}, \tag{1.1}$$

where $\Omega \subset \mathbf{R}^n$ is a domain. Note that $X = \Omega + \sqrt{-1}T$. The T-action on X is given by addition on $y \in \mathbf{R}^n/\mathbf{Z}^n$ in (1.1). We shall see in Section 2 that strict pseudoconvexity of X leads to nice convexity properties of Ω. From now on, X, T, Ω

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shall always be as given in (1.1). The present article deals with arbitrary strictly convex domain Ω, which extends the special case $\Omega = \mathbf{R}^n$ studied in [3].

In Section 3 we study the Hamiltonian T-invariant Kähler forms ω on X as well as their quantization (cf. [5; 7]). They have the expression $\omega = \sqrt{-1}\partial\bar{\partial}F$ (Theorem 3.1), where $\partial, \bar{\partial}$ are the Dolbeault operators on X. Then F, being a T-invariant function on X, is nothing other than a function on Ω. We show that $F \in C^\infty(\Omega)$ is strictly convex, meaning its Hessian matrix is always positive definite. This allows us to apply the convex analysis developed in Section 2. The Kähler form ω can be associated to a (topologically trivial) line bundle $\mathbf{L}$ over X, and $\mathbf{L}$ is equipped with a connection whose curvature is ω. From the connection, we obtain the notion of holomorphic sections on $\mathbf{L}$. The line bundle also carries a natural Hermitian structure $\langle\cdot,\cdot\rangle$. By (1.1), X has a natural measure obtained from the Lebesgue measure of Ω and the Haar measure of T. We let dV denote the natural measure on X or Ω. A holomorphic section s of $\mathbf{L}$ is said to be *square-integrable* if

$$\int_X \langle s, s\rangle \, dV < \infty. \tag{1.2}$$

The Hilbert space of square-integrable holomorphic sections is denoted by H_ω, and these sections constitute a unitary T-representation.

We now set up miscellaneous notation and terminology needed to formulate the main theorem of this article. A *ray* is a subset of Ω of the form

$$R = \{p + tv : t > 0\} \cap \Omega \tag{1.3}$$

for some $p \in \Omega$ and $v \in \mathbf{R}^n$. Here p is called the *initial point* of R. Let (θ, r) be the polar coordinates centered at $p \in \Omega$, where $\theta \in S^{n-1}$ and r measures the distance from p. Then the Lebesgue measure on Ω is

$$dV = r^{n-1}\,dr\,d\theta.$$

So we shall be interested in the measure $r^{n-1}\,dr$ on a ray with initial point p.

The *epigraph* of $F \in C^\infty(\Omega)$ consists of the points above the graph of F. Namely,

$$\mathrm{epi}(F) = \{(x, y) \in \Omega \times \mathbf{R} : y > F(x)\} \subset \mathbf{R}^n \times \mathbf{R}. \tag{1.4}$$

The gradient function $F' : \Omega \to \mathbf{R}^n$ is injective when F is strictly convex. Let Δ be the image of F'. In fact,

$$F' : \Omega \to \Delta, \qquad F'(x) = \left(\frac{\partial F}{\partial x_i}(x)\right)_i$$

is a diffeomorphism, owing to the inverse function theorem. Here Δ may not be convex. Boundary points q of Δ that make Δ nonconvex are called *concave* points; namely,

$$q = tr + (1-t)s \quad \text{for some } r, s \in \Delta,\ 0 < t < 1. \tag{1.5}$$

The rest of the boundary points are said to be nonconcave. Clearly Δ is (geometrically) convex if and only if its boundary consists entirely of nonconcave points. With this notation and terminology in hand, we present our main theorem as follows.

MAIN THEOREM. *Let ω be a T-invariant Kähler form on X, so that $\omega = \sqrt{-1}\partial\bar{\partial}F$ and $F \in C^\infty(\Omega)$ is strictly convex. The following conditions are equivalent*:

(A) *H_ω is a toric model*;
(B) *$\int_R e^{-F(r)+\lambda r} r^{n-1}\, dr < \infty$ for all rays $R \subset \Omega$ and all $\lambda \in \mathbf{Z}^n$*;
(C) *if $L \subset \mathbf{R}^n \times \mathbf{R}$ is a nonvertical line, then $L \cap \operatorname{epi}(F)$ is bounded*;
(D) *F' maps every unbounded subset of Ω to an unbounded subset of Δ*;
(E) *if $\{q_i\} \subset \Delta$ converges to a nonconcave point, then $\{(F')^{-1}(q_i)\}$ is bounded*.

Conditions (B), (C), (D), and (E) are studied in Section 4, 5, 6, and 7, respectively. Condition (B) says that the study of $F \in C^\infty(\Omega)$ can be reduced to the restriction of F to the rays, which are 1-dimensional objects. Condition (C) follows the same spirit, since a nonvertical line is simply the graph of a 1-variable function. Condition (D) says that $(F')^{-1}$ sends bounded subsets of Δ to bounded subsets of Ω, so obviously (D) implies (E). Therefore, the equivalence of (D) and (E) means that, in the study of boundedness property of $(F')^{-1}(U)$, we can ignore the concave limit points of U.

Concerning the sizes of Ω and Δ, conditions (C) and (D) say that a smaller Ω or a bigger Δ increases the likelihood of H_ω to be a toric model. In particular, if Ω is bounded then (C) says that H_ω is always a toric model. Also, (D) says that if $\Delta = \mathbf{R}^n$ then H_ω is always a toric model. The compromise where both Ω and Δ have maximal size is reached in [3]: If $\Omega = \mathbf{R}^n$, then H_ω is a toric model if and only if $\Delta = \mathbf{R}^n$. This can be recovered from (D), since a diffeomorphism $\mathbf{R}^n \to \Delta$ maps every unbounded set to an unbounded set if and only if $\Delta = \mathbf{R}^n$.

In Section 8, we provide several examples of strictly convex functions to demonstrate the equivalent conditions of the main theorem. Finally, in Section 9, we discuss the Bergman kernel $K(z, w)$ [8, Sec. 1] of the Kähler manifold (X, ω), where $\omega = \sqrt{-1}\partial\bar{\partial}F$. We eliminate the coefficient $\mathbf{L}$ and identify H_ω with the Bergman space $\mathcal{H}$ of holomorphic functions on X that are square-integrable with respect to the measure $e^{-F}\, dV$ (Proposition 9.1). In Theorem 9.2, we show that if H_ω is a toric model then the Bergman kernel of $\mathcal{H}$ is

$$K(z, w) = \sum_{\lambda \in \mathbf{Z}^n} \frac{e^{\lambda(z+\bar{w})}}{\int_{x\in\Omega} e^{2\lambda x - F(x)}\, dV}. \tag{1.6}$$

2. Notions of Convexity

In this section, we recall some familiar notions of convexity for sets and functions and also gather some properties to be used later. Recall that our strictly pseudoconvex Reinhardt domain $X = \Omega + \sqrt{-1}T$ is as given by (1.1). The symbol ∂ shall denote boundary (as well as partial derivative—there is no confusion), and the "bar" sign shall denote closure. For example, $\bar{\Omega}\backslash\Omega = \partial\Omega$ because Ω is open. The boundaries of X and Ω are related by $\partial X = \partial\Omega + \sqrt{-1}T$. Pick $z = x + \sqrt{-1}y \in \partial X$, so that $x \in \partial\Omega$ and $y \in T$. The real tangent space of z at ∂X is of real dimension $2n-1$. It contains a codimension-1 linear subspace $T_z^{\mathbf{C}}(\partial X) \subset \mathbf{C}^n$, which is stable under multiplication by $\sqrt{-1}$. This is the complex tangent space of z, and $\dim_{\mathbf{C}} T_z^{\mathbf{C}}(\partial X) = n-1$. Since X is strictly pseudoconvex, there exists a defining function τ for X (see e.g. [8]) such that the Levi form $\left(\frac{\partial^2\tau}{\partial z_i \partial \bar{z}_j}(z)\right)_{ij}$ is Hermitian positive definite on $T_z^{\mathbf{C}}(\partial X)$. The real tangent space $T_x^{\mathbf{R}}(\partial\Omega) \subset \mathbf{R}^n$ of $x \in \partial\Omega$ has real dimension $n-1$. We say that Ω is *strictly convex* if there exists a defining function ρ for Ω such that, for all $x \in \partial\Omega$, the Hessian matrix $\left(\frac{\partial^2\rho}{\partial x_i \partial x_j}(x)\right)_{ij}$ is positive definite on $T_x^{\mathbf{R}}(\partial\Omega)$.

PROPOSITION 2.1. *The Reinhardt domain X is strictly pseudoconvex if and only if the domain Ω is strictly convex.*

Proof. Let ρ be a defining function for Ω. Then it extends to a defining function $\tilde\rho$ for X by $\tilde\rho(z) = \rho(x)$ for all $z = x + \sqrt{-1}y$. Since $\frac{\partial\tilde\rho}{\partial y_i} = 0$, we obtain

$$\frac{\partial^2\tilde\rho}{\partial z_i \partial \bar{z}_j}(z) = \frac{1}{4}\frac{\partial^2\rho}{\partial x_i \partial x_j}(x). \tag{2.1}$$

For $z = x + \sqrt{-1}y \in \partial X$, we have

$$T_z^{\mathbf{C}}(\partial X) = T_x^{\mathbf{R}}(\partial\Omega) + \sqrt{-1}T_x^{\mathbf{R}}(\partial\Omega). \tag{2.2}$$

From (2.1) and (2.2), it follows that $\left(\frac{\partial^2\tilde\rho}{\partial z_i \partial \bar{z}_j}(z)\right)_{ij}$ is Hermitian positive definite on $T_z^{\mathbf{C}}(\partial X)$ if and only if $\left(\frac{\partial^2\rho}{\partial x_i \partial x_j}(x)\right)_{ij}$ is positive definite on $T_x^{\mathbf{R}}(\partial\Omega)$. The proposition follows. $\square$

The classical notion of convexity (i.e., a line segment joining two points) is called “geometric” convexity to distinguish it from the analytic notion of convexity used above. A subset $A \subset \mathbf{R}^n$ is said to be geometrically convex if $p, q \in A$ implies $tp + (1-t)q \in A$ for all $0 < t < 1$. In particular, if A satisfies

$$p, q \in \bar{A} \implies tp + (1-t)q \in \bar{A}\backslash\partial A \quad \text{for all } 0 < t < 1, \tag{2.3}$$

then we say that A is geometrically strictly convex. For example, the unit disk is geometrically strictly convex whereas the square in $\mathbf{R}^2$ is merely geometrically convex.

If a domain Ω is strictly convex then it is geometrically strictly convex [8, Sec. 3]. Therefore, by Proposition 2.1, we obtain the following corollary.

COROLLARY 2.2. *Let $X = \Omega + \sqrt{-1}T$ be a strictly pseudoconvex Reinhardt domain. Then Ω is geometrically strictly convex.*

We next consider the strictly convex functions. Let Ω be strictly convex. A smooth function $F : \Omega \to \mathbf{R}$ is said to be strictly convex if its Hessian matrix is positive definite everywhere on Ω. Strictly convex functions have either 0 or 1 critical point, and the critical point (if it exists) is a global minimum.

PROPOSITION 2.3. *If $F \in C^\infty(\Omega)$ is strictly convex, then $F^{-1}(-\infty, m]$ is geometrically convex.*

Proof. Let $r, s \in F^{-1}(-\infty, m] \in \Omega$, and let q lie between r and s. Let L be the line joining r and s. Since the restricted function $F|_{L\cap\Omega}$ is strictly convex and since $q, r, s \in L \cap \Omega$, it follows that $F(q) < \max\{F(r), F(s)\}$. So $q \in F^{-1}(-\infty, m)$, which implies that $F^{-1}(-\infty, m]$ is geometrically convex. The proposition follows. □

In Proposition 2.3, $F^{-1}(-\infty, m]$ is in fact geometrically *strictly* convex. With strictness in the statement, the proof needs to be modified to let r, s be in the closure of $F^{-1}(-\infty, m]$ relative to $\mathbf{R}^n$. Nevertheless, for the purpose of applications in later sections, we only need $F^{-1}(-\infty, m]$ to be geometrically convex.

Another feature of the strictly convex function F is that its epigraph (1.4) is geometrically strictly convex. The converse fails, and a counter-example is $F(x) = x^4$. Since $\operatorname{epi}(F) \subset \Omega \times \mathbf{R}$ is geometrically strictly convex, it follows that $L \cap \operatorname{epi}(F)$ is connected whenever $L \subset \mathbf{R}^n \times \mathbf{R}$ is a line. As indicated in condition (C) of the main theorem, the boundedness property of $L \cap \operatorname{epi}(F)$ determines whether H_ω is a toric model. We shall take up this topic in Section 5.

Given a strictly convex function, its restriction to a line is also strictly convex. For this reason, it is helpful to study strictly convex functions of one variable. Let $-\infty < a < b \leq \infty$. We say that $F\colon [a, b) \to \mathbf{R}$ is strictly convex if it extends to a smooth function on $(a - \varepsilon, b)$ and if $F''(x) > 0$ for all $x > a$. The following results will be useful later.

LEMMA 2.4. *Let $F\colon [a, b) \to \mathbf{R}$ be a strictly convex function, and let $n \in \{0, 1, 2, \dots\}$. The following conditions are equivalent:*

(i) *$\int_a^b e^{-F(x)} x^n\, dx$ diverges;*
(ii) *$b = \infty$ and F is strictly decreasing.*

Proof. Suppose that condition (ii) holds. Then $e^{-F} x^n$ is strictly increasing for $x > 0$, so its integral diverges.

Conversely, suppose that condition (ii) fails. Then either $b < \infty$ or $F'(c) \geq 0$ for some $c \in [a, b)$. If $b < \infty$ then F (being strictly convex) does not tend to $-\infty$ near b. That is, F is bounded below on $[a, b)$ and hence $e^{-F} x^n$ is bounded above on $[a, b)$. Hence the integral of $e^{-F} x^n$ converges. If $b = \infty$ but $F'(c) \geq 0$ for some c, then $F'(x) > 0$ for all $x > c$ because F' is increasing. In particular, $0 < m < F'(d)$ for some m and d. Thus, for all $x > d$,

$$F(d) + m(x - d) < F(x).$$

Since $0 < m$, we have

$$\int_d^\infty e^{-F(x)} x^n\, dx < \int_d^\infty e^{-F(d) - m(x-d)} x^n\, dx < \infty.$$

Once again (i) fails. This proves the lemma. □

We write $\text{l.u.b.} F'$ to denote the least upper bound of F'. In the event that F' is not bounded above, $\text{l.u.b.} F' = \infty$. The lemma leads to the following corollary.

COROLLARY 2.5. *Let $a \in \mathbf{R}$, let $F : [a, \infty) \to \mathbf{R}$ be strictly convex, and let $n \in \{0, 1, 2, \dots\}$. Then $\int_a^\infty e^{-F(x)+cx} x^n \, dx < \infty$ if and only if $c <$ l.u.b.F'.*

Proof. The proof is a simple computation:

$$\begin{aligned}
&\int_a^\infty e^{-F(x)+cx} x^n \, dx < \infty \\
&\qquad \iff F(x) - cx \text{ is increasing for some } x \quad \text{(by Lemma 2.4)} \\
&\qquad \iff F'(x) - c > 0 \text{ for some } x \\
&\qquad \iff c < \text{l.u.b.} F'.
\end{aligned}$$

This proves the corollary. □

Let $\Omega \subset \mathbf{R}^n$ be strictly convex, and fix $p \in \Omega$. Let (θ, r) be the polar coordinates centered at p, where $\theta \in S^{n-1}$ and r measures the distance from p. Recall from (1.3) that the rays with initial point p are of the form $\{p + tv : t > 0\} \cap \Omega$. By the polar coordinates, we can parameterize these rays by $\theta \in S^{n-1}$, denoted by R_θ accordingly. Let $d(\theta) = |R_\theta| \in (0, \infty]$ denote the length of the ray R_θ. Namely, $d(\theta)$ is the distance between p and the boundary $\partial\Omega$ in the θ-direction. Intuitively we expect d to vary continuously with the angle θ, and this is made precise in the next proposition. We say that $K \subset S^{n-1}$ is a compact neighborhood of $\phi \in S^{n-1}$ if K is compact and ϕ lies in the interior of K.

PROPOSITION 2.6. *Let $\Omega \subset \mathbf{R}^n$ be strictly convex, and fix $p \in \Omega$. For $\theta \in S^{n-1}$, let $d(\theta) = |R_\theta| \in (0, \infty]$ be as given previously. If $d(\phi) < \infty$, then ϕ has a compact neighborhood $K \subset S^{n-1}$ such that $d(\theta) < \infty$ for all $\theta \in K$ and d is continuous on K.*

Proof. Let $c = |R_\phi| < \infty$. Suppose that there exists a sequence $\{\phi_i\} \subset S^{n-1}$ converging to ϕ and that $|R_{\phi_i}| = \infty$. Using the polar coordinates (θ, r) centered at p, we have that $(\phi_i, r) \in \Omega$ for all $r > 0$. Then $(\phi_i, r) \to (\phi, r) \in \bar{\Omega}$ for all $r > 0$. In particular, $(\phi, c), (\phi, c+2) \in \bar{\Omega}$. Since Ω is strictly convex, by (2.3) it follows that $(\phi, c+1) \in \Omega$. This contradicts $c = |R_\phi|$, so $|R_\theta| < \infty$ for all θ sufficiently near ϕ. There exists an open set $U \subset S^{n-1}$ containing ϕ such that $|R_\theta| < \infty$ for all $\theta \in U$. Let K be a compact set satisfying $K \subset U$, and let ϕ lie in the interior of K. Then $d(\theta) < \infty$ for all $\theta \in K$.

It remains to show that d is continuous on K. Pick $\tau \in K$. Let $\{\tau_i\} \subset K$ be a sequence that converges to τ. Write $d(\tau_i) = c_i < \infty$ and $d(\tau) = c < \infty$. To complete the proof, we need to show that $c_i \to c$. Suppose otherwise. Then, taking a subsequence of $\{c_i\}$ if necessary, either $c_i < c - \varepsilon$ for some $\varepsilon > 0$ and all i, or $c + \varepsilon < c_i$ for some $\varepsilon > 0$ and all i. We show that both cases lead to contradiction.

Case 1: $c_i < c - \varepsilon$ for some $\varepsilon > 0$ and all i. Since $0 < c_i < c - \varepsilon$, there exists a subsequence, still denoted by $\{c_i\}$, such that $c_i \to b$ for some b. Then $(\tau_i, c_i) \to (\tau, b)$. But $(\tau_i, c_i) \in \partial\Omega$ and $\partial\Omega$ is closed, so $(\tau, b) \in \partial\Omega$. Since $b < c$, this contradicts $|R_\tau| = c$.

Case 2: $c+\varepsilon < c_i$ for some $\varepsilon > 0$ and all i. For all $x \le c+\varepsilon$, we have $(\tau_i, x) \in \Omega$ and so $(\tau_i, x) \to (\tau, x) \in \bar{\Omega}$. In particular $(\tau, c), (\tau, c+\varepsilon) \in \bar{\Omega}$. Since Ω is strictly convex, by (2.3) this implies that $\left(\tau, c+\frac{\varepsilon}{2}\right) \in \Omega$. This contradicts $|R_\tau| = c$.

By contradictions in both cases, we conclude that $c_i \to c$. So $d(\tau_i) \to d(\tau)$; that is, d is continuous at τ. Because $\tau \in K$ is arbitrary, the proposition follows. □

3. Geometric Quantization

In this section, we study T-invariant Kähler forms ω on the Reinhardt domain X and construct the corresponding unitary T-representations H_ω.

An interesting class of ω consists of those for which the T-action is Hamiltonian [6, Sec. 26]. In this case ω has moment map $\Phi\colon X \to \mathbf{R}^n$, where $\mathbf{R}^n$ is regarded as the Lie algebra of T as well as its dual space. The next theorem shows that this condition is equivalent to $\omega = \sqrt{-1}\partial\bar{\partial}F$ for some F. We shall always use the coordinates $z = x + \sqrt{-1}y$ on X as introduced in (1.1).

THEOREM 3.1. *Let ω be a T-invariant Kähler form on X. The following conditions are equivalent*:

(i) *$\omega = d\beta$ for some real 1-form β*;
(ii) *$\omega = \sqrt{-1}\partial\bar{\partial}F$ for some real-valued function F*;
(iii) *the T-action preserving ω is Hamiltonian*.

Proof. We first show that (i) implies (ii), so suppose that $\omega = d\beta$. Since β is real, we can write $\beta = \alpha + \bar{\alpha}$, where α is a $(0,1)$-form on X. Then $\omega = d\beta = d\alpha + d\bar{\alpha}$. But since ω is of type $(1,1)$, $\bar{\partial}\alpha = \partial\bar{\alpha} = 0$. Hence

$$\omega = \partial\alpha + \bar{\partial}\bar{\alpha}. \tag{3.1}$$

We claim that

$$\alpha = \bar{\partial} f \tag{3.2}$$

for some complex-valued function f. Write $\alpha = \sum_i h_i d\bar{z}_i$, where $h_i(z) = h_i(x)$ by T-invariance. Then

$$0 = \bar{\partial}\alpha = \sum_{ij} \frac{\partial h_i}{\partial \bar{z}_j} d\bar{z}_j \wedge d\bar{z}_i = \frac{1}{2}\sum_{ij} \frac{\partial h_i}{\partial x_j} d\bar{z}_j \wedge d\bar{z}_i.$$

Hence $\frac{\partial h_i}{\partial x_j} = \frac{\partial h_j}{\partial x_i}$ for all $i \neq j$. Equivalently, $\gamma = 2\sum_i h_i(x)dx_i$ is a closed 1-form on Ω. Since Ω is strictly convex, its de Rham cohomology satisfies $H^1(\Omega) = 0$. So $\gamma = df$ for some f, namely $2h_i = \frac{\partial f}{\partial x_i}$. Extend f to X by T-invariance, and let $h_i = \frac{1}{2}\frac{\partial f}{\partial x_i} = \frac{\partial f}{\partial \bar{z}_i}$. It follows that $\bar{\partial} f = \sum_i \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i = \sum_i h_i d\bar{z}_i = \alpha$. This proves (3.2) as claimed.

By the way, (3.2) says that the T-invariant Dolbeault cohomology of X vanishes at degree $(0,1)$. The argument would have been simpler in the special case $\Omega = \mathbf{R}^n$, since then $X = \mathbf{R}^n \times (\mathbf{R}^n/\mathbf{Z}^n)$ is equivalent to the complex group $(\mathbf{C}^\times)^n$, which is a Stein manifold and thus has trivial Dolbeault cohomology.

Let F be the real-valued function $F = \sqrt{-1}(-f + \bar{f})$. Then

$$\begin{aligned}\sqrt{-1}\partial\bar{\partial}F &= \partial\bar{\partial}f - \partial\bar{\partial}\bar{f} \\ &= \partial\alpha + \bar{\partial}\bar{\alpha} \quad \text{(by (3.2))} \\ &= \omega \qquad\qquad \text{(by (3.1)).}\end{aligned}$$

We have proved that (i) implies (ii).

To show that (ii) implies (i), suppose $\omega = \sqrt{-1}\partial\bar{\partial}F$ for some real-valued function F. Let β be the real part of the $(0,1)$-form $\sqrt{-1}\bar{\partial}F$, namely $\beta = \big(\sqrt{-1}/2\big)(\bar{\partial}F - \partial F)$. Then $d\beta = \big(\sqrt{-1}/2\big)d(\bar{\partial}F - \partial F) = \sqrt{-1}\partial\bar{\partial}F$. This proves that (ii) implies (i).

We next show that (i) implies (iii), so suppose $\omega = d\beta$. Since T is compact, we can take β to be T-invariant. By [1, Thm. 4.2.10], the T-action is Hamiltonian. In fact, a moment map Φ is given by $(\Phi(z), \xi) = -(\beta, \xi^\sharp)(z)$, where $z \in X$, $\xi \in \mathbf{R}^n$, and $\xi^\sharp$ is the infinitesimal vector field on X. Hence (i) implies (iii).

To complete the proof of the theorem, it remains to show that (iii) implies (i). If $\xi_1, \ldots, \xi_n$ is the standard basis of the Lie algebra $\mathbf{R}^n$ of T, then their infinitesimal vector fields on X are $\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}$. Suppose (iii) holds; that is, suppose the T-action preserving ω is Hamiltonian. Then, for each i, we have that $\iota\big(\frac{\partial}{\partial y_i}\big)\omega$ is an exact 1-form. Hence there exists $\phi_i \in C^\infty(X)$ such that $d\phi_i = \iota\big(\frac{\partial}{\partial y_i}\big)\omega$. Therefore, ω has the expression

$$\omega = \sum_i dy_i \wedge d\phi_i + \sum_{kl} a_{kl}\, dx_k \wedge dx_l. \tag{3.3}$$

Since T is abelian, $\frac{\partial}{\partial y_i}$ is T-invariant and so is $d\phi_i$. By compactness of T, we may take ϕ_i to be T-invariant, too. Thus $\phi_i(z) = \phi_i(x)$, and so (3.3) becomes

$$\omega = -\sum_{ij} \frac{\partial\phi_i}{\partial x_j} dx_j \wedge dy_i + \sum_{kl} a_{kl}\, dx_k \wedge dx_l. \tag{3.4}$$

Since ω is of type $(1,1)$ and (3.4) has no term involving $dy_k \wedge dy_l$, it follows that $\sum_{kl} a_{kl}\, dx_k \wedge dx_l = 0$. We get

$$\omega = -\sum_{ij} \frac{\partial\phi_i}{\partial x_j} dx_j \wedge dy_i = d\Big(-\sum_i \phi_i dy_i\Big),$$

and (i) follows. This proves the theorem. □

This theorem provides the correction for [3, Thm. 1.1], which mistakenly assumes that these equivalent conditions are always valid. We illustrate Theorem 3.1 by the following examples with $n = 2$:

$$\begin{aligned}\omega_1 &= \sqrt{-1}\textstyle\sum_1^2 dz_i \wedge d\bar{z}_i \\ &= 2dx_1 \wedge dy_1 + 2dx_2 \wedge dy_2; \\ \omega_2 &= \omega_1 + \tfrac{1}{2}(dz_1 \wedge d\bar{z}_2 + d\bar{z}_1 \wedge dz_2) \\ &= 2dx_1 \wedge dy_1 + 2dx_2 \wedge dy_2 + dx_1 \wedge dx_2 + dy_1 \wedge dy_2.\end{aligned}$$

Computations show that ω_1 and ω_2 are Kähler. Here ω_1 satisfies the equivalent conditions of Theorem 3.1: $\omega_1 = d\big(\sum_1^2 2x_i\, dy_i\big) = \sqrt{-1}\partial\bar{\partial}\big(2\sum_1^2 x_i^2\big)$. Also, an infinitesimal vector field ξ of T on X is a linear combination of $\frac{\partial}{\partial y_i}$, so $\iota(\xi)\omega_1$ is a linear combination of dx_i; it is exact and so the action is Hamiltonian. On the other hand, ω_2 does not satisfy the conditions of Theorem 3.1: the term $dy_1 \wedge dy_2$ is not exact. Similarly $\iota\big(\frac{\partial}{\partial y_1}\big)\omega_2 = -2dx_1 + dy_2$ is not exact owing to dy_2, so the action is not Hamiltonian.

It is convenient to work with the class of Kähler forms given by Theorem 3.1, so from now on we shall always assume that ω belongs to this class. Thus $\omega = \sqrt{-1}\partial\bar{\partial}F$ and by T-invariance, F is a function on Ω; namely, $F(z) = F(x)$. Then $\frac{\partial^2 F}{\partial z_i \partial \bar{z}_j} = \frac{1}{4}\frac{\partial^2 F}{\partial x_i \partial x_j}$, and positivity of ω implies that the Hessian matrix $\big(\frac{\partial^2 F}{\partial x_i \partial x_j}\big)_{ij}$ is positive definite, so F is strictly convex. This allows us to apply the convex analysis developed in Section 2 to F.

We next describe the construction of a unitary T-representation H_ω out of ω. This is the scheme outlined in [5] and [7]. There exists a holomorphic line bundle $\mathbf{L}$ over X whose Chern class is the integral de Rham cohomology class of ω. Because ω is exact, $[\omega] = 0$ and so $\mathbf{L}$ is a trivial line bundle. Further, $\mathbf{L}$ is equipped with a connection ∇ (whose curvature is ω) and with an invariant Hermitian structure $\langle\cdot,\cdot\rangle$. A smooth section s of $\mathbf{L}$ is said to be holomorphic if $\nabla_v s = 0$ for all antiholomorphic vector fields v. The arguments in [3, Sec. 3] extend to the following proposition.

Proposition 3.2. *There exists a unique nonvanishing T-invariant holomorphic section s_0 of $\mathbf{L}$ that satisfies $\langle s_0, s_0\rangle = e^{-F}$.*

The uniqueness of s_0 in the statement is up to multiplication by complex numbers of absolute value 1. From (1.1), X has a natural T-invariant measure obtained from the product of the Lebesgue measure on Ω and the Haar measure on T. We write dV for the canonical measure on X or Ω. If a section s of $\mathbf{L}$ satisfies (1.2), we say that it is square-integrable. The square-integrable holomorphic sections are denoted by H_ω. Since $\langle\cdot,\cdot\rangle$ and dV are T-invariant, H_ω becomes a unitary T-representation. Since T is abelian, the irreducible subrepresentations of H_ω are 1-dimensional. In view of Proposition 3.2, they are characterized by $\lambda \in \mathbf{Z}^n$ via $\{ce^{\lambda z}s_0 : c \in \mathbf{C}\}$. Here and in what follows, if $z \in \mathbf{C}^n$ then we write $\lambda z \in \mathbf{C}$ to denote $\sum_i \lambda_i z_i$. For $s = e^{\lambda z}s_0$, the condition (1.2) on the square-integrability becomes

$$\int_X \langle e^{\lambda z}s_0, e^{\lambda z}s_0\rangle\, dV = \int_\Omega e^{2\lambda x - F(x)}\, dV.$$

Therefore, H_ω is a toric model if and only if

$$\int_\Omega e^{2\lambda x - F(x)}\, dV < \infty \quad \text{for all } \lambda \in \mathbf{Z}^n. \tag{3.5}$$

For the rest of this article, we utilize condition (3.5) to study the conditions for H_ω to be a toric model. Namely, for a strictly convex function $F \in C^\infty(\Omega)$, we consider the necessary and sufficient conditions for (3.5).

4. Rays

In this section we show that conditions (A) and (B) of the main theorem are equivalent. Let $\Omega \subset \mathbf{R}^n$ be a strictly convex domain. Recall from (1.3) that a ray in Ω is a subset of the form $\{p + tv : t > 0\} \cap \Omega$. We say that it is bounded or unbounded depending on whether $p + tv$ gets out of Ω for large t. For example, if $\Omega \subset \mathbf{R}^2$ consists of all the points above the graph of $y = x^2$, then a ray is unbounded if and only if it is parallel to the vector $(0, 1)$. If $F \in C^\infty(\Omega)$ is a strictly convex function, then so is its restriction $F|_R$ to a ray R. In this case $F|_R$ resembles a function $F\colon [a, b) \to \mathbf{R}$, where b can be a number or ∞, depending on whether the ray is bounded or unbounded. For this reason, Lemma 2.4 and Corollary 2.5 can be rephrased in terms of rays in Ω. We say that F is decreasing along the ray $\{p + tv : t > 0\} \cap \Omega$ if $F(p + t_1 v) > F(p + t_2 v)$ whenever $t_1 < t_2$.

Observe that, in both Lemma 2.4 and Corollary 2.5, one side of the equivalent conditions is independent of n. This means that if the corresponding condition holds for one n then it holds for every other n. The expression x^n reflects the effect of the Lebesgue measure dV of Ω. Namely, if $(\theta, r) \in S^{n-1} \times \mathbf{R}^+$ are the polar coordinates centered at some $p \in \Omega$, then

$$dV = r^{n-1}\, dr\, d\theta,$$

where $d\theta$ is the measure on S^{n-1} that is invariant under the orthogonal group $O(n)$. So if $R \subset \Omega$ is a ray with initial point p, we shall be interested in the measure $r^{n-1}\, dr$ on R, where r is the distance from p. For our purpose, it is actually irrelevant whether we use dr or $r^{n-1}\, dr$ when we integrate over a ray R. This is because we shall often consider functions of the type e^{-F} over R, where F is strictly convex. Then Lemma 2.4 says that $\int_R e^{-F}\, dr$ and $\int_R e^{-F} r^{n-1}\, dr$ either converge or diverge simultaneously. Lemma 2.4 can be rewritten as follows.

LEMMA 4.1. *Let $F \in C^\infty(\Omega)$ be strictly convex, and let $R \subset \Omega$ be a ray. The following conditions are equivalent*:

(i) *$\int_R e^{-F} r^{n-1}\, dr$ diverges*;

(ii) *R is unbounded and F strictly decreases along R.*

If F is restricted to R then the corresponding directional derivative is denoted by $(F|_R)'$. In other words, $(F|_R)'\colon R \to \mathbf{R}$ is given by $(F|_R)'(r) = (u \cdot F')(r)$ for all $r \in R$, where u is the unit vector parallel to R. Corollary 2.5 also takes on the following format.

COROLLARY 4.2. *Let $R \subset \Omega$ be an unbounded ray. Then $\int_R e^{-F(r)+cr} r^{n-1}\, dr < \infty$ if and only if $c <$ l.u.b.$(F|_R)'$.*

Two rays $R, S \subset \Omega$ are said to be parallel if there exist $p, q \in \Omega$ and $v \in \mathbf{R}^n$ such that $R = \{p + tv : t > 0\} \cap \Omega$ and $S = \{q + tv : t > 0\} \cap \Omega$.

PROPOSITION 4.3. *Let R and S be parallel rays in Ω, and let $F \in C^\infty(\Omega)$ be strictly convex. Then the two equivalent conditions of Lemma 4.1 hold for R if and only if they hold for S. When this happens,* l.u.b.$(F|_R)' =$ l.u.b.$(F|_S)' < \infty$.

Proof. Suppose that R is unbounded and that F is strictly decreasing along R. We want to show that S is unbounded and F is strictly decreasing along S.

Let p, q denote (respectively) the initial points of R, S. Since R is unbounded, there exists $v \in \mathbf{R}^n$ such that $p + tv \in R$ for all $t > 0$. Define $S_1 \subset \mathbf{R}^n$ by

$$S_1 = \{q + tv : t > 0\}, \quad S_1 \cap \Omega = S. \tag{4.1}$$

We claim that $S = S_1$. Let C be the convex hull of $R \cup \{q\}$. Since Ω is convex, $C \subset \Omega$. Since R is unbounded,

$$S_1 \subset \bar{C} \subset \bar{\Omega}. \tag{4.2}$$

Pick $b \in S_1$. It lies in the line segment joining some $a, c \in S_1$. By (4.2), $a, c \in \bar{\Omega}$. By Corollary 2.2, Ω is geometrically strictly convex, so $b \in \Omega$. This implies that $b \in S$. Hence $S = S_1$ as claimed. We conclude that S is unbounded.

We also want to show that F decreases along S. Recall from Proposition 2.3 that $F^{-1}(-\infty, m]$ is geometrically convex. Let $m = \max\{F(p), F(q)\}$. Let C again be the convex hull of $R \cup \{q\}$. Then

$$\begin{aligned}
&\{p, q\} \subset F^{-1}(-\infty, m] \\
&\quad \Longrightarrow \; R \cup \{q\} \subset F^{-1}(-\infty, m] && \text{(since } F \text{ decreases along } R\text{)} \\
&\quad \Longrightarrow \; C \subset F^{-1}(-\infty, m] && \text{(since } F^{-1}(-\infty, m] \text{ is geometrically convex)} \\
&\quad \Longrightarrow \; \bar{C} \subset F^{-1}(-\infty, m] && \text{(since } F^{-1}(-\infty, m] \text{ is closed)} \\
&\quad \Longrightarrow \; S \subset F^{-1}(-\infty, m] && \text{(by (4.1) and (4.2)).}
\end{aligned}$$

This means that F does not tend to ∞ along S. Since S is unbounded and F is strictly convex, this can happen only if F is strictly decreasing along S. This proves the first part of the proposition.

For the rest of the proof, suppose that the equivalent conditions of Lemma 4.1 hold for R and S. Since F is decreasing along R, it follows that $(F|_R)' < 0$ everywhere on R. Hence $\text{l.u.b.}(F|_R)' < \infty$. Similarly, $\text{l.u.b.}(F|_S)' < \infty$. It remains to show that the least upper bounds are equal. Given $c \in \mathbf{R}$, we have

$$\begin{aligned}
&c < \text{l.u.b.}(F|_R)' \\
&\qquad \Longleftrightarrow \; \int_R e^{-F(r)+cr} r^{n-1}\, dr < \infty && \text{(by Corollary 4.2)} \\
&\qquad \Longleftrightarrow \; \int_S e^{-F(r)+cr} r^{n-1}\, dr < \infty && \text{(by the first part of this proposition)} \\
&\qquad \Longleftrightarrow \; c < \text{l.u.b.}(F|_S)' && \text{(by Corollary 4.2).}
\end{aligned}$$

We conclude that $\text{l.u.b.}(F|_R)' = \text{l.u.b.}(F|_S)'$, and the proposition follows. $\square$

Fix $p \in \Omega$, and let (θ, r) be the polar coordinates centered at p. For $\theta \in S^{n-1}$, let $R_\theta \subset \Omega$ be the ray with initial point p in the θ-direction. Given $U \subset S^{n-1}$, let $C(U)$ denote the cone

$$C(U) = \bigcup_{\theta \in U} R_\theta = \{(\theta, r) \in \Omega : \theta \in U\}. \tag{4.3}$$

Proposition 4.4. *Suppose that $\int_R e^{-F} r^{n-1}\, dr < \infty$ for some ray R with initial point p. Then there exists an open set $U \subset S^{n-1}$ such that $R \subset C(U)$ and $\int_{C(U)} e^{-F}\, dV < \infty$.*

Proof. Write $R = R_\phi$ for $\phi \in S^{n-1}$, and suppose that $\int_{R_\phi} e^{-F} r^{n-1} dr < \infty$. By Lemma 4.1, either R_ϕ is bounded or F eventually increases along R_ϕ. We discuss these two cases separately. Recall that a compact neighborhood K of ϕ is a compact set K such that p lies in the interior of K.

Case 1: R_ϕ is bounded. By Proposition 2.6, ϕ has a compact neighborhood $K \subset S^{n-1}$ such that $|R_\theta| < \infty$ for all $\theta \in K$. Let $c > 0$ be small enough so that the ball $B_c(p) = \{x \in \mathbf{R}^n : |x - p| \leq c\}$ is contained in Ω. Let $S_c(p) = \partial B_c(p)$ be its boundary. By compactness of $S_c(p)$, we can define

$$-\infty < m = \min\left\{(F|_{R_\theta})'(c) = \frac{\partial F}{\partial r}(\theta, c) : \theta \in S^{n-1}\right\}. \tag{4.4}$$

By compactness of $B_c(p)$, there exists a sufficiently large b such that

$$mr - b < F(\theta, r), \quad (\theta, r) \in B_c(p). \tag{4.5}$$

Define $G(\theta, r) = mr - b$. It is continuous and is smooth everywhere except at p. By (4.4) and (4.5), $G(\theta, r) < F(\theta, r)$ for all $(\theta, r) \in \Omega$, because F is strictly convex along each ray R_θ. Then

$$\int_{C(K)} e^{-F}\, dV < \int_{C(K)} e^{-G}\, dV < e^b \int_{C(K)} e^{-mr}\, dV. \tag{4.6}$$

Since K is compact and $\theta \mapsto |R_\theta|$ is continuous on K (Proposition 2.6), we can define $M = \max\{|R_\theta| : \theta \in K\} < \infty$. The last integral of (4.6) becomes

$$e^b \int_{C(K)} e^{-mr}\, dV \leq e^b \int_K d\theta \int_0^M e^{-mr} r^{n-1} dr < \infty.$$

Let U be the interior of K, and the proposition is proved for Case 1.

Case 2: F eventually increases along R_ϕ. There exists a $c > 0$ such that $\frac{\partial F}{\partial r}(\phi, c) > 0$. Let $K \subset S^{n-1}$ be a compact neighborhood of ϕ such that $(\theta, c) \in \Omega$ and $\frac{\partial F}{\partial r}(\theta, c) > 0$ for all $\theta \in K$. By compactness of K, we can define

$$0 < m = \min\left\{\frac{\partial F}{\partial r}(\theta, c) : \theta \in K\right\}. \tag{4.7}$$

We repeat the arguments of (4.5) through (4.6). Namely, by compactness of the set $\{(\theta, r) \in \Omega : \theta \in K, r \leq c\}$, there exists a b such that $mr - b < F(\theta, r)$ whenever $\theta \in K$ and $r \leq c$. This, together with (4.7) and strict convexity of F along each R_θ, implies that $mr - b < F(\theta, r)$ for all $(\theta, r) \in C(K)$. Then

$$\begin{aligned}\int_{C(K)} e^{-F}\, dV &< e^b \int_{C(K)} e^{-mr}\, dV \\ &\leq e^b \int_{S^{n-1}\times\mathbf{R}^+} e^{-mr}\, dV \\ &= e^b \int_{S^{n-1}} d\theta \int_0^\infty e^{-mr} r^{n-1} dr.\end{aligned} \tag{4.8}$$

The last expression converges because $0 < m$. Let U be the interior of K, and the proposition is proved for Case 2. □

Propositions 4.3 and 4.4 lead to the following corollary. It implies that conditions (A) and (B) of the main theorem are equivalent.

Corollary 4.5. *$\int_\Omega e^{-F}\, dV < \infty$ if and only if $\int_R e^{-F} r^{n-1}\, dr < \infty$ for all rays $R \subset \Omega$.*

Proof. Suppose that $\int_R e^{-F} r^{n-1}\, dr < \infty$ for all rays $R \subset \Omega$. Pick $p \in \Omega$. Let (θ, r) be the polar coordinates centered at p, and denote the rays with initial point p by R_θ, where $\theta \in S^{n-1}$. Given $U \subset S^{n-1}$, we define the cone $C(U)$ as in (4.3). Since $\int_{R_\theta} e^{-F} r^{n-1}\, dr < \infty$, by Proposition 4.4 it follows that θ has an open neighborhood $U_\theta \subset S^{n-1}$ such that $\int_{C(U_\theta)} e^{-F}\, dV < \infty$. Since S^{n-1} is compact, there exist $\theta_1, \dots, \theta_k$ such that $U_{\theta_1} \cup \cdots \cup U_{\theta_k} = S^{n-1}$, and so $C(U_{\theta_1}) \cup \cdots \cup C(U_{\theta_k}) = \Omega$. Then $\int_\Omega e^{-F}\, dV \le \sum_{i=1}^k \int_{C(U_{\theta_i})} e^{-F}\, dV < \infty$.

Conversely, suppose that $\int_R e^{-F} r^{n-1}\, dr$ diverges for a ray R. By Lemma 4.1, R is unbounded and F decreases along R. By Proposition 4.3, the same situations occur for all rays parallel to R. We write Ω as a union of the rays parallel to R and then find that e^{-F} is increasing along each of these unbounded rays. Therefore, $\int_\Omega e^{-F}\, dV$ diverges. This proves the corollary. □

Thus, for every strictly convex function $F \in C^\infty(\Omega)$, the integrability of $e^{-F}\, dV$ over Ω is equivalent to the integrability of $e^{-F} r^{n-1}\, dr$ over every ray in Ω. This means that checking convergence of (3.5) for a toric model is equivalent to checking condition (B) of the main theorem.

5. Epigraph

In this section, we consider the epigraph (1.4) of F and show that conditions (B) and (C) of the main theorem are equivalent.

Since F is strictly convex, $\operatorname{epi}(F)$ is geometrically strictly convex; namely, it satisfies (2.3). So for every line $L \subset \mathbf{R}^n \times \mathbf{R}$, either $L \cap \operatorname{epi}(F)$ is empty or it is connected. However, $L \cap \operatorname{epi}(F)$ may or may not be bounded. The next proposition shows that this boundedness property determines whether H_ω is a toric model. The line L is said to be nonvertical if it is not parallel to the axis $0 \times \mathbf{R} \subset \mathbf{R}^n \times \mathbf{R}$.

Proposition 5.1. *There exists a nonvertical line $L \subset \mathbf{R}^n \times \mathbf{R}$ such that $L \cap \operatorname{epi}(F)$ is unbounded if and only if condition* (B) *of the main theorem fails.*

Proof. Suppose that $L \cap \operatorname{epi}(F)$ is unbounded for some line L. Since L is nonvertical, it can be regarded as the graph of an affine function. Namely, there exists an unbounded ray $R \subset \Omega$ and an affine function $G\colon \Omega \to \mathbf{R}$ such that the graph of the restricted function $G|_R$ is L. Let r be the distance from the initial point of R. Then, for large r, $G(r)$ corresponds to the unbounded portion of $L \cap \operatorname{epi}(F)$. That is, there exists an r_0 such that $G(r) > F(r)$ for all $r \in (r_0, \infty)$. Since G is an affine function, there exist m, c such that $G(r) = mr + c$ for all $r \in R$. Thus $F(r) < mr + c$ for all $r > r_0$. Equivalently, there exists a $\lambda \in \mathbf{Z}^n$ such that $F(r) < \lambda r + c$ for all $r > r_0$ in R. Then

$$\int_R e^{-F(r)+\lambda r} r^{n-1}\,dr > \int_{r_0}^{\infty} e^{-F(r)+\lambda r} r^{n-1}\,dr = e^{-c}\int_{r_0}^{\infty} e^{-F(r)+\lambda r+c} r^{n-1}\,dr.$$

The final expression diverges because $e^{-F(r)+\lambda r+c} > 1$ for all $r > r_0$. Therefore, condition (B) of the main theorem fails.

Conversely, suppose that condition (B) of the main theorem fails. Then there exist a ray $R \subset \mathbf{R}^n$ and a weight $\lambda \in \mathbf{Z}^n$ such that $\int_R e^{-F(r)+\lambda r} r^{n-1}\,dr = \infty$. By Lemma 4.1, R is unbounded and the function $F(r)-\lambda r$ is strictly decreasing along R. For suitable $m > 0$ and c, we can construct an affine function $G(r) = mr + c$ on R such that $G(r_0) = F(r_0)$ for some r_0. Since G is increasing along R, it follows that $G(r) > F(r) - \lambda r$ for all $r > r_0$. Then the graph of G is a line whose intersection with $\operatorname{epi}(F)$ is unbounded. This proves the proposition. □

By Proposition 5.1, it follows that conditions (B) and (C) of the main theorem are equivalent.

6. Gradient Function

In this section we investigate condition (D) of the main theorem. Because $F \in C^\infty(\Omega)$ is strictly convex, the gradient function F' is a diffeomorphism of Ω onto its image Δ. Condition (D) is equivalent to saying that $(F')^{-1}$ maps bounded sets to bounded sets. We shall show that $\int_R e^{-F(r)+\lambda r} r^{n-1}\,dr < \infty$ for every ray R and weight λ if and only if $(F')^{-1}$ maps every bounded sequence to a bounded sequence. This will then establish the equivalence of conditions (B) and (D). The two directions of these equivalent conditions are given by Propositions 6.1 and 6.3.

PROPOSITION 6.1. *Suppose that $\int_R e^{-F(r)+\lambda r} r^{n-1}\,dr$ diverges for some $\lambda \in \mathbf{Z}^n$ and ray R. Then there exists a bounded sequence $\{s_i\} \subset \Delta$ such that the corresponding sequence $\{(F')^{-1}(s_i)\}$ is unbounded.*

Proof. Suppose that $\int_R e^{-F(r)+\lambda r} r^{n-1}\,dr$ diverges for some weight λ and ray R. For simplicity, write $G(x) = F(x) - \lambda x$. Clearly G' is also one-to-one, and its image is $\Delta - \lambda$. By Lemma 4.1, R is unbounded and G decreases along R. We rotate and shift the coordinates so that R is the positive x^1-axis; thus, for this proposition (only), we write $x \in \mathbf{R}^n$ as $x = (x^1, x^2) \in \mathbf{R} \times \mathbf{R}^{n-1}$. The rays parallel to R are called horizontal rays.

By Proposition 4.3, all the horizontal rays S are unbounded and their directional derivatives $(G|_S)'$ have a common least upper bound $u^1 \in \mathbf{R}$. Hence there exists a $u^2 \in \mathbf{R}^{n-1}$ such that $u = (u^1, u^2) \in \mathbf{R}^n$ is a limit point of $\Delta - \lambda$. Let $\{q_i\} \subset \Delta - \lambda$ be a sequence converging to u. If we now write $q_i = (q_i^1, q_i^2)$, then

$$\lim_{i\to\infty} q_i^1 = u^1. \tag{6.1}$$

Let $p_i = (p_i^1, p_i^2) \in \Omega$ be the corresponding sequence satisfying $G'(p_i) = q_i$.

We claim that $\{p_i\} \subset \Omega$ is an unbounded sequence. Suppose otherwise; we derive a contradiction from here. If $\{p_i\}$ is bounded, then so are $\{p_i^1\}$ and $\{p_i^2\}$. Hence there exists an $m \in \mathbf{R}$ such that

$$p_i^1 < m \tag{6.2}$$

for all i, and

$$A = \{p_i^2\}_i \subset \mathbf{R}^{n-1} \tag{6.3}$$

is bounded. Recall that, by Proposition 4.3, every horizontal ray is unbounded. Therefore, the infinite horizontal cylinder $C = \{(t, A) : t > m\}$ is contained in Ω. By Corollary 2.2, Ω is geometrically strictly convex and so

$$\bar{C} = \{(t, \bar{A}) : t \geq m\} \subset \Omega.$$

Let $a \in \bar{A}$. Since G is strictly convex, the function $t \mapsto \frac{\partial G}{\partial x^1}(t, a)$ is increasing. By Proposition 4.3, $\frac{\partial G}{\partial x^1}(t, a) \to u^1$ as $t \to \infty$. Since $\bar{A}$ is compact, there exists a $c > 0$ such that

$$t < m,\ a \in \bar{A} \implies \frac{\partial G}{\partial x^1}(t, a) < u^1 - c. \tag{6.4}$$

Then (6.2)–(6.4) imply that $\frac{\partial G}{\partial x^1}(p_i) < u^1 - c$ for all i. In other words,

$$q_i^1 < u^1 - c$$

for all i. This contradicts (6.1). By this contradiction, we conclude that $\{p_i\}$ is an unbounded sequence as claimed.

Recall that $q_i \in \Delta - \lambda$. Let $s_i = q_i + \lambda \in \Delta$. Then $F'(p_i) = G'(p_i) + \lambda = q_i + \lambda = s_i \in \Delta$. So $\{s_i\}$ is a convergent sequence in Δ such that $\{(F')^{-1}(s_i) = p_i\}$ is unbounded. This proves the proposition. $\square$

To obtain the converse of Proposition 6.1, we need the following lemma.

LEMMA 6.2. *Suppose that there exists an unbounded sequence $\{p_i\} \subset \Omega$ such that $F'(p_i) \to 0$. Then there exists an unbounded ray $R \subset \Omega$ such that $\int_R e^{-F} r^{n-1}\, dr$ diverges.*

Proof. Let $p_1, p_2, \ldots$ be an unbounded sequence in Ω, and let $F'(p_i) \to 0$. Fix $p \in \Omega$, where $p \neq p_i$. We shall construct an unbounded ray with initial point p. Since $\{p_i\}$ is unbounded, we may assume (taking a subsequence if necessary) that $|p_i - p| \to \infty$. Let (θ, r) be the polar coordinates centered at p, so that we can parameterize the rays with initial point p by R_θ, $\theta \in S^{n-1}$. Let R_{ϕ_i} be the ray with initial point p and containing p_i. By compactness of S^{n-1}, a subsequence of $\{\phi_i\} \subset S^{n-1}$ converges to some $\phi \in S^{n-1}$. We may assume that $\phi_i \to \phi$.

We claim that R_ϕ is unbounded. For all $r > 0$, (ϕ, r) is a limit point of $\bigcup_i R_{\phi_i} \subset \Omega$ because $|R_{\phi_i}| \to \infty$. Hence $(\phi, r) \in \bar{\Omega}$. Since Ω is geometrically convex, it follows from $(\phi, r \pm \varepsilon) \in \bar{\Omega}$ that $(\phi, r) \in \Omega$. So R_ϕ is unbounded as claimed.

We also claim that F decreases along R_ϕ. Suppose otherwise. Then there exists a sufficiently large t such that $\frac{\partial F}{\partial r}(\phi, t) > 0$. Let $K \subset S^{n-1}$ be a compact neighborhood of ϕ such that $(\theta, t) \in \Omega$ and $\frac{\partial F}{\partial r}(\theta, t) > 0$ for all $\theta \in K$. By compactness of K, we can define

$$0 < m = \min\left\{\frac{\partial F}{\partial r}(\theta, t) : \theta \in K\right\}.$$

Since F is strictly convex along each R_θ, it follows that

$$0 < m \leq \frac{\partial F}{\partial r}(\theta, r) \quad \text{for all } \theta \in K,\ r \geq t. \tag{6.5}$$

Write $p_i = (\phi_i, t_i)$. Since $\phi_i \to \phi$, it follows that $\phi_i \in K$ eventually. Also, $t_i \geq t$ eventually as $|p_i - p| \to \infty$. Then the condition $F'(p_i) \to 0$ contradicts (6.5). We conclude that F decreases along R_ϕ as claimed.

Since R_ϕ is unbounded and F decreases along R_ϕ, it follows by Lemma 4.1 that $\int_{R_\phi} e^{-F} r^{n-1}\, dr$ diverges. □

This lemma leads to the converse of Proposition 6.1. Let $F' \colon \Omega \to \Delta$ be the diffeomorphism as before.

PROPOSITION 6.3. *Suppose that there exists a bounded sequence $\{q_i\} \subset \Delta$ in which $\{(F')^{-1}(q_i)\}$ is unbounded. Then there exists a ray $R \subset \Omega$ and a $\lambda \in \mathbf{Z}^n$ such that $\int_R e^{-F(r)+\lambda r} r^{n-1}\, dr$ diverges.*

Proof. Let $\{q_i\} \subset \Delta$ be a bounded sequence in which $p_i = (F')^{-1}(q_i)$ is unbounded. Taking a subsequence if necessary, we may assume that $q_i \to q \in \bar{\Delta}$. Write $G(x) = F(x) - qx$, so that $G'(p_i) \to 0$. Since $\{p_i\}$ is unbounded, Lemma 6.2 says that there exists an unbounded ray $R \subset \Omega$ such that

$$\int_R e^{-F(r)+qr} r^{n-1}\, dr = \int_R e^{-G(r)} r^{n-1}\, dr = \infty. \tag{6.6}$$

For such R, we can replace $q \in \mathbf{R}^n$ with some $\lambda \in \mathbf{Z}^n$ such that $qr < \lambda r$ for all $r \in R$ that are sufficiently far. Then (6.6) says that $\int_R e^{-F(r)+\lambda r} r^{n-1}\, dr$ diverges. The proposition follows. □

By Propositions 6.1 and 6.3, we have established the equivalence of conditions (B) and (D) of the main theorem.

7. Concavity of Boundary

Recall that, since $F \in C^\infty(\Omega)$ is strictly convex, its gradient $F' \colon \Omega \to \Delta$ is a diffeomorphism. Condition (D) is equivalent to saying that $(F')^{-1}$ sends bounded sets to bounded sets, which clearly implies (E). The purpose of this section is to show that, conversely, (E) also implies (D). Namely, if $\{q_i\}$ is a sequence converging to $q \in \partial\Delta$ then, in applying (D) and checking boundedness of $\{(F')^{-1}(q_i)\}$, we can ignore the case where q is concave. This will be proved in Proposition 7.3.

Note that for Δ we can only discuss geometric convexity in the sense of (1.5) or (2.3); the analytic notion of convexity (as in Proposition 2.1) is not available because Δ may not have smooth boundary. For example, if $F \in C^\infty(\mathbf{R}^2)$ is given by $F(x, y) = e^x + e^y$, then Δ is the open first quadrant and so its boundary is not smooth.

By the way, if $\Omega = \mathbf{R}^n$ then Δ has no concave boundary point.

PROPOSITION 7.1. *If $\Omega = \mathbf{R}^n$, then Δ is geometrically convex.*

Proof. Let $\Omega = \mathbf{R}^n$. We introduce the notation $F_c \in C^\infty(\mathbf{R}^n)$ for $c \in \mathbf{R}^n$, given by $F_c(x) = F(x) - cx$. Then F_c is strictly convex. Note that F_c has a global minimum if and only if $c \in \Delta$. Also, F_c has a global minimum if and only if $F_c(x) \to \infty$ whenever $|x| \to \infty$. This observation makes use of $\Omega = \mathbf{R}^n$.

Suppose that $r, s \in \Delta$ and $q = tr + (1-t)s$ for some $0 < t < 1$. We want to show that $q \in \Delta$. Since $r, s \in \Delta$, the strictly convex functions F_r and F_s have global minima. The same is true for tF_r and $(1-t)F_s$. Hence, for $x \in \mathbf{R}^n$,

$$|x| \to \infty \implies tF_r(x),\ (1-t)F_s(x) \to \infty. \tag{7.1}$$

Direct computation shows that

$$F_q = tF_r + (1-t)F_s. \tag{7.2}$$

Then (7.1) and (7.2) imply that, for $x \in \mathbf{R}^n$,

$$|x| \to \infty \implies F_q(x) \to \infty.$$

Therefore, F_q has a global minimum. Equivalently $q \in \Delta$, which proves that Δ is geometrically convex. $\square$

However, for general strictly convex domains Ω, the image set Δ may not be geometrically convex. The following lemma will be needed for Proposition 7.3.

LEMMA 7.2. *If $q \in \Delta$, then $\int_\Omega e^{-F(x)+qx}\, dV < \infty$.*

Proof. Since $q \in \Delta$, we write $F'(p) = q$ for some $p \in \Omega$. Let (θ, r) be the polar coordinates centered at p. Let $G(x) = F(x) - qx$. Then $G \in C^\infty(\Omega)$ is strictly convex and $G'(p) = 0$. Thus p is the global minimum of G. Therefore, G increases along every ray R with initial point p, namely $(G|_R)'(r) > 0$. Pick $c > 0$ small enough so that the ball $B_c(p) = \{x \in \mathbf{R}^n : |x - p| \leq c\}$ is contained in Ω. Let $S_c(p) = \partial B_c(p)$ be its boundary. The rest of the proof imitates the arguments of (4.7) through (4.8) in Proposition 4.4. By compactness of $S_c(p)$, we can define $0 < m < \min\{(G|_{R_\theta})'(c) : \theta \in S^{n-1}\}$. By compactness of $B_c(p)$, there exists a b such that $mr - b < G(\theta, r)$ for all $(\theta, r) \in B_c(p)$. By strict convexity of G along each R_θ, it follows that $mr - b < G(\theta, r)$ for all $(\theta, r) \in \Omega$. Then

$$\begin{aligned}\int_\Omega e^{-G}\, dV &< e^b \int_\Omega e^{-mr}\, dV \\ &\leq e^b \int_{S^{n-1}\times\mathbf{R}^+} e^{-mr}\, dV \\ &= e^b \int_{S^{n-1}} d\theta \int_0^\infty e^{-mr} r^{n-1}\, dr.\end{aligned}$$

The last expression converges because $m > 0$. This proves the lemma. $\square$

The converse of Lemma 7.2 is not true. For example, $e^{-F(x)+qx}$ is always integrable if Ω is bounded. In the special case where $\Omega = \mathbf{R}^n$, the converse holds [3].

PROPOSITION 7.3. *Let $q \in \partial\Delta$ be a concave boundary point. If $\{q_i\} \subset \Delta$ converges to q, then $(F')^{-1}(q_i)$ is bounded.*

Proof. Since q is concave, there exist $u, v \in \Delta$ such that q lies in the line segment joining u and v. Since $u, v \in \Delta$, Lemma 7.2 implies that

$$\int_\Omega e^{-F(x)+ux}\,dV < \infty, \qquad \int_\Omega e^{-F(x)+vx}\,dV < \infty.$$

Then Corollary 4.5 says that, for all rays R,

$$\int_R e^{-F(r)+ur} r^{n-1}\,dr < \infty, \qquad \int_R e^{-F(r)+vr} r^{n-1}\,dr < \infty. \tag{7.3}$$

We claim that, for any ray $R \subset \Omega$,

$$\int_R e^{-F(r)+qr} r^{n-1}\,dr < \infty. \tag{7.4}$$

If R is bounded, then Lemma 4.1 immediately gives (7.4) and there is nothing to prove. Suppose then that R is unbounded. Since q lies between u and v, either $qr < ur$ or $qr < vr$ for $r \in R$ that are far away. Equivalently,

$$e^{-F(r)+qr} < e^{-F(r)+ur} \quad \text{or} \quad e^{-F(r)+qr} < e^{-F(r)+vr} \tag{7.5}$$

for all $r \in R$ that are far away. By (7.3) and (7.5), $\int_R e^{-F(r)+qr} r^{n-1}\,dr < \infty$. This proves (7.4) for all the rays $R \subset \Omega$.

Now let $p_i = (F')^{-1}(q_i)$ and write $G(x) = F(x) - qx$. Then $G'(p_i) \to 0$. Suppose that $\{p_i\}$ is unbounded. Then, along with $G'(p_i) \to 0$, Lemma 6.2 says that there exists a ray R such that $\int_R e^{-G} r^{n-1}\,dr = \infty$. But this contradicts (7.4). Therefore, $\{p_i\}$ must be bounded, and the proposition follows. $\square$

By Proposition 7.3, we conclude that the boundedness of $\{(F')^{-1}(q_i)\}$ is automatic if $\{q_i\}$ approaches a concave point. Thus, conditions (D) and (E) of the main theorem are equivalent.

8. Examples

We provide the following three examples of strictly convex functions F on 1-dimensional domains Ω. Thus,

$$F : \Omega \to \mathbf{R}, \qquad \Omega = (a, b) \subset \mathbf{R}, \ -\infty \le a < b \le \infty.$$

These examples illustrate the spirits of conditions (B), (C), and (D) of the main theorem. The image Δ of the gradient function is 1-dimensional, so it has no concave boundary point. Therefore, condition (E) is irrelevant here.

EXAMPLE 8.1 $F(x) = x^2$. Whether a, b are numbers or $\pm\infty$ is not important here. Note that $\int_a^b e^{-x^2+\lambda x}\,dx$ converges for all λ. Equivalently, $L \cap \mathrm{epi}(F)$ is bounded whenever $L \subset \mathbf{R}^2$ is a nonvertical line. Since $F'(x) = 2x$, clearly $F'(U) = 2U$ is bounded if and only if U is bounded, and $(F')^{-1}$ has the same property. Hence H_ω is always a toric model.

EXAMPLE 8.2 $F(x) = e^x$.

Case 1: $a > -\infty$. Here $\int_a^b \exp(-e^x + \lambda x)\,dx$ converges for all λ. Equivalently, whenever $L \subset \mathbf{R}^2$ is a nonvertical line, $L \cap \operatorname{epi}(F)$ is always bounded. Further, $F'(U) = e^U$ is bounded if and only if U is bounded, so in this case H_ω is a toric model.

Case 2: $a = -\infty$. For $\lambda < 0$, we have that $\int_{-\infty}^{\infty} \exp(-e^x + \lambda x)\,dx$ blows up near $-\infty$. Equivalently, if L has negative slope, then $L \cap \operatorname{epi}(G)$ is unbounded toward $x \to -\infty$. The gradient function $F'(x) = e^x$ sends an unbounded set $(-\infty, c)$ to a bounded set $(0, e^c)$, so here H_ω is not a toric model.

EXAMPLE 8.3. Let $\Omega = \mathbf{R}$ and let F be a strictly convex function whose graph lies above the diagonals $\{x = y\}$ and $\{x = -y\}$, having them as asymptotes. In this case H_ω fails to be a toric model because of obstructions at both ends, $\pm\infty$. From the integral viewpoint, $\int_{-\infty}^{\infty} e^{-F(x)+\lambda x}\,dx$ blows up near ∞ if $\lambda \geq 1$ and blows up near $-\infty$ if $\lambda \leq 1$. From the epigraph viewpoint, let m be the slope of $L \subset \mathbf{R}^2$. Then $L \cap \operatorname{epi}(F)$ is unbounded toward $x \to \infty$ if $m \geq 1$ and is unbounded toward $x \to -\infty$ if $m \leq 1$. From the gradient viewpoint, observe that F' is a diffeomorphism from $\mathbf{R}$ onto $(-1, 1)$. Therefore, F' maps the unbounded sets $(-\infty, u)$ and (v, ∞) to some bounded sets $(-1, F'(u))$ and $(F'(v), 1)$.

The various obstructions at the $-\infty$ end will be removed if $a > -\infty$; similarly, the obstructions at the ∞ end will be removed if $b < \infty$.

9. Bergman Kernel

In this section we discuss the Bergman kernel associated to the Kähler form $\omega = \sqrt{-1}\partial\bar{\partial}F$ on X and then show that it is given by (1.6).

To avoid coefficients in $\mathbf{L}$, we trivialize it by the nonvanishing holomorphic section s_0 of Proposition 3.2. Consider the Bergman space $\mathcal{H}$ of holomorphic functions on X that are square-integrable with respect to the measure $e^{-F}\,dV$, namely,

$$\mathcal{H} = \left\{ h\colon X \to \mathbf{C} : h \text{ is holomorphic and } \int_X h(z)\overline{h(z)}e^{-F(z)}\,dV < \infty \right\}. \tag{9.1}$$

The inner product of the Hilbert space $\mathcal{H}$ is given by the integral, and we let $\|\cdot\|_2$ denote the corresponding norm.

PROPOSITION 9.1. *The trivialization $hs_0 \leftrightarrow h$ defines an isomorphism between the T-invariant Hilbert spaces H_ω and $\mathcal{H}$.*

Proof. By direct computation, we have

$$\begin{aligned} \|hs_0\|^2 &= \int_X |h(z)|^2 \langle s_0, s_0\rangle\,dV && \text{(by (1.2))}\\ &= \int_X |h(z)|^2 e^{-F(z)}\,dV && \text{(by Proposition 3.2)}\\ &= \|h\|_2^2 && \text{(by (9.1)).} \end{aligned}$$

This proves the proposition. ☐

We use $\mathcal{H}$ to compute the Bergman kernel $K : X \times X \to \mathbf{C}$, which is defined by $K(z, w) = \sum_\lambda e_\lambda(z)\overline{e_\lambda(w)}$, where $\{e_\lambda\}_\lambda$ is an orthonormal basis of $\mathcal{H}$. The infinite sum converges and is independent of the choice of orthonormal basis [8, Sec. 1].

THEOREM 9.2. *If H_ω is a toric model, then the Bergman kernel is*

$$K(z, w) = \sum_{\lambda \in \mathbf{Z}^n} \frac{e^{\lambda(z+\bar{w})}}{\int_{x\in\Omega} e^{2\lambda x - F(x)}\, dV}.$$

Proof. Suppose that H_ω is a toric model. Then $e^{\lambda z} s_0 \in H_\omega$ for all $\lambda \in \mathbf{Z}^n$. Equivalently, by Proposition 9.1, $e^{\lambda z} \in \mathcal{H}$ for all $\lambda \in \mathbf{Z}^n$. By the Peter–Weyl theorem (see [2, Thm. 5.10]), $\{e^{\lambda z} : \lambda \in \mathbf{Z}^n\}$ is an orthogonal basis of $\mathcal{H}$. Since

$$\|e^{\lambda z}\|_2^2 = \int_X e^{\lambda z} e^{\lambda \bar{z}} e^{-F(z)}\, dV = \int_{x\in\Omega} e^{2\lambda x} e^{-F(x)}\, dV,$$

the Bergman kernel is

$$K(z, w) = \sum_{\lambda \in \mathbf{Z}^n} \frac{e^{\lambda z}}{\|e^{\lambda z}\|_2} \frac{e^{\lambda \bar{w}}}{\|e^{\lambda w}\|_2} = \sum_{\lambda \in \mathbf{Z}^n} \frac{e^{\lambda(z+\bar{w})}}{\int_{x\in\Omega} e^{2\lambda x - F(x)}\, dV}.$$

The theorem follows. □

If H_ω is not a toric model, then some $e^{\lambda z}$ do not lie in $\mathcal{H}$, and the expression in the theorem contains some denominators $\int_{x\in\Omega} e^{2\lambda x - F(x)}\, dV$ that diverge. In this case we need to disregard such summands in order to obtain the Bergman kernel.

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