REMARKS CONCERNING DOMAINS OF SMIRNOV TYPE

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A Jordan domain G with rectifiable boundary is said to be a *Smirnov domain* (or *domain of Smirnov type*) if the derivative of the mapping function f(z) from $|z| < 1$ onto G is an "outer" function, that is, if $\log |f'(re^{i\theta})|$ is the Poisson integral of the boundary values $\log |f'(e^{it})|$. For a fuller discussion of this property, which is of decisive significance for many questions regarding function theory in G, the reader is referred to Privalov [13, p. 159]. A number of sufficient conditions (in terms of geometrical properties of G) that G be of Smirnov type are known, although the picture is far from complete. Some criteria of this kind, due to Smirnov, Keldysh, and Lavrentiev, are given in [13]. More recently, Tumarkin [15] has established a powerful sufficiency criterion. In [7], the concept of Smirnov domain was extended to multiply connected domains; but here we shall be concerned only with Jordan domains.

The main result achieved by Tumarkin in [15] may be roughly expressed this way: if every point of the boundary F of G, with countably many exceptions, is contained in a "nice" arc of F, then G is of Smirnov type (for a more precise formulation, see below). A weaker criterion of this kind was given in [10], the authors being at the time unaware of Tumarkin's results. The method employed in [10] may however be extended to yield a stronger result in which the allowable set of "bad" boundary points need not be countable, provided it is "sparse" in a certain precise metric sense. Section 1 of the present paper is devoted to this result. (Actually, in Theorem 1 we prove a slightly more general result, which does not presuppose rectifiability of the boundary.)

Section 2 of the present paper is essentially independent of Section 1, the only common link being that the remarkable type of pathological curve (which we call a pseudocircle) discussed there was first constructed by Keldysh and Lavrentiev to show that there exist domains not of Smirnov type. Pseudocircles may be characterized independently of conformal mapping (see Theorem 2). The direct construction of some pseudocircle without function theoretic methods would be of interest; but we have not been able to achieve it. Section 2 summarizes the known information concerning pseudocircles.

Notation and conventions. In this paper, D shall always denote the unit disk $|z| < 1$, and C its boundary. An (open) *smooth arc* denotes a homeomorphic image of the interval $0 < t < 1$ by a complex-valued function w(t) having a continuous derivative that is different from zero for all t $(0 < t < 1)$. This is equivalent to saying that the oriented arc parametrized by w(t) admits a continuously varying unit tangent vector. Concerning H^p-spaces and general function-theoretic background, the reader may consult [13]. An *inner function* or *function of class* U *in the sense of Seidel* is a function f of class H^∞ in D with $|f(e^{i\theta})| = 1$ a.e. The *outer factor* of an $f \in H^p$ is the quotient of f by its inner factor in the canonical factorization, and f is an *outer function* if it is equal to its outer factor. (The terms "inner" and "outer," introduced by Beurling, are not particularly suggestive; but they correspond to a distinction that is vital in many questions.)

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1. A SUFFICIENT CONDITION FOR SMIRNOV DOMAINS

THEOREM 1. *Suppose that* $f(z)$ *maps* $|z| < 1$ *conformally onto the interior of a Jordan curve* F *and that* $f'(z) \in H^p$, *for some* $p > 0$. *Suppose that for every* $\varepsilon > 0$ *it is possible to choose a (possibly empty) collection* F_i *of closed sub-arcs of* F *with the two properties*

(i) $F \setminus \bigcup F_i$ *consists of smooth arcs,*

(ii) $\sum d_i^{1/2} \log(1/d_i) < \varepsilon$, *where* d_i *denotes the diameter of* F_i $(d_i \leq 1/2)$. *Then* f' *is an outer function, that is,* $\log|f'(re^{i\theta})|$ *is the Poisson integral of* $\log|f'(e^{it})|$. *In particular, if* F *is rectifiable (so that* $f' \in H^1$*), then* F *bounds a Smirnov domain.*

Remark. The restriction $d_i \leq 1/2$ is of course not essential, and it can be removed if in (ii) we replace $\log\frac{1}{d_i}$ by $1 + \log^+\left(\frac{1}{d_i}\right)$.

The proof is based on several lemmas. The first two are essentially contained in [10, p. 474], and we include them here for the reader's convenience.

LEMMA 1. *Let* $f(z)$ *be univalent in* $|z| < 1$, *and let* $f'(z) \in H^p$, *where* $p > 0$. *Then, if* $U(z)$ *denotes the inner factor of* $f'(z)$,

$$|U(re^{i\theta})| \geq A(1 - r)^{2+1/p}, \tag{1}$$

where A *is a positive number depending only on* f.

Proof. By a well-known distortion theorem on univalent functions, there exists a positive constant A_1, independent of θ, such that $|f'(re^{i\theta})| \geq A_1(1 - r)^2$. If $f' = Ug$, then $g \in H^p$, and hence $|g(re^{i\theta})| \leq A_2(1 - r)^{-1/p}$. These inequalities together yield the required lower bound on U.

LEMMA 2. *Under the hypotheses of Lemma* 1, *if*

$$\log U(z) = \int \frac{z + w}{z - w}\, dm(w) \tag{2}$$

(here m *is a nonnegative measure on* C, *singular with respect to arc length), and* I *is an arc of* C *having length* h, *then*

$$m(I) \leq A_3 h \log 1/h. \tag{3}$$

Proof. We may clearly suppose $h \leq 1/2$. By (2),

$$\log|U(re^{i\theta})| = -\int \frac{1 - r^2}{|re^{i\theta} - w|^2}\, dm(w). \tag{4}$$

In particular, if we choose $z = re^{i\theta}$ on the radius vector of the midpoint of I, and write $r = 1 - h$, then the right side of (4) is not greater than $-m(I)/2h$. This, together with the lower bound for $\log|U(re^{i\theta})|$ obtained from (1), gives (3).

LEMMA 3. *For* $a \leq t \leq b$, *let* $F(t) = F_1(t) + F_2(t)$, *where* $F_1(t)$ *is real and absolutely continuous, and* $F_2(t)$ *is continuous, nondecreasing, and purely singular with respect to Lebesgue measure, with* $F_2(b) > F_2(a)$. *Then* $F'(c) = +\infty$, *for some* c $(a < c < b)$.

This lemma follows from a theorem of de la Vallée-Poussin. See for instance [14, pp. 125-128] or [8, Chapitre IX, especially p. 185] (there is a tendency to ignore these important results, in modern texts).

LEMMA 4. *If* f(z) *is univalent in* D *and maps* D *conformally onto a Jordan domain* G *such that an arc about* $e^{i\theta_0}$ *is mapped on a smooth arc on the boundary of* G, *then* $\lim_{r\to 1} f'(re^{i\theta_0})$ *exists and is not zero.*

This follows from a theorem of Lindelöf (see [6, p. 372]).

LEMMA 5. *Let* G *be a Jordan domain,* F_0 *an arc on its boundary,* d *the diameter of* F_0, *and* b *a point of* G *having distance at least* c *from* F_0 $(c > d)$. *If* D *is mapped conformally on* G *so that* 0 *is taken into* b, *then the pre-image of* F_0 *on* C *has length not greater than* $15(d/c)^{1/2}$.

Proof. The length of the pre-image is 2π times the harmonic measure of F_0 relative to b, and the lemma is a consequence of known estimates of harmonic measures (see for example [5, p. 42, Theorem III]). We can easily obtain a direct proof by using the Carleman "enlargement of domain" principle of majorization, replacing G by a suitable domain G' whose boundary is a circle enclosing F_0, joined to infinity along a radial slit. The harmonic measure of the circular part of the boundary of G' relative to a point in G is then easily estimated (see [11, p. 419, Theorem XXI] or [5, p. 39, Theorem VI]).

The referee has remarked that Lemma 5 can also be deduced from the Milloux-Schmidt inequality [9, p. 104].

Proof of Theorem 1. In the proof, there is no essential loss of generality if we suppose that f(0) has distance at least 225 from F. Suppose first that I is an arc of C mapped into $F \setminus \bigcup F_i$ by f(z). We assert that $m(I) = 0$. Indeed, $\log|f'(re^{i\theta})|$ is the Poisson integral of the measure

$$d\alpha(t) = \log|f'(e^{it})|\,dt - 2\pi\,dm(e^{it}).$$

Suppose $m(I) > 0$. Then, by Lemma 3, $\frac{d\alpha}{dt} = -\infty$ at some point $e^{it_0} \in I$, and therefore

$$\log|f'(re^{it_0})| \to -\infty \qquad \text{as } r \to 1,$$

by a well-known property of the Poisson-Stieltjes integral. But, by hypothesis, f maps I onto a smooth arc, and so, by Lemma 4, $|f'(re^{it_0})|$ cannot tend to zero for $e^{it_0} \in I$. This contradiction establishes the assertion that $m(I) = 0$.

Consider next the arc F_i and its pre-image C_i on C. By Lemma 5, the length of C_i is not greater than $d_i^{1/2}$, and so

$$m(C) \leq \sum m(C_i) \leq A_3 \sum d_i^{1/2} \log d_i^{-1/2}$$

(by Lemma 3). By hypothesis (ii), it follows that $m(C) \leq \varepsilon A_3/2$. Since ε is arbitrary, $m(C) = 0$; in other words, U(z) is constant. This completes the proof.

Remark 1. Concerning the problem of Smirnov domains, Tumarkin introduced in [15] the notion of an (oriented, rectifiable) *arc of class* S (that is, of Smirnov type): this is an arc F_1 such that there exists a Smirnov domain whose (oriented) boundary admits F_1 as a sub-arc. From the analysis given in [15], it follows, in terms of our notation above, that if f maps I onto a Smirnov arc, then $m(I) = 0$. Hence, in the case $p = 1$, Theorem 1 remains true if in the hypothesis (i) "smooth arcs" is replaced by "Smirnov arcs." This gives a more general result, since every smooth arc is (with either orientation) a Smirnov arc.

Remark 2. The requirement of smoothness in Theorem 1, or of belonging to class S, as discussed in Remark 1, cannot be replaced by the requirement that the arc be of class Lip α, for some $\alpha < 1$ (by an arc of class Lip α we mean one that admits a parametrization $w = w(t)$ $(0 \leq t \leq 1)$ with $w(t) \in \text{Lip}\,\alpha$). Indeed, it follows from [3] that for any $\alpha < 1$ there exists a rectifiable Jordan curve of class Lip α that bounds a domain not of Smirnov type (in this connection, see also the next section).

Remark 3. It would be of interest if one could determine geometrical properties of a domain G sufficient to guarantee that the derivative of the mapping function from D onto G be of class H^p (or, alternatively, of bounded characteristic, or that it belong to some other natural class). The only implications of this kind known to the author are

(a) rectifiability of the boundary implies $f'(z) \in H^1$ and

(b) finite area of G implies $\int_D \int |f'(re^{i\theta})|^2 \, r\,dr\,d\theta < \infty$.

These implications are also reversible; results of such simplicity and precision are probably not to be expected in general.

2. PROPERTIES OF PSEUDOCIRCLES

Definition. Let F be a rectifiable Jordan curve, and w_0 a complex number. F is a *pseudocircle with center* w_0, if for every harmonic polynomial $P(w)$

$$P(w_0) = \frac{1}{L} \int_F P(w)\,ds, \tag{1}$$

where s denotes arc-length on F and $L = \int_F ds$. Clearly, w_0 is uniquely determined by F, since taking $P(w)$ in turn to be u and v (where $w = u + iv$), we can deduce from (1) that w_0 lies at the center of gravity of F (assuming a uniform mass distribution on F). It is also easy to see that w_0 lies interior to F. It cannot lie outside, because for any w_0 exterior to F we can construct a harmonic polynomial that vanishes at w_0 and is positive on F; contrary to (1). Likewise, w_0 cannot lie on F, since (1) must in fact hold for all functions $P(w)$ continuous in $G \cup F$ and harmonic in G (where G is the domain interior to F), and among such functions we can find one vanishing at a given point $w_0 \in F$ and positive elsewhere on F. We also state several alternative definitions (2), (3), (4), of pseudocircles, leaving to the reader the trivial verifications of their equivalence with (1):

$$\int_F (w - w_0)^n\,ds = 0 \qquad (n = 1, 2, \cdots). \tag{2}$$

$$\int_F \frac{ds}{w - a} = \frac{L}{w_0 - a} \qquad (a \notin G \cup F). \tag{3}$$

$$\int_F e^{aw}\,ds = Le^{aw_0}. \tag{4}$$

Clearly every circle is a pseudocircle whose "center" (in the sense of our definition) coincides with its center.

The following theorem gives some less obvious properties of pseudocircles; in particular, (i) is the basis of their relationship to the Smirnov condition.

THEOREM 2. *Each of the following three conditions is necessary and sufficient that the Jordan domain* G *with rectifiable boundary* F *be a pseudocircle with center at* $w = 0$:

(i) *If* $f(z)$ *maps* D *conformally on* G *and* $f(0) = 0$, *then* $|f'(e^{i\theta})|$ *is constant almost everywhere.*

(ii) *For all* w *exterior to* $\overline{G}$, *the logarithmic potential at* w *due to a unit mass uniformly distributed along* F *is the same as that due to a unit mass at* $w = 0$.

(iii) *Let* a *be any point exterior to* $\overline{G}$; *then for each continuous determination of* $\arg(a - w)$, *where* $w \in \overline{G}$,

$$\frac{1}{L}\int_F \arg(a-w)\,ds = \arg a \qquad (\textit{where } L = \text{length of } F).$$

(Qualitatively, this condition expresses a kind of symmetry "on the average" with respect to lines through 0.)

Proof. Consider first (i). Suppose $|f'(e^{i\theta})| = A$ a.e. Then, for $n \geq 1$,

$$\int_F w^n\,ds = \int_0^{2\pi} [f(e^{i\theta})]^n\,|f'(e^{i\theta})|\,d\theta = 2\pi A[f(0)]^n = 0,$$

so that (2) holds and F is a pseudocircle. Conversely, if F is a pseudocircle, then $\int_F w^n\,ds = 0$ for $n \geq 1$, and by a theorem of Walsh, $\int_F g(w)\,ds = 0$ for every g continuous in $\overline{G}$, holomorphic in G, and vanishing at 0. Choosing in particular $g(w) = h(w)^m$, where h is the function inverse to f, and where m is a positive integer, we see that

$$0 = \int_F h(w)^m\,ds = \int_0^{2\pi} e^{im\theta}\,|f'(e^{i\theta})|\,d\theta \qquad (m = 1, 2, \cdots),$$

which implies that $|f'(e^{i\theta})|$ is constant almost everywhere.

Now consider (ii). Suppose F is a pseudocircle with center at 0. Since $w \notin \overline{G}$, $\log|\lambda - w|$ is a continuous function of λ, for $\lambda \in \overline{G}$, and harmonic for $\lambda \in G$, and therefore

$$\frac{1}{L}\int_F \log|\lambda - w|\,ds = \log|w|,$$

as was asserted. To prove the converse, it is clearly sufficient to show that every function $k(w)$ harmonic on $\overline{G}$ can be uniformly approximated on $\overline{G}$ by linear combinations of the harmonic functions $\log|w - a|$ $(a \notin \overline{G})$. Let $\phi(w)$ be analytic on $\overline{G}$, and let $\Re\,\phi(w) = k(w)$. Then $\phi_1(w) = e^{\phi(w)}$ is analytic on $\overline{G}$ and $|\phi_1(w)| \geq c > 0$ for $w \in \overline{G}$. Therefore we can find a sequence of (analytic) polynomials $P_n(w)$ such that $|P_n(w)| \geq c/2$ and $P_n \to \phi_1$ uniformly on $\overline{G}$. Therefore,

$$\log|P_n(w)| \to \log|\phi_1(w)| = k(w).$$

But $\log|P_n(w)|$ is a real constant plus a finite sum of functions of the form $\log|w - a_i|$, where the a_i, the roots of P_n, lie outside $\overline{G}$. Hence we need only verify that constants can be approximated by functions of the required form, and this is so because $\log\left|\dfrac{w + ne^t}{w+n}\right|$ tends to t uniformly for $w \in \overline{G}$, as $n \to \infty$.

The necessity of the condition (iii) is evident, since $\arg(a - w)$ is harmonic on $\overline{G}$. For the sufficiency, we must show that every function $k(w)$ harmonic on $\overline{G}$ can be approximated by linear combinations of functions $\arg(a - w)$. Note first that constants can be approximated, since $\lim_{r\to\infty} \arg(ra - w) = \arg a$, uniformly for $w \in \overline{G}$. Suppose now that $k(w)$ is harmonic in $\overline{G}$ and (without loss of generality) that $|k(w)| \leq \pi/2$. Letting $\psi(w)$ be analytic in $\overline{G}$, with $\Im\,\psi(w) = k(w)$, and $\psi_1(w) = e^{\psi(w)}$, we find, for $w \in \overline{G}$, that

$$|\psi_1(w)| \geq c > 0 \quad \text{and} \quad \Re\,\psi_1(w) \geq 0.$$

Therefore, if the $Q_n(w)$ are polynomials converging uniformly to $\psi_1(w)$ on $\overline{G}$, then from some n onward the zeros of Q_n are outside $\overline{G}$, and also, for a suitable determination, $\log Q_n(w)$ converges to $\psi(w)$ uniformly on $\overline{G}$. Since the imaginary part of $\log Q_n(w)$ is a constant plus a sum of functions of the form $\arg(a_i - w)$, the proof is complete. We remark that in (ii) and (iii) it would have been enough to assume the relevant properties only in certain subsets of the complement of $\overline{G}$, as a more refined analysis shows.

Keldysh and Lavrentiev (see [13, p. 166 ff.]) constructed the first known domains not of Smirnov type. For each of their domains, the mapping function $f(z)$ from D to the domain satisfies the condition $|f'(e^{i\theta})| = 1$ a.e.; thus in our terminology the domains constructed by Keldysh and Lavrentiev are pseudocircles. Keldysh and Lavrentiev proved the existence of pseudocircles having perimeter 2π and arbitrarily small diameter. Each of these curves consists of a countable union of analytic arcs, plus a perfect set of linear measure zero. Because

$$\log|f'(0)| < 0 = \frac{1}{2\pi}\int_0^{2\pi} \log|f'(e^{i\theta})|\,d\theta,$$

the domains violate the Smirnov condition.

For a long time, these were the only known domains not of Smirnov type. Recently [3], the following was shown: Let Z_A denote the class of (continuous) measures m on C such that $|m(I_1) - m(I_2)| \leq Ah$ for each pair of adjacent arcs I_1 and I_2 of length h. Then there exists a positive constant A_1 such that, if $m \in Z_A$ with $A \leq A_1$, the function

$$g(z) = \exp \int \frac{z+w}{z-w}\,dm(w)$$

is the derivative of a function univalent in D and mapping D on a Jordan domain G. Moreover, for each $\alpha < 1$, there exists a positive number B_α such that if $m \in Z_A$ with $A \leq B_\alpha$, then the boundary of G is of class Lip α. Piranian [12] showed that for every $A > 0$ the class Z_A contains purely singular positive measures m; for such m, the associated functions g have the property $|g(e^{i\theta})| = 1$ a.e., and the curves G are nontrivial pseudocircles. Kahane [12, Section 3] showed that for every $A > 0$, Z_A contains a purely singular positive measure carried on a perfect subset of C. In this case, the boundary of the corresponding pseudocircle consists of a countable union of analytic arcs, plus a perfect set of linear measure zero, as in the case of the original domains constructed by Keldysh and Lavrentiev. Since the classes Z_A also contain positive measures having both nonvanishing singular part and nonvanishing absolutely continuous part, the results of [3] show also the existence of non-Smirnov domains that are not pseudocircles.

Up to the present time, not much is known of a purely geometrical character regarding the Smirnov condition, and for this reason it is possible that further study of pseudocircles, where the Smirnov condition fails in an extreme way, so to speak, might prove suggestive.

In closing, we suggest that aside from their relation to the Smirnov condition, pseudocircles have a certain independent interest from the standpoint of their mean-value relationship to harmonic functions. One interesting question is whether there exists a "pseudosphere" in 3-space, that is, a surface homeomorphic (but not congruent) to a sphere with respect to which the average of each harmonic function equals the value of the function at some fixed point.

We also remark that there exist noncircular *analytic* Jordan curves F for which

$$\int_F w^n ds = 0 \qquad (n = 2, 3, \cdots).$$

An example of such a curve is the image of the unit circle under the map $f(z) = (z - a)^3 + a^3$, with $a > \sqrt{3}$, as the reader may easily verify. Nevertheless, the only curves satisfying the additional moment condition corresponding to $n = 1$ are circles centered at 0 and pseudocircles centered at 0; the latter, being non-Smirnov curves, are of necessity quite pathological.

Finally, we remark that with respect to *areal averages* a domain analogous to the (noncircular) pseudocircle does not exist. Indeed, Epstein [4] showed that if G is any simply connected plane domain of finite area and w_0 a point of G such that

$$k(w_0) = \iint_G k(w)\, d\sigma \qquad (d\sigma = \text{Lebesgue areal measure})$$

for every function k harmonic in G and integrable over G, then G is a circular disk and w_0 is its center. Closely related problems had been studied earlier by Brödel [2]. On the other hand, J. Wolff constructed a purely discrete positive measure in $|w| < 1$, without mass at 0, that has the reproducing property at 0 for harmonic functions. (His construction is formulated not in terms of reproducing measures for harmonic functions, but in terms of the equivalent property analogous to (4).) For references and further discussion on this, see [1].

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