SOME POINT SETS ASSOCIATED WITH TAYLOR SERIES

Fritz Herzog and George Piranian

1. INTRODUCTION

Let P be a property which a function f(z) (regular in $|z| < 1$) or its Taylor series $\Sigma a_n z^n$ may or may not possess at a point $z = e^{i\theta}$ on the unit circle C. And let K(P) be the class of point sets E on C for which there exists a function $f = f_E$ (again regular in $|z| < 1$) possessing the property P at each point of E and at no point of C — E. In the case where P is the property of convergence of the Taylor series of f, Sierpiński [11] (see also Steinhaus [13, pp. 435-436]) pointed out that for reasons of cardinality the class K(P) cannot contain every set on C: indeed, there are 2^c point sets on C, but only $c^{\aleph_0} = c$ analytic functions f; in the application of this argument it is, of course, quite irrelevant what the property P may be.

The problem of obtaining a characterization of the sets of convergence of Taylor series was attacked with considerable success by Lusin [7], Steinhaus [13] and [14], Neder [10], and Mazurkiewicz [9]. Recently, further advances have been made by the present authors, partly in collaboration with Paul Erdös [1], [4], [5]. In particular, it was shown in [4, I] that K(P) contains every set of type F_σ, that is, every set which can be represented as the union of countably many closed sets. The same is true (see [5]) if P denotes the property that the function f possesses a radial limit on the radius vector of the point $e^{i\theta}$.

In the present paper, we consider the problem of determining the class K(P) in the case in which P denotes boundedness of the sequence $\{s_m(e^{i\theta})\}$, where $s_m(z) = \Sigma_{k=0}^{m} a_k z^k$; also, in the case in which P denotes boundedness of f on the radius $z = re^{i\theta}$ $(0 \leq r < 1)$. In both cases, the problem is amenable to complete solution: K(P) consists of all sets of type F_σ.

In both of these instances, the proof that every set of type F_σ on C is in K(P) could be obtained by appropriate modifications of the construction in [5]. However, we prefer giving a proof based on a new construction. This construction avoids the appeal which had previously been made to Hermite's and Hurwitz's theorem on the approximation of real numbers by rationals (see [3] and [6]), and it is simpler in detail than the earlier constructions. Its simplicity is due largely to the introduction of the polynomials $[(z^n - 1)/(z - 1)]^2$; the substitution of these functions for Lusin's polynomials $(z^n - 1)/(z - 1)$, which had previously been used, was suggested by Paul Erdös in a conversation on sets of convergence.

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2. SOME PRELIMINARY ESTIMATES

The basic functions in our construction are polynomials of the form g(z,
where n is a positive integer, t denotes a point on the unit circle C, and

$$g(z, n) = [(z^n - 1)/(z - 1)]^2 = (1 + z + z^2 + \cdots + z^{n-1})^2$$
$$= 1 + 2z + 3z^2 + \cdots + nz^{n-1} + (n - 1)z^n + \cdots + z^{2n-2}.$$

From the relation

$$|g(z, n)| = \frac{\sin^2 (n\theta/2)}{\sin^2 (\theta/2)} \quad (z = e^{i\theta},\ 0 < |\theta| \leq \pi)$$

it follows that

$$(1) \qquad |g(z, n)| > A_1 n^2 \quad (z = e^{i\theta},\ 0 \leq |\theta| \leq \pi/n).$$

(Here and in what follows, A_1, A_2, $\cdots$ denote positive universal constants.)
other direction we have the inequality

$$(2) \qquad |g(z, n)| < A_2/\theta^2 \quad (z = e^{i\theta},\ 0 < |\theta| \leq \pi).$$

We also need an estimate on the modulus of the sum $g^*(z, n)$ of an arbitrar
consecutive terms from the polynomial $g(z, n)$. To obtain this, we use Abe
mation: for $1 \leq m \leq m + k \leq n$ we have

$$m + (m + 1)z + \cdots + (m + k)z^k$$

$$= (m + k)(1 + z + \cdots + z^k) - \sum_{\mu=1}^{k} (1 + z + \cdots + z^{\mu-1})$$

$$= (z - 1)^{-1}\{(m + k)(z^{k+1} - 1) - \sum_{\mu=1}^{k} (z^\mu - 1)\}.$$

Thus the modulus of the sum of $k + 1$ consecutive terms taken from the fir
of $g(e^{i\theta}, n)$ does not exceed $A_3 n/|\theta|$. And since the same estimate holds
second "half" of $g(e^{i\theta}, n)$, we conclude that

$$(3) \qquad |g^*(z, n)| < A_4 n/|\theta| \quad (z = e^{i\theta},\ 0 < |\theta| \leq \pi).$$

The functions which we shall construct in the following sections will be

$$(4) \qquad \sum_{\nu=1}^{\infty} b_\nu z^{h_\nu} g(z/t_\nu, n_\nu),$$

where the t_ν are points on C, the n_ν are positive integers, and the b_ν ar
coefficients. The exponents h_ν are nonnegative integers selected in such
no two terms on the right side of (4) have like powers of z.

3. THE BASIC TECHNIQUE

In this section we exhibit our simplest result. In later sections we indicate modifications in the construction which lead to more general theorems.

THEOREM 1. *Let* E *be a set of type* F_σ *on* C. *Then there exists a function* $f(z) = \Sigma a_m z^m$ *with the following properties:*

(i) *at each point of* E, *the sequence* $\{s_m(z)\}$ *converges;*

(ii) *at each point of* C - E, *the sequence* $\{s_m(z)\}$ *is unbounded.*

In the proof we disregard the trivial case where E = C. We write $C - E = \bigcap G_j$, where G_j is open and $G_j \supset G_{j+1}$ $(j = 1, 2, \cdots)$. Each of the sets G_j is the union of non-overlapping open arcs I_{jk}. It will be convenient to assume that the lengths $|I_{jk}|$ satisfy the restriction

$$(5) \qquad |I_{jk}| \leq 2^{-j}.$$

If that is not the case, we replace each arc I_{jk} which is of length greater than 2^{-j} by the union of a finite number of adjacent open arcs of length less than or equal to 2^{-j}. This is accomplished by dropping a finite number of points from G_j, which can be done in such a manner that no point on C is dropped from more than one of the sets G_j. The notations G_j and I_{jk} will be used in the following for the sets and arcs altered in the manner just described. The fact that the condition $G_j \supset G_{j+1}$ may no longer hold will not affect the validity of the proof.

Let ω_1 and ω_2 be the endpoints of the arc I_{jk}. Then, for each positive integer p for which $p^{-1/2} \leq |I_{jk}|/2$, we denote by t_{jkp} that point of I_{jk} which lies at an angular distance $p^{-1/2}$ from ω_1 or from ω_2, according as p is odd or even. We conclude from (5) that each p occurring in this construction satisfies the inequality

$$(6) \qquad p \geq 2^{2j+2} \geq 16.$$

If z is any point in I_{jk}, there exists an integer p such that the angular distance from z to t_{jkp} is less than $(p-2)^{-1/2} - p^{-1/2}$, which by the use of (6) is easily seen to be less than $2p^{-3/2}$. It follows from (1) that, for each z in I_{jk}, there exists an integer p such that

$$(7) \qquad |g(z/t_{jkp}, [p^{3/2}])| > A_1 [p^{3/2}]^2 > A_5 p^3.$$

On the other hand, for each z in $C - I_{jk}$ and for each p selected for I_{jk}, the inequality (2) gives the estimate

$$(8) \qquad |g(z/t_{jkp}, [p^{3/2}])| < A_2 p,$$

and the inequality (3) implies that

$$(9) \qquad |g^*(z/t_{jkp}, [p^{3/2}])| < A_4 p^2,$$

for z in $C - I_{jk}$ and any block g^* of consecutive terms in g.

We now define our function f by the formula

$$f(z) = \sum_j j \sum_k \sum_p p^{-3} z^h g(z/t_{jkp}, [p^{3/2}]). \tag{10}$$

Here the exponents $h = h_{jkp}$ are to be chosen in accordance with the remarl the end of Section 2.

To show that the function f has the property (ii), we note that if z_1 lies it lies in each of the sets G_j with one possible exception. This implies that index j, except possibly one, there exist integers k and p and a block of cc terms in the Taylor series of f, namely, the corresponding polynomial jp^{-3} whose sum has, by (7), a modulus greater than $A_5 j$ at the point z_1.

To show that f has the property (i), let z_2 be a point in E. Then z_2 lie most finitely many of the arcs I_{jk}. The sum of the moduli of the terms in (correspond to arcs I_{jk} that do not contain z_2 is by (8) not greater than

$$A_2 \sum_j j \sum_k \sum_p p^{-2} \leq A_6 \sum_j j \sum_k |I_{jk}|^2 . \tag{11}$$

It follows from (5) that, for each index j,

$$\sum_k |I_{jk}|^2 \leq 2^{-j} \sum_k |I_{jk}| \leq (2\pi) 2^{-j}.$$

Hence the series in (11) converge.

We now consider the sum of those terms in (10) which correspond to the many arcs I_{jk} that contain z_2. We only have to show that each correspondi $\Sigma_p p^{-3} z^h g$ converges at z_2, and this follows at once from the fact that, for large p, the inequality (2) implies that $|g| < A_2/(\delta/2)^2$, where δ is the ang tance of z_2 from the nearer of the two endpoints of I_{jk}. This completes the the convergence of the triple series in (10) at the point z_2.

Finally, in order to show that also the Taylor series of f converges at z_2, we use the inequality (9), which shows that, for all but finitely many ter series in (10), the modulus of the sum of any block of consecutive terms in does not exceed $A_4 jp^{-1}$. Since

$$jp^{-1} \leq j |I_{jk}|^2/4 \leq j 2^{-2j-2},$$

we conclude that $A_4 jp^{-1}$ can exceed a given positive number for at most fir of the triplets (j, k, p) entering in (10). This completes the proof of Theor

For the terminology used in the following Theorem, we refer to the Intr

THEOREM 2. *If* P *denotes boundedness of the sequence* $\{s_m(e^{i\theta})\}$, *th set* E *on* C *belongs to* K(P) *if and only if* E *is of type* F_σ.

Theorem 1 implies the sufficiency of the condition, and we refer to [2,] a proof of the necessity. As a matter of fact, only the continuity of the func $s_m(e^{i\theta})$ is needed in that proof.

4. EXTENSION TO TOEPLITZ TRANSFORMS OF $\{s_m(z)\}$

Mazurkiewicz [8] showed that if B is a sequence-to-sequence transformation represented by a regular Toeplitz matrix, there exists a Taylor series $\Sigma a_m z^m$ with $a_m \to 0$ and with the transform $B\{s_m(z)\}$ diverging at every point of C. Minor modifications of his proof yield the corresponding proposition for regular sequence-to-function transformations

$$t(x) = \sum_{m=o}^{\infty} a(x, m) s_m \tag{12}$$

such as the Abel transformation. We will discuss briefly an extension of Theorem 1 in the direction of these results.

Let B denote a regular Toeplitz transformation or a transformation of the form (12), and let f be the function (4) constructed in the proof of Theorem 1. If the exponents h_ν in the series (4) are chosen so as to create sufficiently long gaps between the polynomials under the summation sign, then f has the property that at each point of C - E the sequence $B\{s_m(z)\}$ is unbounded. The proof is similar to the proofs of Steinhaus' classical theorem [12] on sequences of 0's and 1's, of Mazurkiewicz's theorem mentioned above, and of the authors' main result in [5]. We omit details, since they are typographically tedious but conceptually elementary.

A further generalization can be made by means of the diagonal process:

THEOREM 3. *Let* E *be a set of type* F_σ *on* C, *and let* $\{B_k\}$ *be a denumerable set of regular Toeplitz transformations. Then there exists a function* $f(z) = \Sigma a_m z^m$ *with the property that at each point of* E *the sequence* $\{s_m(z)\}$ *converges and, for each* k, *the sequence* $B_k\{s_m(z)\}$ *exists and is unbounded at every point of* C - E.

Again, the theorem remains true if the Toeplitz transformations B_k are replaced by regular sequence-to-function transformations of the form (12).

5. FUNCTIONS OF BOUNDED MEAN SQUARE MODULUS

THEOREM 4. *If the set* E *in Theorems* 1 *and* 3 *has measure* 2π, *then the function* f(z) *can be constructed in such a way that* $\Sigma |a_m|^2 < \infty$.

To prove this, we begin by finding an upper bound for the sum $S_{jkp} = \Sigma |a_m|^2$, extended over those terms that arise from one of the polynomials

$$jp^{-3} z^h g(z/t_{jkp}, [p^{3/2}]).$$

From the relations

$$S_{jkp} = j^2 p^{-6} (1^2 + 2^2 + \cdots + [p^{3/2}]^2 + \cdots + 1^2)$$

and

$$1^2 + 2^2 + \cdots + q^2 + \cdots + 1^2 = q(2q^2 + 1)/3 \le q^3$$

it follows that

$$S_{jkp} \leq j^2 p^{-6}[p^{3/2}]^3 \leq j^2 p^{-3/2}.$$

The sum S_{jk} of all these quantities associated with the interval I_{jk} is not g than

$$j^2 \Sigma p^{-3/2} < 2j^2 (p_{jk} - 1)^{-1/2} < 3j^2 p_{jk}^{-1/2},$$

where p_{jk} is the least of the integers p associated with I_{jk}. Since $p_{jk}^{-1/2} \leq$ it follows that

$$S_{jk} \leq 3j^2 |I_{jk}|/2,$$

and if each set G_j is chosen so that its measure is less than j^{-4}, then Σ_0^∞ This concludes the proof.

6. TRIGONOMETRIC SERIES

For the purpose of transferring some of our results on power series to metric series, we apply the "doubling process" which we first used in Secti 4 of [1]. In other words, we replace (4) by the series

$$\Sigma b_\nu (z^{h_\nu} + z^{2h_\nu}) g(z/t_\nu, n_\nu),$$

and we obtain immediately the following result.

THEOREM 5. *Let* E *be a set of type* F_σ *on* $I = [0, 2\pi)$. *Then there ex series* $\Sigma a_m \cos mx$ *which converges at each point of* E *and has unbounded sums at each point of* I - E. *If* E *has measure* 2π, *the series can be chose* $\Sigma a_m^2 < \infty$.

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Added in proof. K. Zeller [*Über Konvergenzmengen von Fourierreihen*, Arch. Math. 6 (1955), 335-340] has shown that if the set E on the interval I (see Section 6) is of type F_σ, then there exists a Lebesgue integrable function whose Fourier series converges everywhere on E and has unbounded partial sums everywhere on I - E. This result is not contained in Theorem 5 of the present paper, nor does it contain that theorem. On the one hand, the trigonometric series constructed in connection with Theorem 5 is not necessarily a Fourier series; on the other hand, if E has measure 2π, then our construction yields the Fourier series of a function which belongs not only to L_1, but also to L_2.

Michigan State University
and
University of Michigan