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# Infinitely Many Grand Orbits

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## 1. Introduction

Sullivan's non-wandering theorem is one of the best-known and most fundamental results of classical rational iteration. We exhibit a counterexample which shows that a natural consequence of this theorem no longer holds if one is allowed to choose a different polynomial at each stage of the iterative process. The proof relies heavily on properties of the local dynamics near a parabolic fixed point.

We begin by considering a sequence of rational functions $\{R_n\}_{n=1}^{\infty} = \{R_1, R_2, R_3, \dots\}$ of some fixed degree $d \geq 2$. Let $Q_n(z)$ be the composition of the first $n$ of these functions in the natural order; that is,

$$Q_n = R_n \circ R_{n-1} \circ \cdots \circ R_2 \circ R_1.$$

We will also be interested in the compositions

$$Q_{m,n} = R_n \circ R_{n-1} \circ \cdots \circ R_{m+2} \circ R_{m+1}.$$

Define the *Fatou set* $\mathcal{F}$ for such a sequence of rational functions as

$$\mathcal{F} = \{z \in \bar{\mathbb{C}} : \{Q_n\}_{n=1}^{\infty} \text{ is a normal family on some neighbourhood of } z\};$$

the *Julia set* $\mathcal{J}$ is then simply the complement of the Fatou set in $\bar{\mathbb{C}}$. Note that, if $\{R_n\}_{n=1}^{\infty}$ is a constant sequence $\{R, R, R, \dots\}$, then these definitions coincide with the standard ones. One of the reasons for this definition of Julia and Fatou sets is that we can formulate an analogue of the principle of complete invariance in standard rational iteration. In order to do this, we shall introduce the following terminology.

We start by fixing as before a sequence $\{R_n\}_{n=1}^{\infty} = \{R_1, R_2, R_3, \dots\}$ of rational functions of fixed degree $d \geq 2$. With this in mind, for any $n \geq 0$, let us define the $n$th *Julia set* $\mathcal{J}_n$ to be the Julia set for the sequence $\{R_{n+1}, R_{n+2}, R_{n+3}, \dots\}$ that we obtain from our original sequence simply by deleting the first $n$ members. The $n$th *Fatou set* is similarly defined as the Fatou set for $\{R_{n+1}, R_{n+2}, R_{n+3}, \dots\}$. Note that, with these definitions, $\mathcal{J}_0 = \mathcal{J}$ and $\mathcal{F}_0 = \mathcal{F}$. We now state the principle of complete invariance for random iteration as follows.

Theorem 1.1. *For any $0 \leq m < n$ we have $Q_{m,n}(\mathcal{J}_m) = \mathcal{J}_n$ and $Q_{m,n}(\mathcal{F}_m) = \mathcal{F}_n$, with Fatou components of $\mathcal{F}_m$ being mapped surjectively onto those of $\mathcal{F}_n$ by $Q_{m,n}$.*

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The proof is a straightforward adaptation of the standard classical proof.

The notation introduced previously can also be extended in the obvious way to cover sets and points. For a set $U$ that we introduce at stage $m$, we set $U_n = Q_{m,n}(U)$; for a point $x$ that is introduced at stage $m$, we set $x_n = Q_{m,n}(x)$.

It turns out that this scenario of using rational functions is somewhat too general for proving significant results. The most natural restriction one can probably make was introduced by Fornæss and Sibony [6], who considered sequences of monic polynomials with uniformly bounded coefficients—that is, sequences of the form

$$R_n(z) = P_n(z) = z^d + a_{d-1,n} z^{d-1} + \cdots + a_{1,n} z + a_{0,n},$$

where we can find some $M \geq 0$ such that $|a_{i,n}| \leq M$ for $1 \leq i \leq d-1$ and all $n \in \mathbb{N}$. From now on, we shall call such sequences *bounded polynomial sequences.*

One of the advantages of this definition is that we can find some radius $R$ depending only on the coefficient bound $M$ just described so that, for any sequence $\{P_n\}_{n=1}^{\infty}$ as before, it is easy to see that

$$|Q_n(z)| \to \infty \text{ as } n \to \infty, \quad |z| > R,$$

which shows in particular that, as for classical polynomial Julia sets, there will be a basin at infinity $\mathcal{A}_\infty$ on which all points escape to infinity under iteration. Such a radius will be called an *escape radius* for the coefficient bound $M$.

## 2. Grand Orbits in the Classical Case

We start with the definition of a grand orbit for random iteration.

DEFINITION 2.1. Let $\{P_n\}_{n=1}^{\infty}$ be a bounded polynomial sequence and let $U$ be some subset of $\mathbb{C}$ that is introduced at some stage $n \geq 0$. We say that a set $V$ appearing at some stage $m \geq 0$ is in the *grand orbit* of $U$ if we can find $N \geq \max\{m, n\}$ such that $Q_{m,N}(V) = Q_{n,N}(U)$, and we write $\mathcal{G}(U)$ for the set of all such $V$.

For $m \geq 0$ fixed, the set of all $V$ at stage $m$ with the above property is called *the grand orbit of $U$ at stage $m$*, and we write $\mathcal{G}_m(U)$. If $m = n$ then we call $\mathcal{G}_n(U)$ the *immediate* grand orbit of $U$ (at stage $n$).

Basically, saying two sets $U$ and $V$ are in the same grand orbit means that (at some stage) they are mapped to the same set and so the dynamical behavior on each of them is eventually the same. One should note that this definition is slightly different from the classical one. For example, a rational function whose (classical) Fatou set contains a cycle of period 3 would give rise to three distinct grand orbits according to our definition but to only one grand orbit using the standard definition such as is given in [7; 9].

We now state some well-known results from classical complex dynamics. For proofs and terminology, the reader is referred to the standard references [5; 7; 8; 9]. We start with Sullivan's famous non-wandering theorem.

THEOREM 2.1 (Sullivan). *Let $R$ be a rational function and let $U$ be a (classical) Fatou component for $R$. Then $U$ is eventually periodic under iteration by $R$.*

THEOREM 2.2 (Classification of Periodic Fatou Components). *Let $R$ be a rational function and let $U$ be a (classical) Fatou component for $R$ that is periodic (i.e., $R^{\circ n}(U) = U$ for some $n \geq 1$). Then we have one of the following four possibilities for $U$:*

(1) *$U$ contains an attracting or superattracting cycle;*
(2) *$U$ is a basin of a parabolic periodic point lying on $\partial U$;*
(3) *$U$ is a Siegel disk; or*
(4) *$U$ is a Herman ring.*

For polynomials, the last possibility cannot occur because all Fatou components associated with polynomials must be simply connected (in view of the maximum principle) whereas Herman rings are doubly connected. The other result we need to state is equally well known.

THEOREM 2.3 (Fatou, Shishikura, Epstein). *Let $R$ be a rational function of degree $d$. Then the number of nonrepelling cycles associated with $R$ is at most $2d-2$.*

By combining these last two results with Sullivan's non-wandering theorem, we obtain the following corollary.

COROLLARY 2.1. *The number of grand orbits of Fatou components associated with a constant sequence arising from a polynomial $P$ is finite.*

*Proof*. By Theorem 2.2, each periodic Fatou component is associated with a nonrepelling cycle, and the number of grand orbits associated with that component and its iterates is then equal to the period of that cycle. (Note that here we are using the *random* definition of grand orbit as given in Definition 2.1.) However, Theorem 2.3 shows that we can have only finitely many grand orbits that are associated with some periodic Fatou component, whereas Sullivan's non-wandering theorem (Theorem 2.1) tells us that every grand orbit of Fatou components must be associated with a periodic Fatou component. The result now follows. □

It is worth noting that, since we can find nonrepelling cycles of arbitrarily long period even for quadratic polynomials of the form $z^2 + c$, the actual number of grand orbits of Fatou components—though finite—may be arbitrarily large. In the random case, however, we can say more.

## 3. Grand Orbits in the Random Case

We start by stating the principal result of this paper.

THEOREM 3.1. *There exists a bounded sequence of degree-2 polynomials whose corresponding Fatou set has infinitely many grand orbits.*

The proof will rely heavily on properties of the behavior of the iterates near a parabolic periodic point. The reader is referred to [5] for an exposition that contains proofs of the relevant facts that we shall make use of in the sequel.

We start by considering the polynomial $P(z) = z^2 - z$. This has a parabolic fixed point at 0 and, if we look at the second iterate $P^{\circ 2}(z) = z - 2z^3 + z^4$, we see that there are (as expected) two petals that are interchanged by $P$. Also, the expanding directions coincide with the imaginary axis and the contracting directions with the real axis. Finally, we have a "repelling tongue" that consists of the two parts of $\mathcal{A}_\infty$ (the attracting basin at infinity) that approach 0 from above and below the real axis. Points in the repelling tongues will lie in repelling petals but not in attracting ones. Because, near the origin, the attracting petals will contain a region lying outside that bounded by curves of the form $x = \pm ky^2 + \mathcal{O}(|y^{5/2}|)$ for some constant $k$ depending on $P$, we see that it will be easy to choose points whose iterates are guaranteed to remain in the attracting petals and avoid the repelling tongues.

The family $\mathcal{P}$ we will consider is based on $P^{\circ 2}$ and contains (in addition to $P^{\circ 2}$ itself) translates of $P^{\circ 2}$ of the form $z^4 - 2z^3 + z + c_n$, where the constants $c_n$ are real, bounded, and at our disposal (any small bound such as 1/4 will do). (We could simply use translates of $P$ instead, but using $P^{\circ 2}$ is slightly simpler as it preserves the order of points on the real axis near 0; since all polynomials in $\mathcal{P}$ are simply compositions of $P$ and translates of $P$, the result will clearly follow if we use these degree-4 polynomials instead of translates of $P$.) Now let $R$ be an escape radius for the coefficient bound 2, so that (denoting by $C(0, R)$ the circle about 0 of radius $R$) the iterates of points lying outside $C(0, R)$ will escape to infinity, regardless of which polynomials from $\mathcal{P}$ we iterate with. Our first main task is to prove the following simple lemma concerning the dynamics associated with sequences of polynomials taken from $\mathcal{P}$.

LEMMA 3.1. *Let $\{P_n\}_{n=1}^{\infty}$ be a sequence of polynomials chosen from $\mathcal{P}$ and let $x_1$ and $x_2$ be two points on the real axis that lie in bounded Fatou components associated with this sequence. Then $x_1$ and $x_2$ lie in the same Fatou component if and only if the line segment joining them also lies in this Fatou component.*

*Proof.* The "if" part of the result is trivial. For the "only if" part, suppose $x_1$ and $x_2$ lie in the same Fatou component $U$ and let $\gamma$ be a path in $U$ joining $x_1$ to $x_2$. If we denote the image of $\gamma$ under complex conjugation by $\bar{\gamma}$, then symmetry and the fact that all polynomials involved have real coefficients imply that $\bar{\gamma}$ is also a path in $U$ joining $x_1$ and $x_2$. It then follows easily from the maximum principle that the straight line segment connecting the two points also must lie within $U$, which completes the proof. □

Before we start the construction proper, we need some observations concerning distortion. We begin by considering a simply connected domain $U$ that is small, approximately circular, and symmetric about the real axis. We will also require that $U$ be small in comparison to its distance from the point 0. Having fixed such a neighborhood, one of our requirements will be to construct our sequence so that (among other things) the iterates of such a neighborhood will be bounded, thereby ensuring that $U$ will lie inside a Fatou component for such a sequence. Clearly, this will be the case if the iterates of $U$ remain small and also remain small compared to their distance from 0; this will guarantee that each iterate will lie in one of the two attracting petals for $P^{\circ 2}$, which (as we shall see) is something we will need.

In practice, in order to generate infinitely many grand orbits of Fatou components, we will need to introduce infinitely many such neighborhoods. Our aim will be to show that, at every stage, each one of these will lie in a Fatou component and that different neighborhoods will lie in different components, thereby yielding infinitely many grand orbits. These neighborhoods will be introduced as we construct our sequence of polynomials, and we will need to consider them in adjacent pairs. Toward that end, let $U^1$ and $U^2$ denote such a pair and let us also consider a point $x^1$ lying on the real axis at approximately the midpoint between them (see Figure 1). We are interested in the distortion of this picture because we want to ensure our sequence is such that, if we move the point $x^1$ at some stage $N$ to 0, then the corresponding iterates of $U^1$ and $U^2$ will each lie in one of the attracting petals for $P^{\circ 2}$; this is something we need in order to ensure that $U^1$ and $U^2$ lie in bounded Fatou components for the sequence of polynomials that we will construct. We need to consider the following two cases.

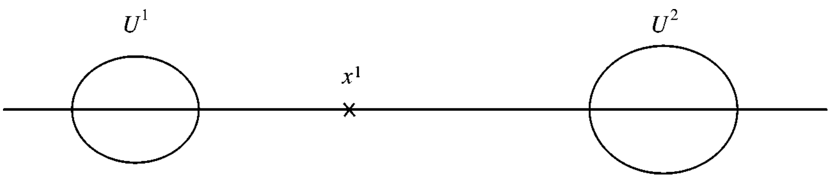


**Figure 1** Geometry for the Fatou domains

*Case 1.* Set $x = 0$. In order to investigate how the picture distorts, we need to consider the local behavior of $P^{\circ 2}$ near the origin. So suppose $x + iy$ is small and, in addition, assume that $|y| \le |x|$. If we now let $u + iv = P^{\circ 2}(x + iy)$, then

$$u = x - 2(x^3 - 3xy^2) + (x^4 - 6x^2y^2 + y^4),$$
$$v = y - 2(3x^2y - y^3) + (4x^3y - 4xy^3);$$

from this it follows (provided $x + iy$ is sufficiently small) that

$$\left|\frac{v}{u}\right| = \left|\frac{y}{x}\right|\left|\frac{1 - 6x^2 + 2y^2}{1 - 2x^2 + 6y^2} + \mathcal{O}(x^3)\right| < \left|\frac{y}{x}\right|.$$

This shows that, if we consider the angle of the narrowest sector centered at $x^1$ that contains $U^1$ and $U^2$, then this angle will become smaller under iteration by $P^{\circ 2}$, which will clearly be necessary in order to ensure that we can always place the iterates of $U^1$ and $U^2$ in the two attracting petals for $P^{\circ 2}$ by moving $x^1$ to 0.

Next we suppose that $x_1 + iy_1$ and $x_2 + iy_2$ are again two points close to 0 for which $|x_1| < |x_2|$ and that also satisfy $y_1/x_1 = y_2/x_2 = k$, where $|k| < 1/\sqrt{3}$ (i.e., the two points lie on the same line through the origin). Let $u_1 + iv_1$ and $u_2 + iv_2$ be the images of these points under $P^{\circ 2}$. Then, arguing as before, we have

$$\begin{aligned}\left|\frac{u_1}{u_2}\right| &= \left|\frac{x_1}{x_2}\right|\left|\frac{1-2x_1^2+6y_1^2}{1-2x_2^2+6y_2^2} + \mathcal{O}(x_1^3)\right| \\ &= \left|\frac{x_1}{x_2}\right|\left|\frac{1-x_1^2(2-6k^2)}{1-x_2^2(2-6k^2)} + \mathcal{O}(x_1^3)\right| > \left|\frac{x_1}{x_2}\right|,\end{aligned}$$

provided $x_2$ is small enough. Combined with our earlier remark concerning angles of sectors, this shows that the ratios $\operatorname{diam}(U^1)/\operatorname{dist}(U^1, 0)$ and $\operatorname{diam}(U^2)/\operatorname{dist}(U^2, 0)$ become smaller under iteration by $P^{\circ 2}$ and also shows that the ratio $\operatorname{dist}(U^1, 0)/\operatorname{dist}(U^2, 0)$ becomes closer to 1.

These calculations show that the image of our original picture will look something like Figure 2 after we map $x^1$ to 0 and apply a high iterate of $P^{\circ 2}$. In particular, we see that the ratios $\operatorname{diam}(U^1)/\operatorname{dist}(U^1, 0)$ and $\operatorname{diam}(U^2)/\operatorname{dist}(U^2, 0)$ will decrease under iteration by $P^{\circ 2}$ and will, in fact, tend to zero as the number of iterations with $P^{\circ 2}$ approaches infinity.

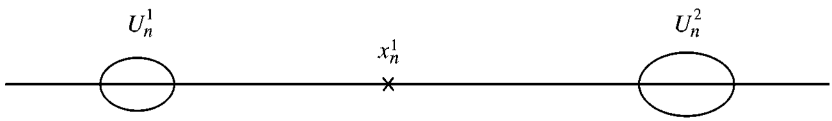


**Figure 2** After treatment with $P^{\circ 2n}$

*Case 2.* Here we have that the corresponding point $x^1$ is 0 for another pair of domains (one of which could easily be either $U^1$ or $U^2$) and that the diameters of $U^1$ and $U^2$ are again small compared to their distances from 0—that is, smaller than some fairly small constant (which, as we saw before, should be smaller than $1/\sqrt{3}$). That this can always be done will be shown later, but for now let us assume that it is possible. To investigate the distortion under this situation, we apply the standard analysis for the behavior of a function near a parabolic point. On each petal, $P^{\circ 2}(z)$ is conjugate via $1/4z^2$ to the transformation

$$w' = g(w) = w + 1 + \mathcal{O}(|w|^{-1/2}).$$

Also, if the absolute values of $w^1$ and $w^2$ are bigger than some large integer $M$, then we can apply the chain rule to obtain

$$|g(w^1) - g(w^2)| = |w^1 - w^2|(1 + \mathcal{O}(M^{-3/2})).$$

The conjugacy to $g$ above shows that iterates of points near 0 under $P^{\circ 2}$ approach 0 at a rate comparable to that of the sequence $\{1/\sqrt{n}\}$. Let $z_n^1, z_n^2$ be the iterates of two such points under $P^{\circ 2}$. Then we can certainly say that they will eventually lie within a distance less than (say) $n^{-3/8}$ of 0. Let $w_n^1, w_n^2$ denote the image of these sequences of points under the conjugacy $w = 1/4z^2$. If we choose $z_1^1$ and $z_1^2$ to be less than $N^{-3/8}$ in absolute value for large $N$, it follows that there are positive constants $C$ and $K$ (depending only on $P$) such that

$$\prod_{n=N}^{\infty}\left(1 - \frac{C}{n^{9/8}}\right) \leq \frac{|w_n^1 - w_n^2|}{|w_1^1 - w_1^2|} \leq \prod_{n=N}^{\infty}\left(1 + \frac{C}{n^{9/8}}\right),$$

and from this we may conclude that

$$1 - \frac{K}{N^{1/8}} \leq \frac{|w_n^1 - w_n^2|}{|w_1^1 - w_1^2|} \leq 1 + \frac{K}{N^{1/8}}.$$

This shows (provided we choose $U^1$ and $U^2$ to be small enough initially) that all ratios of the four quantities $\operatorname{diam}(U^i)$ and $\operatorname{dist}(U^i, x^i)$, $i = 1, 2$—as well as the angle of the narrowest sector at $x^1$ that contains $U^1$ and $U^2$—can be made to change as little as we wish by making $U^1 \cup U^2$ lie within a sufficiently small disk about 0.

The result of examining these two cases, where 0 lies either at the point $x$ between $U^1$ and $U^2$ or at the corresponding point for another such pair of domains, is that we can be sure Figure 1 will distort either by an amount we can control or in such a way as to actually improve—in the sense of ensuring that $U^1$ and $U^2$ will always lie in attracting petals. We can summarize all we need in the following lemma.

LEMMA 3.2. *Let $k$ be a small positive constant that we may take to be less than $1/3$, and let $\varepsilon > 0$. Let $U^1, U^2, \dots, U^n, U^{n+1}$ be simply connected domains that are symmetric about the real axis and are numbered from left to right and separated by points $x^1, x^2, \dots, x^n$ on the real axis, so that $x^i$ lies between $U^i$ and $U^{i+1}$. Then there exists a $\delta_\varepsilon > 0$ such that, if*

$$\operatorname{diam}(U^1 \cup U^2 \cup \cdots \cup U^{n+1}) < \delta_\varepsilon$$

*and, for $i = 1, \dots, n$,*

$$\operatorname{diam}(U^i)/\operatorname{dist}(U^i, x^i) < k$$
$$\operatorname{diam}(U^{i+1})/\operatorname{dist}(U^{i+1}, x^i) < k,$$

*then for any positive integer $N$ and $i = 1, \dots, n$ we have*

$$\operatorname{diam}(U_N^i)/\operatorname{dist}(U_N^i, x_N^i) < k(1+\varepsilon),$$
$$\operatorname{diam}(U_N^{i+1})/\operatorname{dist}(U_N^{i+1}, x_N^i) < k(1+\varepsilon),$$

*where $U_N^i$ and $x_N^i$ denote the images of $U^i$ and $x^i$ under $P^{\circ 2N}$.*

We now turn to proving the main result.

*Proof of Theorem 3.1.* The theorem will be proved by running the inductive construction that follows. This will generate, on the one hand, a sequence of polynomials and, on the other, a sequence of domains each of which lies in a Fatou component for this sequence of polynomials and with different domains guaranteed to lie in different components at every stage. Before we start the induction, we first fix $\varepsilon > 0$ small enough so that

$$\frac{1}{4}\prod_{n=1}^{\infty}(1+\varepsilon 2^{-n}) < \frac{1}{3}.$$

*Induction: Stage 1.* We are now ready to start the construction proper. Begin with two small domains $U^1$ and $U^2$, symmetric about the real axis, which we may as well take to be small disks symmetrically placed about 0 so that they lie in the two attracting petals for $P^{\circ 2}$. We will also require (a) that $U^1 \cup U^2$ have diameter less than $\delta_{\varepsilon/2}$ (using the notation from Lemma 3.2) and (b) that $\mathrm{diam}(U^1)/\mathrm{dist}(U^1, 0)$ and $\mathrm{diam}(U^2)/\mathrm{dist}(U^2, 0)$ each be less than $1/4$ in value.

We now iterate $m_{1,1}$ times with $P^{\circ 2}$, where $m_{1,1}$ is large enough so that the inverse image of $C(0, R)$ at stage $m_{1,1}$ under $P^{\circ 2m_{1,1}}$ has points above and below the real axis that are less than 1 in absolute value.

This takes care of the "separation" part of the first induction step. For the "distortion" part, we note that we can make $m_{1,1}$ larger if needed so that $\mathrm{diam}(U^1_{m_{1,1}} \cup U^2_{m_{1,1}}) < \frac{1}{5}\delta_{\varepsilon/16}$. Now apply a suitable polynomial from $\mathcal{P}$ whose constant term $c_{1,1}$ has been chosen so as to move $U^1$ and $U^2$ to the left of 0 and such that $\mathrm{diam}(U^2_{m_{1,1}+1})/\mathrm{dist}(U^2_{m_{1,1}+1}, 0) < 1/4$. By making $\delta_{\varepsilon/48}$ smaller if necessary, we can introduce another small disk $U^3$ whose center lies to the right of 0 so that $\mathrm{diam}(U^3_{m_{1,1}+1})/\mathrm{dist}(U^3_{m_{1,1}+1}, 0)$ is also less than $1/4$ while $\mathrm{diam}(U^1_{m_{1,1}+1} \cup U^2_{m_{1,1}+1} \cup U^3_{m_{1,1}+1}) < \delta_{\varepsilon/16}$.

Lastly, to be definite, let us label the sequence of polynomials generated so far by using $P_n$ to denote the polynomial that is the $n$th to be applied. In other words, $P_n = P^{\circ 2}$ for $n \le n \le m_{1,1}$, and $P_{m_{1,1}+1} = z^4 - 2z^3 + z + c_{1,1}$.

*Induction Hypothesis: Stage $n$.* Suppose now that the first $n$ steps have been carried out. We assume that, at the start of stage $n+1$, we have already constructed and applied the first $N_n$ members of our sequence of polynomials, which of course we will label $P_1, P_2, \ldots, P_{N_n}$. We now have $n+2$ domains $U^1_{N_n}, U^2_{N_n}, \ldots, U^{n+2}_{N_n}$ that we label from left to right as well as points $x^1_{N_n}, x^2_{N_n}, \ldots, x^{n+1}_{N_n}$, where $x^i_{N_n}$ lies between $U^i_{N_n}$ and $U^{i+1}_{N_n}$. We first make an assumption concerning the closeness of points in $\mathcal{A}_{\infty,m}$ (the basin at infinity at stage $m$) to each of the points $x^1_m, x^2_m, \ldots, x^{n+1}_m$. If $1 \le m \le N_{n-1}$ (where $N_{n-1}$ denotes the start of stage $n$), then we assume there are suitable points in $\mathcal{A}_{\infty,m}$ each of which is within a distance of less than $1/n$ of the real axis. In other words, for each $1 \le i \le n+1$ there are points within a distance of less than $1/n$ of $x^i_m$, lying both above and below the real axis, whose orbits have escaped outside $B(R, 0)$ by stage $N_n$ and are thus guaranteed to escape to infinity, regardless of which polynomials are chosen for our sequence after $P_{N_n}$, the last to be chosen so far. Note that even though $x^i_m$ may not have actually been introduced by stage $m$, there is no real problem. Either we simply ignore this case and make no assumption, or we consider the preimage of $x^i_m$ closest to 0 taken from the stage at which it was introduced.

We assume with regard to the distortion of this picture that, for each $1 \le i \le n+1$, $\operatorname{diam}(U^i_{N_n})/\operatorname{dist}(U^i_{N_n}, x^i_{N_n})$ and $\operatorname{diam}(U^{i+1}_{N_n})/\operatorname{dist}(U^{i+1}_{N_n}, x^i_{N_n})$ are both less than or equal to

$$\frac{1}{4}\prod_{j=1}^{n}\left(1+\frac{\varepsilon 2^{-j-1}}{j}\right)^{j} < \frac{1}{4}\prod_{j=1}^{n}(1+\varepsilon 2^{-j}) < \frac{1}{3},$$

provided we choose $\varepsilon$ small enough. In order to guarantee that we can control future distortions, we will also need to ensure that $U^1_{N_n} \cup U^2_{N_n} \cup \cdots \cup U^{n+1}_{N_n}$ is sufficiently small, with diameter less than $\delta_{\varepsilon 2^{-(n+2)}/(n+1)}$.

*Induction Step: Stage* $n+1$. We start by choosing from $\mathcal{P}$ the polynomial $P_{N_n+1}$ that moves $x^1_{N_n}$ to 0. Now apply $P^{\circ 2}$ $m_{n+1,1}$ times; because the first $N_n+1$ polynomials are now fixed, we may choose $m_{n+1,1}$ to be large enough to ensure the existence of points in the preimage under $Q_{m,N_n+m_{n,1}+1}$ of $C(0,R)$, both above and below the real axis, within a distance of $1/(n+1)$ of $x^1_m$ and where $0 \le m \le N_n$. This ensures there exist points in $\mathcal{A}_{\infty,m}$ lying above and below the real axis and within a distance of $1/(n+1)$ of $x^1_m$ for $0 \le m \le N_n$, that is, from the point at which we started our sequence until the beginning of stage $n$.

We have now constructed the next $m_{n,1}+1$ members of our sequence of polynomials. Explicitly, $P_{N_n+1} = z - 2z^3 + z^4 - x^1_{N_n}$ and $P_m = P^{\circ 2}$ for $N_n+2 \le m \le N_n + m_{n,1} + 1$. Our assumption in the induction hypothesis (that $U^1_{N_n} \cup U^2_{N_n} \cup \cdots \cup U^{n+2}_{N_n}$ has diameter less than $\delta_{\varepsilon 2^{-(n+2)}/(n+1)}$) guarantees that, for each $i$ from 2 to $n$, $\operatorname{diam}(U^i_m)/\operatorname{dist}(U^i_m, x^i_m)$ and $\operatorname{diam}(U^{i+1}_m)/\operatorname{dist}(U^{i+1}_m, x^i_m)$ are both bounded by

$$\frac{1}{4}\left(\prod_{j=1}^{n}(1+\varepsilon 2^{-j})\right)\left(1+\varepsilon\frac{2^{-n-2}}{n+1}\right) < \frac{1}{3}$$

for $N_n+1 \le m \le N_n + m_{n,1} + 1$. Also, for $i=1$ we have the stronger conclusion that $\operatorname{diam}(U^1_m)/\operatorname{dist}(U^1_m, x^1_m)$ and $\operatorname{diam}(U^2_m)/\operatorname{dist}(U^2_m, x^1_m)$ are both less than $\frac{1}{4}\prod_{j=1}^{n}(1+\varepsilon 2^{-j}) < \frac{1}{3}$. In view of our previous remarks, we see that the distortion remains small.

This completes the first step of stage $n+1$ for the first pair of domains. We now apply a polynomial $P_{n,2}$ from $\mathcal{P}$ that moves $x^2_{N_n+m_{n,1}+1}$ to 0 and then iterate $m_{n,2}$ times with $P^{\circ 2}$. As before, by considering preimages of the circle $C(0,R)$ we may guarantee that, for $0 \le m \le N_n$, there will be points in $\mathcal{A}_\infty$ that are above and below $x^2_m$ and that lie within a distance $1/(n+1)$ of $x^2_m$.

Similarly, that $U^1_{N_n} \cup U^2_{N_n} \cup \cdots \cup U^{n+2}_{N_n}$ has diameter less than $\delta_{\varepsilon 2^{-(n+2)}/(n+1)}$ ensures that $\operatorname{diam}(U^i_m)/\operatorname{dist}(U^i_m, x^i_m)$ and $\operatorname{diam}(U^{i+1}_m)/\operatorname{dist}(U^{i+1}_m, x^i_m)$ are both bounded by

$$\frac{1}{4}\left(\prod_{j=1}^{n}(1+\varepsilon 2^{-j})\right)\left(1+\varepsilon\frac{2^{-n-2}}{n+1}\right)^{2} < \frac{1}{3}$$

for $N_n + m_{n,1} + 1 \le m \le N_n + m_{n,1} + m_{n,2} + 2$ and for $3 \le i \le n+2$. For $i = 1, 2$, the corresponding assumption is that $\operatorname{diam}(U^i_m)/\operatorname{dist}(U^i_m, x^i_m)$ and $\operatorname{diam}(U^{i+1}_m)/\operatorname{dist}(U^{i+1}_m, x^i_m)$ are both less than

$$\frac{1}{4}\left(\prod_{j=1}^{n}(1+\varepsilon 2^{-j})\right)\left(1+\varepsilon\frac{2^{-n-2}}{n+1}\right) < \frac{1}{3}$$

for $N_n + m_{n,1} + 1 \leq m \leq N_n + m_{n,1} + m_{n,2} + 2$. Again, this ensures that the distortion remains small and under control.

We now continue this process for all the other pairs of domains from left to right. For the last pair the argument is the same, except we make the stronger assumption that $m_{n+1,n+1}$ is big enough to guarantee that, if we let $N'_{n+1} = N_n + \sum_{i=1}^{n+1} m_{n,i} + n + 1$, then

$$\operatorname{diam}(U^1_{N'_{n+1}} \cup U^2_{N'_{n+1}} \cup \cdots \cup U^{n+2}_{N'_{n+1}}) < \tfrac{1}{5}\delta_{\varepsilon 2^{-(n+3)}/(n+2)}.$$

This enables us to apply another polynomial $P'_{N_{n+1}+1}$ that moves all our domains to the left of 0 and then introduce a new disc $U_{n+3}$ at stage $N'_{n+1}+1$, which we shall simply relabel $N_{n+1}$ so that

$$\operatorname{diam}(U^1_{N_{n+1}} \cup U^2_{N_{n+1}} \cup \cdots \cup U^{n+3}_{N_{n+1}}) < \delta_{\varepsilon 2^{-(n+3)}/(n+2)}$$

and also $\operatorname{diam}(U^{n+2}_{N_{n+1}})/\operatorname{dist}(U^{n+2}_{N_{n+1}}, 0)$ and $\operatorname{diam}(U^{n+3}_{N_{n+1}})/\operatorname{dist}(U^{n+3}_{N_{n+1}}, 0)$ are both less than $1/4$. We have now reached stage $N_{n+1}$ and, as the notation suggests, stage $n+1$ is finished and we are now at the start of stage $n+2$.

This completes the induction. For each pair of neighborhoods $U^i_m$ and $U^{i+1}_m$ (where $m$ is any natural number), the ratios $\operatorname{diam}(U^i_m)/\operatorname{dist}(U^i_m, x^i_m)$ and $\operatorname{diam}(U^{i+1}_m)/\operatorname{dist}(U^{i+1}_m, x^i_m)$ remain bounded by $1/3$; hence the iterates of each domain will lie in one of the attracting petals for $P^{\circ 2}$ and thus will certainly be bounded. This guarantees that each $U^i$ will lie in a Fatou component for the sequence of polynomials generated by our inductive scheme (suitably shifted to take into account when each domain is introduced). On the other hand, the points $x^i_m$ have bounded orbits but are approached arbitrarily closely by points in $\mathcal{A}_{\infty,m}$, which implies that they must all lie in the corresponding iterated Julia set $\mathcal{J}_m$. It follows in view of Lemma 3.1 that, at each stage, all the domains will lie in different Fatou components. Therefore, we do indeed obtain infinitely many grand orbits as desired. □

## References

[1] R. Brück, *Connectedness and stability of Julia sets of the composition of polynomials of the form $z^2 + c_n$*, J. London Math. Soc. (2) 61 (2000), 462–470.

[2] ———, *Geometric properties of Julia sets of the composition of polynomials of the form $z^2 + c_n$*, Pacific J. Math. 198 (2001), 347–372.

[3] R. Brück, M. Büger, and S. Reitz, *Random iterations of polynomials of the form $z^2 + c_n$: Connectedness of Julia sets,* Ergodic Theory Dynam. Systems 19 (1999), 1221–1231.

[4] M. Büger, *Self-similarity of Julia sets of the composition of polynomials,* Ergodic Theory Dynam. Systems 17 (1997), 1289–1297.

[5] L. Carleson and T. W. Gamelin, *Complex dynamics,* Springer-Verlag, New York, 1993.

[6] J. E. Fornæss and N. Sibony, *Random iterations of rational functions,* Ergodic Theory Dynam. Systems 11 (1991), 687–708.

[7] J. Milnor, *Dynamics in one complex variable*, Vieweg & Sohn, Braunschweig, 1999.
[8] N. Steinmetz, *Rational iteration. Complex analytic dynamical systems,* de Gruyter, Berlin, 1993.
[9] D. Sullivan, *Quasiconformal homeomorphisms and dynamics I. Solution of the Fatou–Julia problem on wandering domains,* Ann. of Math. (2) 122 (1985), 401–418.

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