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Examples Relating to the Crossing Number, Writhe, and Maximal Bridge Length of Knot Diagrams

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1. Introduction

The "Perko pair" knot $10_{161} = 10_{162}$ [R, p. 415; identification noted in second printing] is exceptional in at least two distinct ways. It first achieved its name and fame when Perko [P] discovered that the knot refutes a conjecture of Tait: it possesses two 10-crossing diagrams (the minimum possible) with distinct writhes. All previous knot tables of that depth had incorrectly listed the diagrams as representing distinct knots.

The same knot refutes a much shorter-lived conjecture concerning one of the link polynomials discovered in the 1980s. Let $Q(z)$ be the polynomial of Brandt, Lickorish, and Millett [BLM] and Ho [Ho]. Define an *overbridge* in a link diagram to be a consecutive sequence of overcrossing segments, and define an *underbridge* (the natural word "tunnel" has a different meaning in low-dimensional topology) to be a consecutive sequence of undercrossing segments. A *bridge* is the common term for both under- and overbridges. Define the *length* of a bridge to be the number of segments overcrossed or undercrossed. Kidwell [K] proved that, if a knot has a diagram with c crossings and its longest (over or under-) bridge has length d, then

$$\deg Q(z) \le c - d. \tag{a}$$

(Kidwell was thinking only of overbridges at the time; the dual underbridges were more recently pointed out to him by Stoimenow.) Kidwell asked (see [M2]) whether every knot or link has a diagram for which (a) is an equality, knowing that this was probably too much to hope for, since it would imply that the unknot is characterized as the unique knot with $Q(z) = 1$ (still an intractable open question). Equality in (a) can be achieved for alternating knots and for all knots with minimal crossing number below 10. The Perko pair, however, became a leading candidate to refute the conjecture. It has recently been demonstrated [SK] that relation (a) is a strict inequality for every diagram of the Perko pair.

Kauffman [Ka] soon generalized $Q(z)$ to his two-variable polynomial $F(a, z)$, and Thistlethwaite [T1] investigated its degree properties. He proved an analogous inequality to (a):

$$\deg_z F(a, z) \le c - d. \tag{b}$$

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Table 1

Minimal crossing number	Number of nonalter-nating knots	Strict inequality in (b)	% of "exceptions"
10	42	1	2.4
11	185	8	4.3
12	888	59	6.6
13	5110	416	8.1
14	27436	2997	10.9

Recently, examples have come to light to show that (b) can be stronger than (a), though $\deg Q(z) = \deg_z F(a, z)$ for the Perko pair. This phenomenon first appears among 12-crossing knots, where there are ten examples. (Other examples with 15 crossings were shown in [SK].)

2. Some Statistics

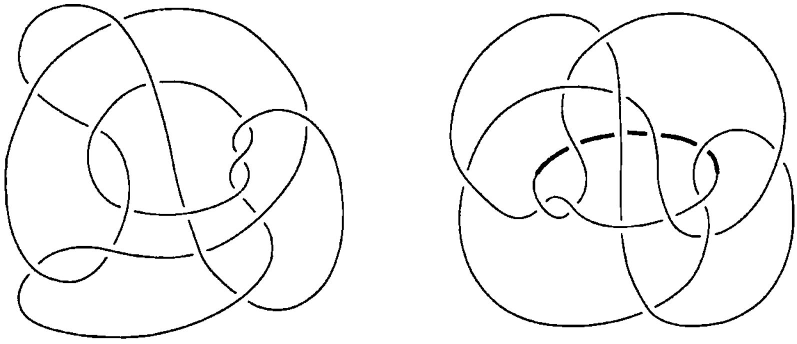
If one looks at the knot tables from the era of Conway, Bailey–Rolfsen, and Perko, the behavior of the Perko pair seems truly exceptional. However, the recent vast expansion of the knot tables [HTW] has taught us that special properties of a knot, such as being alternating or invertible, are seriously overrepresented among knots of low crossing number. This appears to be the case with knots in which relation (b) is an equality for some diagram with minimal crossing number, as Table 1 shows.

These statistics were generated by the computer program KnotScape [HT] and take into account only prime knots and diagrams with minimal crossing number.

A further search on KnotScape has turned up the example 15_{219453} (Figure 1) to show that $c - d$ is not necessarily minimized with minimal c. This knot has a unique 15-crossing diagram with maximal bridge length 2, but it has a 16-crossing diagram with maximal bridge length 4, and $\deg Q = \deg_z F = 12$. Such an example makes it more difficult to assert that (a) or (b) is a strict inequality for all diagrams of a given knot. There is, however, a helpful bound on the size of c. Results of [SK] show that, if a knot diagram has a bridge of length more than one third its crossing number, then it can be turned into a diagram of smaller crossing number without increasing the difference $c - d$. Suppose, for example, that a knot had minimal $c = 12$, that the longest bridge among 12-crossing diagrams had length 2, and that $\deg_z F = 9$. Then the knot could have a diagram with $c = 13$ and $d = 4$ but, if not, it will not have any diagram with $c - d = 9$ for $c \geq 14$. We are thus searching in a huge but finite domain.

A remark and a warning with regard to the knot tables is in order.

It is understood that alternative work on knot tabulation is being done by Aneziris [A]. Unfortunately, it seems as if every new knot tabulator chooses and insists on his own numbering convention for knots, which will lead to confusion in using the



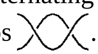
15_{219453}

Figure 1 The knot 15_{219453} has $\deg_z F = \deg Q = 12$. The diagram on the left is its only 15-crossing diagram, and it has no over- or underbridge of length 3. However, some 16-crossing diagrams of the same knot (like the one on the right) have an underbridge of length 4. Thus the Kidwell inequality for this knot is sharp, but the minimal value of $c - d$ is not attained in any minimal-crossing-number diagram. There are at least twenty additional 15-crossing knots for which this phenomenon occurs.

different knot tables. It appears most correct to stick to the convention of the first tabulator for each crossing number. We use here the convention of Rolfsen's tables [R] for ≤ 10 crossing knots and that of [HT] for ≥ 11 crossing knots; this coincides with those of the first tabulators for any crossing number except 11, where the initial tables were compiled by Conway [C]. We apologize for not using his numbering. An excuse is that all calculations have been performed by KnotScape, which does not yet provide a translator between its notation and that of Conway. For uniformity reasons, we will need to continue using this convention in subsequent papers, too.

To generate all minimal-crossing-number diagrams of a given nonalternating knot, we applied Thistlethwaite's diagram move tool `knotfind` and a special list of duplications. Although improvements were made to `knotfind` in February 2002, the program occasionally misses some minimal crossing diagrams. Table 1 and the remark after Question 2 are to be understood modulo this problem. (It occurs rather seldom, so expectedly alters the outcome only insignificantly.)

In the case of the particular examples $K \in \{15_{219453}, 13_{8838}, 13_{9221}\}$, where it is important to have the correct list of minimal-crossing-number diagrams, the completeness of `knotfind`'s output was independently checked in the following manner.

Knots are tabulated in [HT] lexicographically by their smallest Dowker–Thistlethwaite notation [DT] and so, in order to make the diagram of K alternating, it suffices to change crossings, obtain an alternating knot K', and then consider all diagrams that (i) are obtained by crossing changes from the alternating knots after K' in the alternating knot table and (ii) contain no trivial clasps . The Jones polynomial of such diagrams was checked against the Jones polynomial of K, giving only the diagrams rendered by `knotfind`.

3. The Kauffman Polynomial

The Kauffman polynomial is usually defined via a regular isotopy invariant $\Lambda(a, z)$ with the properties

(1) $\Lambda(\diagup\!\!\!\!\diagdown) + \Lambda(\diagdown\!\!\!\!\diagup) = z\big(\Lambda(\asymp) + \Lambda()\,())\big)$.

(2a) $\Lambda(\diagup\!\!\!\!\bigcirc) = a\Lambda(|)$;

(2b) $\Lambda(\diagdown\!\!\!\!\bigcirc) = a^{-1}\Lambda(|)$.

(3) $\Lambda(\bigcirc) = 1$.

The isotopy invariant $F(a, z)$ of the knot K with diagram D is then defined as $a^{-w(D)}\Lambda(a, z)$, where $w(D)$ is the writhe of D. Thus $\deg_z(F) = \deg_z(\Lambda)$ in any of its forms.

The Brandt–Lickorish–Millett–Ho polynomial Q is given by $Q(z) = F(1, z)$. Note that there is a way to distinguish the two smoothings of a crossing, as shown in Figure 2.

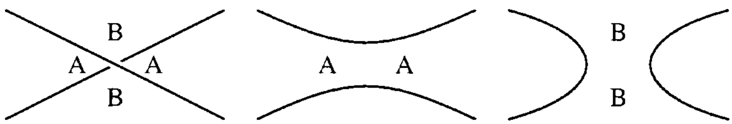


Figure 2 The A- and B-corners of a crossing, and its two smoothings. The corner A (resp. B) is the one passed by the overcrossing strand when rotated counterclockwise (resp. clockwise) toward the undercrossing strand. A type-A (resp. type-B) smoothing is obtained by connecting the A (resp. B) corners of the crossing.

Since the defining relations (1) and (2) provide at most one letter (a or z) for each reduction in crossing number of a diagram, Thistlethwaite [T1] was able to prove that, if $u_{rs}a^r z^s$ is a nonzero term of $\Lambda(a, z)$, then

$$|r| + s \le c. \tag{c}$$

The Perko pair has smaller-than-expected $\deg_z F$, so it would seem natural to guess that variation of minimal writhe is another contributor (besides long bridges) to truncation of the Kauffman polynomial. We now demonstrate that the need to fit two or more versions of $\Lambda(a, z)$ within Thistlethwaite's bounds also restricts $\deg_z F$.

Thistlethwaite [T2] made a substantial study of the terms on the "critical lines" $|r| + s = c$, where c is the minimal crossing number. His main notion was that of an adequate link, which we pause to define. A link diagram D is *plus semiadequate* if the positive (A) smoothing of D yields a link in which the two strands of every former crossing belong to different components. The definition of *minus semiadequate* is similar for the negative (B) smoothing of D. A diagram is *adequate* if it is both plus semiadequate and minus semiadequate and *inadequate* if it is neither. A link is adequate, plus semiadequate, and/or minus semiadequate if it possesses a diagram with the respective properties and *inadequate* otherwise.

Thistlethwaite demonstrated that a link is plus semiadequate if and only if there are nonzero terms $u_{rs}a^r z^s$ on the "positive critical line" where $r+s=c$ and that a link is minus semiadequate if and only if there are nonzero terms on the "negative critical line" where $-r+s=c$. He also proved that nonzero coefficients u_{rs} along the critical lines must be positive.

Let D_1 and D_2 be two minimal crossing diagrams of a knot K, each with crossing number c but with differing writhes. The *variation* $w(D_2)-w(D_1)$ must be an even number $2v$. Without loss of generality, assume $2v>0$. The Λ polynomials Λ_1 and Λ_2 of D_1 and D_2 are related by $a^{-w(D_1)}\Lambda_1(a,z)=F(a,z)=a^{-w(D_2)}\Lambda_2(a,z)$, or

$$\Lambda_2(a,z)=a^{2v}\Lambda_1(a,z). \tag{d}$$

Define the variation of a link to be the maximum of $w(D_2)-w(D_1)$ over all its minimal crossing diagrams.

For any $s>0$, there must be two nonzero terms of the form $a^r z^s$ if there are any at all, since $F(i,z)=(-1)^{|L|-1}$ [L1, p. 573, table].

THEOREM 1. *Let L be a link with Kauffman polynomial $F(a,z)$, minimal crossing number c, and variation v.*

(i) *If L has an adequate diagram, then $v=0$.*

(ii) *If L is not adequate but has a semiadequate diagram with crossing number c, then*

$$\deg_z F \le c-v-2. \tag{e}$$

(iii) *If L is inadequate, then*

$$\deg_z F \le c-v-3. \tag{f}$$

Proof. (i) If L has an adequate minimal crossing diagram D, then $\Lambda(D)$ will have terms on both critical lines. No multiplication by powers of a is permitted by Thistlethwaite's condition. Thus, no writhe other than that of D is possible in a minimal crossing diagram. This part is added for completeness and duplicates [T2, Cor. 3.3].

(ii) Suppose Λ_2 has nonzero terms $a^{c-s}z^s$ (on the positive critical line) and $a^{c-s-2}z^s$. (A parity condition prevents a nonzero $a^{c-s-1}z^s$ term.) Then Λ_1 will have nonzero terms $a^{c-s-2v}z^s$ and $a^{c-s-2v-2}z^s$. If the latter term is on the negative critical line, then

$$-(c-s-2v-2)+s=c$$

or

$$2s=2c-2v-2$$

or

$$s=c-v-1.$$

Other positions for the nonzero terms are less favorable and lead to a smaller s for given c and v. Thus, if the term on the positive critical line is zero then $s\le c-v-2$, and similarly if the term on the negative critical line is zero.

(iii) If both critical-line terms are zero then $s\le c-v-3$. □

THEOREM 2. *Let K be a knot with minimal crossing number c, Kauffman polynomial $F(a, z)$, and Brandt–Lickorish–Millett–Ho polynomial $Q(z)$.*

(1) *If* $\deg_z F = c - 2$, *then* $\deg Q = c - 2$.
(2) *If* $\deg_z F = c-3$ *and* $\deg Q < c-3$, *then K is semiadequate* (*or adequate*).

Curiously, no examples illustrating part (2) have yet turned up in the tables.

Proof. (1) We can assume that $c \geq 3$. By Thistlethwaite's bounds, the terms in Λ of highest z-degree must be of the form $(u_{c-2,-2}a^{-2}+u_{c-2,0}+u_{c-2,2}a^2)z^{c-2}$. By the Lickorish condition on $F(i, z)$, we must have $-u_{c-2,-2}+u_{c-2,0}-u_{c-2,2} = 0$. If $\deg Q < c - 2$ then we must also have $u_{c-2,-2} + u_{c-2,0} + u_{c-2,2} = 0$. Adding these two equations gives $u_{c-2,0} = 0$; subtracting them gives $u_{c-2,-2} + u_{c-2,2} = 0$. But if these two critical-line terms are nonzero then they must be positive, which is impossible.

(2) In this case, the terms in Λ of highest z-degree must be $(u_{c-3,-3}a^{-3} + u_{c-3,-1}a^{-1} + u_{c-3,1}a + u_{c-3,3}a^3)z^{c-3}$. The condition on $F(i, z)$ gives $u_{c-3,-3} - u_{c-3,-1}+u_{c-3,1}-u_{c-3,3} = 0$. If $\deg Q < c-3$, we also have $u_{c-3,-3}+u_{c-3,-1}+u_{c-3,1} + u_{c-3,3} = 0$. Adding and subtracting as before, we obtain $u_{c-3,-3} + u_{c-3,1} = 0$ and $u_{c-3,-1} + u_{c-3,3} = 0$. If the two critical-line terms $u_{c-3,-3}$ and $u_{c-3,3}$ are zero, then all four terms are zero and we could not have $\deg_z F = c-3$. Thus K must be semiadequate or adequate. □

4. Bridge Length and Variation

THEOREM 3. *Any link has a diagram of maximal bridge length* 2. *This diagram can be made to have no nugatory crossings or trivial clasps.*

Proof. This is achieved by the moves shown in Figure 3. Consider an overbridge (a), and add kinks on all its segments (b). When putting the kinks as shown, on the first segment a kink is allowed but not necessary. To construct a reduced diagram, replace each pair of kinks by the tangle in (c) (by putting or not a kink on the first segment, we can adjust the number of kinks always to be even). Notice that, to avoid creating a longer underbridge or overbridge on the underpassing strand d between the two kinks involved in the move, we must take care that the first and last crossings of this strand are underpasses and that the second and second-to-last are overpasses. To avoid the creation of a trivial clasp, modify the tangle in (c) to the one in (d). □

Clearly, this construction is made possible at the cost of high augmentation of the crossing number of the diagram. On the other hand, bridge length 2 sometimes cannot be achieved in *any* minimal diagram, as example 13_{8838} shows (see Figure 4). It has 37 diagrams of 13 crossings, all with bridge length 3.

We have so far considered overbridges and underbridges as equivalent, since we did not care about mirror images. However, the next example shows that there is sometimes a difference when distinguishing mirror images. If a knot K has a minimal diagram with an overbridge of length c, then its mirror image has a

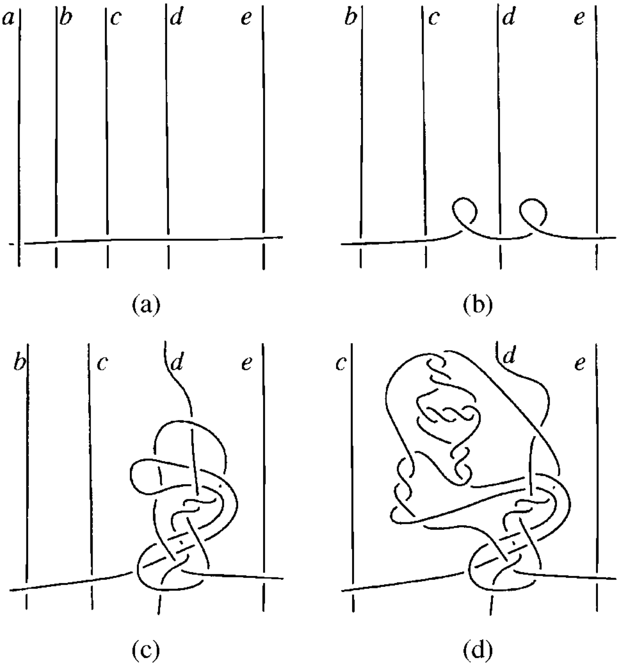
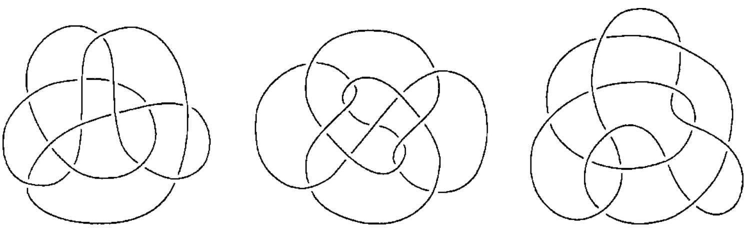


Figure 3 Several ways of modifying a diagram so as to avoid overbridges of length > 2 (the mirrored moves deal with underbridges).



13_{8838} 13_{9221} 13_{8962}

Figure 4

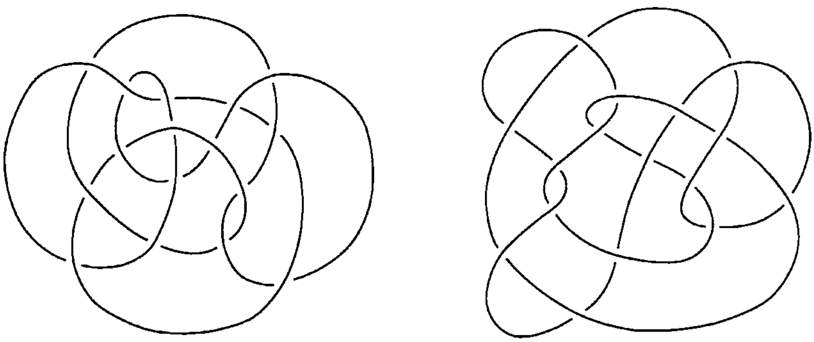
minimal diagram with an underbridge of length c. Thus the knot's maximal overbridge length in minimal diagrams is the same as the maximal underbridge length of the minimal diagrams of its mirror image. However, this may not be equal to the maximal underbridge length of the minimal diagrams of K itself. The knot 13_{9221} has a unique 13-crossing diagram with maximal underbridge length 3 but maximal overbridge length only 2.

Given inequality (b) and Theorem 1, one might hope that a long bridge and variation of writhe could be used simultaneously to bound $\deg_z F$. These hopes are dashed by the knot 13_{8962} (Figure 4). The knot has a plus semiadequate minimal crossing diagram of writhe 7 and an inadequate minimal crossing diagram of writhe 9. Since $\deg_z F = 9$, it follows that inequality (e) of Theorem 1 is strict; one cannot subtract a further term for the bridge of length 3 in the writhe-9 diagram. Inequality (b) fails by 1 to be strict for this knot.

5. Questions

We conclude our exposition with some problems.

When the crossing number increases, so does (in general) the number of minimal diagrams and hence the variation of their writhes. The Perko knot is the first one with variation 2. Among the 16-crossing knots there are several examples with variation 4. Particularly interesting among them is the amphicheiral knot in Figure 5, which features a minimal diagram of nonzero writhe. (This is the first even-crossing-number example of such type after the amphicheiral 15-crossing knot found by Thistlethwaite; see [HTW].) This naturally leads to the following question.



$16_{1184186}$

Figure 5 The knot $16_{1184186}$ is amphicheiral but has a diagram of minimal crossing number and (nonzero) writhe -2 (right) and hence also one of writhe 2. It also has many zero writhe diagrams of minimal crossing number, such as the diagram on the left (which is the one included in the table).

QUESTION 1. Are there knots with arbitrarily large (minimal crossing) writhe variation?

The answer is surely “yes”: take some knot with nonzero variation and consider its iterated connected sums. The problem is how to show that the crossing number is additive under connected sums for such a knot.

An even more difficult question is whether the writhes occurring in minimal diagrams are always consecutive.

QUESTION 2. Is the set of minimal writhes for every knot connected? That is, if $w+2$ and $w-2$ are minimal writhes then is w a minimal writhe?

The answer is positive for all knots in the tables of [HT] up to 16 crossings (modulo the warning in Section 2).

Here is another curious problem.

QUESTION 3. Let K be a nontrivial knot, W_K a Whitehead double of K, P the HOMFLY polynomial, and m the Alexander variable of P. Is then $\deg_m P(W_K) = 2\deg_z F(K) + 2$?

The equality holds for K up to 11 crossings. The origin of this problem is as follows.

The diagrammatic genus g_D of a knot is defined to be the minimum genus obtained by applying Seifert's algorithm to any diagram of the knot. Suppose a diagram achieving such a surface has c crossings and s Seifert circuits. Morton [M1] proved that $\deg_m P(K) \le 2g_D = c - s + 1$. This inequality is not always an equality, but it is often sharp. (For example, Crowell proved that $\deg \Delta = 2g = 2g_D$ for alternating knots.) Satellite knots were exactly the examples Morton used to show that diagrammatic genus can be larger than ordinary genus. Now take the simplest (possibly twisted) Whitehead double diagram D_w with $4c + 2$ crossings of the knot obtained from D. We count c "little square" Seifert circuits (one for each crossing of the original diagram), $c + 2$ larger Seifert circuits (one for each region of the original diagram), and a clasp where the doubling occurs, for a total of $2c + 3$ Seifert circuits. So, for this particular diagram,

$$2g_{D_w} = c_W - s_W + 1 = (4c + 2) - (2c + 3) + 1 = 2c.$$

Thus we have an estimate for the crossing number of K by the diagrammatic genus of the Whitehead double of K, as from $\deg_z F$. (Note that, by a simple skein argument, $\deg_m P(W_K)$ is independent of the twist of W_K if it is > 2.) It is also a simple skein calculation to show that this estimate is 2-subadditive under connected sums, thus suggesting the relation to $2\deg_z F + 2$.

In a related paper, Yamada [Y] proves that the Kauffman polynomial contains the 2-cable Jones polynomial. A like proof won't work here to show a direct relationship between $F(K)$ and $P(W_K)$, because both invariants have two variables. If a variable substitution $R^2 \to R^2$ has a noncritical point, then it would be locally invertible and hence both $P(W_K)$ and $F(K)$ would be interconvertible. But this is not the case—there are examples of knots K with equal $F(K)$ but different $P(W_K)$. One such pair are the knots 11_{30} and 11_{189} mentioned in [L1, p. 573] (a picture may be found in [S]) to have equal Kauffman but distinct Conway polynomials.

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