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L^q-Differentials for Weighted Sobolev Spaces

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1. Introduction

Let $B(x_0, r)$ denote the open ball in $\mathbf{R}^n$ with center x_0 and radius r. Throughout the paper we assume that all measures are Borel and satisfy $0 < \mu(B) < \infty$ for all balls B.

DEFINITION 1.1. Let μ be a measure on $\mathbf{R}^n$. We say that a function u is *differentiable at x_0 in the $L^q(\mu)$ sense* if

$$\lim_{r\to 0} \frac{1}{r}\left(\fint_{B(x_0,r)} |u(x) - u(x_0) - \nabla u(x_0)\cdot(x - x_0)|^q \, d\mu(x)\right)^{1/q} = 0. \qquad (1)$$

Here and in what follows, the symbol $\fint$ stands for the mean-value integral

$$\fint_B f\,d\mu = \frac{1}{\mu(B)}\int_B f\,d\mu.$$

For μ equal to the Lebesgue measure, the following theorem about L^q-differentials of Sobolev functions is well known (see e.g. Theorem 12 in Calderón and Zygmund [3] or Theorem 1, Chapter VIII in Stein [14]).

THEOREM 1.2. *Let u be a function from the Sobolev space $H^{1,p}(\Omega)$, where $\Omega \subset \mathbf{R}^n$ $(n \ge 2)$ and $1 \le p < n$. Then u is differentiable in the L^q sense with $q = np/(n-p)$ a.e. in Ω. If $p = n$, then the same is true for all $q < \infty$. Moreover, if $u \in H^{1,p}(\Omega)$ and $p > n$, then u can be modified on a set of measure zero so that it becomes differentiable a.e. in Ω in the classical sense.*

Theorem 1.2 can be regarded as a higher-order analog of the classical Lebesgue differentiation theorem: If $u \in L^p_{\mathrm{loc}}(\mathbf{R}^n, \mu)$, $1 \le p < \infty$, and μ is a Radon measure, then μ-a.e. $x_0 \in \mathbf{R}^n$ is an $L^p(\mu)$-Lebesgue point of u; that is,

$$\lim_{r\to 0}\left(\fint_{B(x_0,r)} |u(x) - u(x_0)|^p \, d\mu(x)\right)^{1/p} = 0. \qquad (2)$$

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(See e.g. [15, p. 14] for the case $p = 1$; the case $p > 1$ then follows from the inequality $|a - b|^p \le |a^p - b^p|$ for $a, b \ge 0$.)

It has been shown (see e.g. [8; 11; 12]) that weak solutions of certain elliptic partial differential equations are differentiable a.e. in the classical sense provided that, at a.e. x_0, the L^p-norms of their difference quotients $v_h(X) = (u(x_0+hX) - u(x_0))/h$ remain uniformly bounded as $h \to 0$. This, in turn, follows from (1) with $q = p$.

When applying the same method to weighted elliptic partial differential equations and systems (see [1]), the author arrived at the following problem.

PROBLEM 1.3. For which measures is there a weighted analog of Theorem 1.2?

In the affirmative case, the space $H^{1,p}(\Omega)$ in the formulation of Theorem 1.2 is to be replaced by the weighted Sobolev space $H^{1,p}(\Omega, \mu)$, and the exponent q will also depend on the measure μ. In order to be able to define weighted Sobolev spaces, we restrict our considerations to doubling measures admitting a $(1, p)$-Poincaré inequality, for which the theory of Sobolev spaces is well developed (see e.g. [9]).

In this note we prove the following generalization of Theorem 1.2. The proof is surprisingly simple and seems to be of interest even in the unweighted case, when μ is the Lebesgue measure. In Section 4, further generalizations of this theorem are developed. In particular, a two-weighted situation and Sobolev spaces on metric measure spaces are considered.

PROPOSITION 1.4. *Let μ be a doubling measure on $\mathbf{R}^n$ admitting a weak (q, p)-Poincaré inequality, $1 < p < \infty$. Then every $u \in H^{1,p}(\Omega, \mu)$ is differentiable μ-a.e. in Ω in the $L^q(\mu)$ sense.*

The following theorem gives a more geometrical sufficient condition for differentiability of Sobolev functions in the $L^q(\mu)$ sense.

THEOREM 1.5. *Let μ be a doubling measure on $\mathbf{R}^n$ admitting a weak $(1, p)$-Poincaré inequality, $1 < p < \infty$. Assume that there exists a constant $C > 0$ such that*

$$\frac{\mu(B')}{\mu(B)} \ge C\left(\frac{r'}{r}\right)^s \tag{3}$$

whenever $B = B(x, r)$ and $B' = B(x', r')$ are balls, where $x' \in B$ and $0 < r' \le r$. Let $u \in H^{1,p}(\Omega, \mu)$. Then the following are true.

(i) *If $p < s$, then u is differentiable μ-a.e. in Ω in the $L^q(\mu)$ sense for all $q \le sp/(s - p)$. If $p = s$, then the same is true for all $q < \infty$.*

(ii) *If $p > s$, then u can be modified on a set of μ-measure zero so that it becomes differentiable μ-a.e. in Ω in the classical sense.*

It is shown in Example 3.2 that the critical exponent s in Theorem 1.5(ii) cannot be made smaller in general.

Let us also mention the special case $d\mu = w\,dx$, where w is an A_p weight. By [9, Sec. 15.5], the decay condition (3) holds with $s = np$ whenever $d\mu = w\,dx$

and w is an A_p weight. Hence, the following result is an immediate consequence of Theorem 1.5. Note also that the proofs of Theorem 1.2 given in Stein [14] and Ziemer [15] can also be applied to A_p weights.

COROLLARY 1.6. *Let w be an A_p weight in $\mathbf{R}^n$, with $p_0 = \inf\{p : w \in A_p\}$, $d\mu = w\,dx$, and $u \in H^{1,p}(\Omega, \mu)$.*

(i) *If $p < p_0 n$, then u is differentiable a.e. in Ω in the $L^q(\mu)$ sense for all $q < p_0 np/(p_0 n - p)$. If $p = p_0 n$, then the same is true for all $q < \infty$.*
(ii) *If $p > p_0 n$, then u can be modified on a set of measure zero so that it becomes differentiable a.e. in Ω in the classical sense.*

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2. Weighted Sobolev Spaces and A_p Weights

DEFINITION 2.1. We say that a measure μ on $\mathbf{R}^n$ admits a *weak (q, p)-Poincaré inequality* if there are constants $C > 0$ and $\sigma \geq 1$ such that, for all balls $B = B(x_0, r) \subset \mathbf{R}^n$ and all $\varphi \in \mathcal{C}^\infty(B(x_0, \sigma r))$,

$$\left(\fint_{B(x_0,r)} |\varphi - \varphi_B|^q \, d\mu \right)^{1/q} \leq Cr \left(\fint_{B(x_0,\sigma r)} |\nabla\varphi|^p \, d\mu \right)^{1/p},$$

where $\varphi_B = \fint_B \varphi \, d\mu$.

Let $1 < p < \infty$ and let μ be a measure on $\mathbf{R}^n$ admitting a weak $(1, p)$-Poincaré inequality. Assume also that μ is doubling—that is, assume there is a constant $C > 0$ such that

$$\mu(B(x_0, 2r)) \leq C\mu(B(x_0, r))$$

for all balls $B(x_0, r)$. Let Ω be an open subset of $\mathbf{R}^n$ and define a norm on $\mathcal{C}^\infty(\Omega)$ by

$$\|\varphi\|_{H^{1,p}(\Omega,\mu)} = \left(\int_\Omega |\varphi|^p \, d\mu \right)^{1/p} + \left(\int_\Omega |\nabla\varphi|^p \, d\mu \right)^{1/p}.$$

The weighted Sobolev space $H^{1,p}(\Omega, \mu)$ is the closure of

$$\{\varphi \in \mathcal{C}^\infty(\Omega) : \|\varphi\|_{H^{1,p}(\Omega,\mu)} < \infty\}$$

in the $H^{1,p}(\Omega, \mu)$-norm. In other words, $u \in H^{1,p}(\Omega, \mu)$ if and only if $u \in L^p(\Omega, \mu)$ and there exist $\varphi_j \in \mathcal{C}^\infty(\Omega)$ and a vector-valued function ξ such that $\varphi_j \to u$ and $\nabla\varphi_j \to \xi$ in $L^p(\Omega, \mu)$ as $j \to \infty$. By the doubling property of μ and the Poincaré inequality, the "gradient" ξ of u is unique (see [5, Thm. 10]). We shall denote the unique "gradient" by ∇u. If $d\mu = w\,dx$ and $w^{1/(1-p)}$ is locally integrable, then ∇u is the distributional gradient of u. Note that the weak $(1, p)$-Poincaré inequality holds for all functions in $H^{1,p}(\Omega, \mu)$ and all balls $B(x, r)$ with $B(x_0, \sigma r) \subset \Omega$.

DEFINITION 2.2. Let $1 < p < \infty$. Then w is an A_p *weight* ($w \in A_p$) if there is a constant C such that, for all balls $B \subset \mathbf{R}^n$,

$$\fint_B w(x)\,dx \left(\fint_B w(x)^{1/(1-p)}\,dx \right)^{p-1} \le C.$$

It is well known that if $d\mu = w\,dx$ and $p_0 = \inf\{p : w \in A_p\}$, then μ is doubling and admits the (q,p)-Poincaré inequality for all $q < p_0np/(p_0n - p)$ if $p_0 < p < p_0n$ and for all $q < \infty$ if $p \ge p_0n$ (see e.g. [9, Sec. 15]).

3. Proofs and an Example

In this section we give proofs of Proposition 1.4 and Theorem 1.5. It is also shown that the critical exponent in Theorem 1.5(ii) cannot be made smaller in general.

Unless otherwise stated, the letter C will denote a positive constant whose exact value is unimportant and may change even within a line. We allow C to depend on fixed parameters such as the constants in the doubling condition and the Poincaré inequality.

Proof of Proposition 1.4. Let $x_0 \in \Omega$ be an $L^p(\mu)$-Lebesgue point of both u and ∇u. By (2), μ-a.e. $x_0 \in \Omega$ has this property. Let

$$v(x) = u(x) - u(x_0) - \nabla u(x_0) \cdot (x - x_0).$$

Then $\nabla v(x) = \nabla u(x) - \nabla u(x_0)$ and $v \in H^{1,p}_{\mathrm{loc}}(\Omega,\mu)$. Let $0 < \sigma r < \operatorname{dist}(x_0, \partial\Omega)$ and $B_j = B(x_0, 2^{-j}r)$ for $j = 0, 1, \dots$. Since $v_{B_j} = \fint_{B_j} v\,d\mu \to v(x_0) = 0$ as $j \to \infty$, by the doubling property of μ and the weak $(1,p)$-Poincaré inequality we have

$$\begin{aligned} |v_{B_0}| = |v(x_0) - v_{B_0}| &\le \sum_{j=0}^{\infty} |v_{B_{j+1}} - v_{B_j}| \le C \sum_{j=0}^{\infty} \fint_{B_j} |v - v_{B_j}|\,d\mu \\ &\le C \sum_{j=0}^{\infty} \frac{r}{2^j} \left(\fint_{B(x_0, 2^{-j}\sigma r)} |\nabla v|^p\,d\mu \right)^{1/p} \\ &\le Cr \sup_{0<\rho\le r} \left(\fint_{B(x_0,\sigma\rho)} |\nabla v|^p\,d\mu \right)^{1/p}. \end{aligned}$$

Hence, by the weak (q,p)-Poincaré inequality,

$$\begin{aligned} \frac{1}{r}\left(\fint_{B(x_0,r)} |v|^q\,d\mu \right)^{1/q} &\le \frac{1}{r}\left(\fint_{B(x_0,r)} |v - v_{B_0}|^q\,d\mu \right)^{1/q} + \frac{|v_{B_0}|}{r} \\ &\le C \sup_{0<\rho\le r} \left(\fint_{B(x_0,\sigma\rho)} |\nabla v|^p\,d\mu \right)^{1/p} \to 0 \end{aligned}$$

as $r \to 0$, by (2). □

Theorem 1.5(i) follows directly from Proposition 1.4 and the following result, which is a special case of [7, Thm. 5.1].

THEOREM 3.1. *Let μ be a doubling measure on $\mathbf{R}^n$ admitting a weak $(1, p)$-Poincaré inequality, $1 < p < \infty$, and satisfying the decay condition (3). Then the following are true.*

(i) *If $p < s$, then μ admits a weak (q, p)-Poincaré inequality for all $q \le sp/(s-p)$. If $p = s$ then the same is true for all $q < \infty$.*
(ii) *If $p > s$, then every $u \in H^{1,p}(\Omega, \mu)$ can be modified on a set of μ-measure zero so that it becomes locally Hölder continuous and*

$$|u(x) - u(y)| \le Cr^{s/p}|x-y|^{1-s/p}\left(\fint_{B(x_0,5\sigma r)} |\nabla u|^p \, d\mu\right)^{1/p}$$

for all $x, y \in B(x_0, r)$ with $B(x_0, 5\sigma r) \subset \Omega$.

Proof of Theorem 1.5(ii). Modify u as in Theorem 3.1(ii) and let $x_0 \in \Omega$ be an $L^p(\mu)$-Lebesgue point of ∇u. Let $v(x) = u(x) - u(x_0) - \nabla u(x_0) \cdot (x - x_0)$. Then $\nabla v(x) = \nabla u(x) - \nabla u(x_0)$ and $v \in H^{1,p}_{\mathrm{loc}}(\Omega, \mu)$. Let $0 < 5\sigma r < \operatorname{dist}(x_0, \partial\Omega)$. Then v is Hölder continuous in $B(x_0, r)$ and, by Theorem 3.1(ii), for $x \in B(x_0, r)$ we have

$$\begin{aligned}\frac{|v(x)|}{r} = \frac{|v(x) - v(x_0)|}{r} &\le Cr^{s/p-1}|x - x_0|^{1-s/p}\left(\fint_{B(x_0,5\sigma r)} |\nabla v|^p \, d\mu\right)^{1/p} \\ &\le C\left(\fint_{B(x_0,5\sigma r)} |\nabla u(x) - \nabla u(x_0)|^p \, d\mu(x)\right)^{1/p} \to 0\end{aligned}$$

as $r \to 0$, by (2). □

It is well known that, in the unweighted case, the critical exponent $s = n$ in Theorem 1.2 is best possible. On the other hand, the weights $|x|^\alpha$ with $\alpha > 0$ have lower order of decay $s \ge n + \alpha$ but their critical exponent (for differentiability a.e. in the classical sense) is n. We next show that, also in the case $s > n$, the critical exponent in part (ii) of Theorem 1.5 cannot be made smaller in general.

Note that if $d\mu = w\,dx$, then $\lim_{r\to 0} \mu(B(x, r))/|B(x, r)| = w(x) > 0$ for some $x \in \mathbf{R}^n$ and hence $s \ge n$. At the same time, the existence of singular doubling measures in $\mathbf{R}^n$ admitting a $(1, p)$-Poincaré inequality is still an open question.

EXAMPLE 3.2. Given $s > n \ge 2$ and $0 < \delta < 1$, we find a number $p \ge s - \delta$ and construct a p-admissible weight $\tilde{w}$ satisfying the decay condition (3) so that the Sobolev space $H^{1,p}(\mathbf{R}^n, \mu)$ with $d\mu = \tilde{w}\,dx$ contains a function that is not differentiable at any point.

Fix $\alpha > 0$ such that $n - n\delta/(s-n) \le \alpha < n$ and let $\{q_k\}_{k=1}^\infty$ be a countable dense subset of $\mathbf{R}^n$. Consider the weight

$$w(x) = \sum_{k=1}^{\infty} a_k |x - q_k|^{-\alpha}$$

with $a_k > 0$ for all k and $\sum_{k=1}^\infty a_k < \infty$. Each summand $w_k(x) = |x - q_k|^{-\alpha}$ is an A_1 weight; that is, each w_k satisfies the condition

$$\fint_B w_k(x)\,dx \le C \operatorname*{ess\,inf}_{x\in B} w_k(x)$$

for all balls $B \subset \mathbf{R}^n$ and a constant C independent of B and k. Hence, from the Fubini theorem we derive that $w(x)$ is finite for a.e. $x \in \mathbf{R}^n$ and w is also an A_1 weight.

Let $t = s/n < s - \delta$. Then $\tilde{w} = w^{1-t}$ is an A_t weight and the lower decay condition (3) holds with $s = tn$. Let

$$u_k(x) = \log\max\left\{1, \log\frac{1}{|x-q_k|}\right\}\ (k = 1, 2, \dots) \quad \text{and} \quad u(x) = \sum_{k=1}^{\infty} a_k^t u_k(x).$$

Clearly, the function u is unbounded in every neighborhood of every point and, in particular, it is not differentiable at any point. An elementary calculation shows that u belongs to $H^{1,p}(\mathbf{R}^n, \mu)$ for $p = n + \alpha(t-1) \ge s - \delta$.

4. Generalizations

In this section we describe two further generalizations of Theorem 1.2.

4.1. Two-Weighted Situation

Consider a pair (ν, μ) of doubling measures on $\mathbf{R}^n$. We say that the pair (ν, μ) admits a *weak two-weighted (q, p)-Poincaré inequality* if there are constants $C > 0$ and $\sigma \ge 1$ such that, for all balls $B = B(x_0, r) \subset \mathbf{R}^n$ and all $\varphi \in \mathcal{C}^\infty(B(x_0, \sigma r))$,

$$\left(\fint_{B(x_0,r)} |\varphi - \varphi_B|^q\,d\nu\right)^{1/q} \le Cr\left(\fint_{B(x_0,\sigma r)} |\nabla\varphi|^p\,d\mu\right)^{1/p},$$

where $\varphi_B = \fint_B \varphi\,d\nu$.

A slight modification of the proof of Proposition 1.4 leads to the following two-weighted version of Proposition 1.4.

THEOREM 4.1. *Let (ν, μ) be a pair of doubling measures on $\mathbf{R}^n$ admitting a weak two-weighted (q, p)-Poincaré inequality, $1 < p < \infty$. Let $u \in H^{1,p}(\Omega, \mu)$. Then there exists a representative $\bar{u} \in H^{1,p}(\Omega, \mu)$ of u such that $\bar{u} \in L^q_{\mathrm{loc}}(\Omega, \nu)$ and $\bar{u}$ is differentiable μ-a.e. in Ω in the $L^q(\nu)$ sense.*

Proof. Note first that the weak two-weighted (q, p)-Poincaré inequality implies the following Sobolev inequality:

$$\left(\fint_{B(x,r)} |\varphi|^q\,d\nu\right)^{1/q} \le Cr\left(\fint_{B(x,r)} |\nabla\varphi|^p\,d\mu\right)^{1/p}, \tag{4}$$

for all balls $B(x, r)$ and all $\varphi \in \mathcal{C}_0^\infty(B(x, r))$. This is proved in the same way as Theorem 13.1 in [7].

Let $\varphi_k \in \mathcal{C}^\infty(\Omega)$, $k = 1, 2, \dots$, be a sequence converging to u in $H^{1,p}(\Omega, \mu)$. Let $B(x, r)$ be a ball such that $B(x, 2r) \cup B(x, \sigma r) \subset \Omega$, and choose $\eta \in \mathcal{C}_0^\infty(B(x, 2r))$ so that $0 \le \eta \le 1$, $\eta = 1$ on $B(x, r)$, and $|\nabla \eta| \le 2/r$. Then, by (4), for $\varphi_{k,B} = \fint_{B(x,r)} \varphi_k \, d\nu$ we have

$$
\begin{aligned}
&|\varphi_{k,B} - \varphi_{l,B}| \\
&\qquad \le C\left(\fint_{B(x,2r)} |(\varphi_k - \varphi_l)\eta|^q \, d\nu\right)^{1/q} \\
&\qquad \le Cr\left(\fint_{B(x,2r)} |\nabla\varphi_k - \nabla\varphi_l|^p \, d\mu\right)^{1/p} + C\left(\fint_{B(x,2r)} |\varphi_k - \varphi_l|^p \, d\mu\right)^{1/p}.
\end{aligned}
$$

This and the weak two-weighted (q, p)-Poincaré inequality imply that the functions φ_k form a Cauchy sequence in $L^q(B(x, r), \nu)$ and converge to a function $\tilde{u} \in L^q_{\mathrm{loc}}(\Omega, \nu)$. By taking a subsequence converging pointwise both μ-a.e. and ν-a.e, we obtain that $u = \tilde{u}$ on $\Omega \setminus (E_1 \cup E_2)$, where $\mu(E_1) = 0$ and $\nu(E_2) = 0$. Let $\bar{u} = u$ on $\Omega \setminus E_1$ and $\bar{u} = \tilde{u}$ on E_1. Then $\bar{u} \in H^{1,p}(\Omega, \mu) \cap L^q_{\mathrm{loc}}(\Omega, \nu)$, $\nabla\bar{u} = \nabla u$, and $\bar{u} = u$ μ-a.e. in Ω, while $\bar{u} = \tilde{u}$ ν-a.e. in Ω.

Let $x_0 \in \Omega \setminus E_1$ be an $L^p(\mu)$-Lebesgue point of $\nabla\bar{u}$ such that $\varphi_k(x_0) \to \bar{u}(x_0)$. Define $v_k \in \mathcal{C}^\infty(\Omega)$ by $v_k(x) = \varphi_k(x) - \bar{u}(x_0) - \nabla\bar{u}(x_0) \cdot (x - x_0)$. Let $0 < \sigma r < \operatorname{dist}(x_0, \partial\Omega)$ and $B_j = B(x_0, 2^{-j} r)$, $j = 0, 1, \dots$. Since $v_{k,B_j} = \fint_{B_j} v_k \, d\nu \to v_k(x_0)$ as $j \to \infty$, by using the doubling property of ν and the weak two-weighted $(1, p)$-Poincaré inequality (as in the proof of Proposition 1.4) we have

$$
|v_{k,B_0}| \le Cr \sup_{0<\rho\le r}\left(\fint_{B(x_0,\sigma\rho)} |\nabla v_k|^p \, d\mu\right)^{1/p} + |v_k(x_0)|.
$$

Consequently, by the weak two-weighted (q, p)-Poincaré inequality for (ν, μ),

$$
\begin{aligned}
\frac{1}{r}\left(\fint_{B(x_0,r)} |v_k|^q \, d\nu\right)^{1/q} &\le \frac{1}{r}\left(\fint_{B(x_0,r)} |v_k - v_{k,B_0}|^q \, d\nu\right)^{1/q} + \frac{|v_{k,B_0}|}{r} \\
&\le C \sup_{0<\rho\le r}\left(\fint_{B(x_0,\sigma\rho)} |\nabla v_k|^p \, d\mu\right)^{1/p} + \frac{|v_k(x_0)|}{r}.
\end{aligned}
$$

As $v_k(x_0) = \varphi_k(x_0) - \bar{u}(x_0)$ and $\nabla v_k(x) = \nabla\varphi_k(x) - \nabla\bar{u}(x_0)$, letting $k \to \infty$ yields

$$
\begin{aligned}
&\frac{1}{r}\left(\fint_{B(x_0,r)} |\bar{u}(x) - \bar{u}(x_0) - \nabla\bar{u}(x_0) \cdot (x - x_0)|^q \, d\nu(x)\right)^{1/q} \\
&\qquad\qquad \le C \sup_{0<\rho\le r}\left(\fint_{B(x_0,\sigma\rho)} |\nabla\bar{u}(x) - \nabla\bar{u}(x_0)|^p \, d\mu(x)\right)^{1/p},
\end{aligned}
$$

which tends to zero as $r \to 0$, by (2). $\square$

In [2], the following sufficient condition for the validity of a two-weighted (q, p)-Poincaré inequality is proved.

PROPOSITION 4.2. *Let μ be a doubling measure admitting a weak one-weighted $(1, p)$-Poincaré inequality, $1 < p < \infty$. Let $q > p$, and let ν be a doubling measure satisfying the condition*

$$\frac{r'}{r}\left(\frac{\nu(B')}{\nu(B)}\right)^{1/q} \le C\left(\frac{\mu(B')}{\mu(B)}\right)^{1/p} \tag{5}$$

for some $C > 0$ and all balls $B = B(x, r)$ and $B' = B(x', r')$ such that $x' \in B$ and $0 < r' \le r$. Then the pair (ν, μ) admits a weak two-weighted (q, p)-Poincaré inequality.

This, together with Theorem 4.1, immediately gives the following result.

COROLLARY 4.3. *Let μ be a doubling measure admitting a weak one-weighted $(1, p)$-Poincaré inequality, $1 < p < \infty$. Let $q > p$, and let ν be a doubling measure satisfying the condition* (5). *Let $u \in H^{1,p}(\Omega, \mu)$. Then there exists a representative $\bar{u} \in H^{1,p}(\Omega, \mu)$ of u such that $\bar{u} \in L^q_{\mathrm{loc}}(\Omega, \nu)$ and $\bar{u}$ is differentiable μ-a.e. in Ω in the $L^q(\nu)$ sense.*

REMARK. Note that the decay condition (3) is equivalent to the condition (5) with $\nu = \mu$ and $q = sp/(s - p)$.

4.2. Sobolev Spaces on Metric Measure Spaces

Let $X = (X, d, \mu)$ be a metric space equipped with a Borel regular measure μ satisfying $0 < \mu(B) < \infty$ for every ball $B = B(x_0, r) = \{x \in X : d(x, x_0) < r\}$ in X with $0 < r < \infty$.

Recently, there have appeared several different definitions of Sobolev spaces on metric measure spaces (see e.g. [4; 5; 6; 13]). Here we follow Cheeger [4], which is the most convenient for our purposes. Note that the definition given in Shanmugalingam [13] leads to the same space but does not involve the "differential" D.

A Borel function g on X is an *upper gradient* of a real-valued function f on X if, for all rectifiable paths $\gamma : [0, l_\gamma] \to X$ parameterized by the arc length ds,

$$|f(\gamma(0)) - f(\gamma(l_\gamma))| \le \int_\gamma g\, ds.$$

A function $g \in L^p(X, \mu)$, $1 < p < \infty$, is called a *weak* (*generalized*) *upper gradient* of f if there exist sequences f_j and g_j ($j = 1, 2, \dots$) such that g_j is an upper gradient of f_j and $f_j \to f$ and $g_j \to g$ in $L^p(X, \mu)$. By Theorems 2.10 and 2.18 in [4], there exists a minimal weak upper gradient g_f of f satisfying $g_f \le g$ μ-a.e. in X for all weak upper gradients g of f.

We say that X admits a *weak* (q, p)*-Poincaré inequality* if there are constants $C > 0$ and $\sigma \ge 1$ such that, for all balls $B = B(x_0, r) \subset X$ and all measurable functions f on X,

$$\left(\fint_{B(x_0,r)} |f - f_B|^q \, d\mu \right)^{1/q} \le Cr \left(\fint_{B(x_0,\sigma r)} g_f^p \, d\mu \right)^{1/p},$$

where $f_B = \fint_B f \, d\mu$. Equivalently, we can consider all upper gradients g of f. If X is quasi-convex and all closed balls of finite radius are compact, then by [10] it suffices to consider only Lipschitz functions f.

The following theorem about differentiability of Lipschitz functions on X is due to Cheeger [4, Thm. 4.38].

THEOREM 4.4. *Let $X = (X, d, \mu)$ be a metric measure space equipped with a doubling Borel regular measure μ. Assume that X admits a weak $(1, p)$-Poincaré inequality for some $1 < p < \infty$. Then there exists a countable collection (U_α, X^α) of measurable sets U_α and of Lipschitz "coordinate" functions $X^\alpha = (X_1^\alpha, \dots, X_{k(\alpha)}^\alpha) : X \to \mathbf{R}^{k(\alpha)}$ such that $\mu\big(X \setminus \bigcup_\alpha U_\alpha\big) = 0$ and, for all α, the following hold.*

The functions $X_1^\alpha, \dots, X_{k(\alpha)}^\alpha$ are linearly independent on U_α and $1 \le k(\alpha) \le N$, where N is a constant depending only on the doubling constant of μ and the constants from the Poincaré inequality. If $f : X \to \mathbf{R}$ is Lipschitz, then there exist bounded vector-valued functions $d^\alpha f : U_\alpha \to \mathbf{R}^{k(\alpha)}$ such that, for μ-a.e. $x_0 \in U_\alpha$,

$$\lim_{r \to 0} \sup_{B(x_0,r)} \frac{|f(x) - f(x_0) - d^\alpha f(x_0) \cdot (X^\alpha(x) - X^\alpha(x_0))|}{r} = 0. \tag{6}$$

Moreover, the functions $d^\alpha f$ are unique in the sense that the vector $d^\alpha f(x_0)$ in (6) *cannot be replaced by any other vector in $\mathbf{R}^{k(\alpha)}$.*

We can assume that the sets U^α are pairwise disjoint and extend $d^\alpha f$ by zero outside U^α. Regard $d^\alpha f(x)$ as vectors in $\mathbf{R}^N$ and let $Df = \sum_\alpha d^\alpha f$. The "differential" mapping $D : f \mapsto Df$ is linear and, for all Lipschitz functions f and μ-a.e. $x \in X$,

$$C^{-1} g_f(x) \le |Df(x)| \le C g_f(x), \tag{7}$$

see [4; Sec. 4]. Also (by [13] or [4, Prop. 2.2]), $Df = 0$ μ-a.e. on every set where f is constant.

Define the Sobolev space $H^{1,p}(X, d, \mu)$ as the closure in the $H^{1,p}(X, d, \mu)$-norm of the collection of locally Lipschitz functions on X with

$$\|f\|_{H^{1,p}(X,d,\mu)} = \left(\int_X |f|^p \, d\mu \right)^{1/p} + \left(\int_X |Df|^p \, d\mu \right)^{1/p} < \infty.$$

The uniqueness of Du for every $u \in H^{1,p}(X, d, \mu)$ is guaranteed by [5, Thm. 10].

We can now adapt the proof of Proposition 1.4 to this setting and obtain the following result.

THEOREM 4.5. *Let $X = (X, d, \mu)$ be a metric measure space equipped with a doubling Borel regular measure μ. Assume that X admits a weak (q, p)-Poincaré inequality, $1 < p < \infty$. Let $X^\alpha : X \to \mathbf{R}^{k(\alpha)} \subset \mathbf{R}^N$ be the "coordinate" functions provided by Theorem 4.4 and let $u \in H^{1,p}(X, d, \mu)$. Then, for μ-a.e. $x_0 \in U^\alpha$,*

$$\lim_{r\to 0}\frac{1}{r}\biggl(\fint_{B(x_0,r)}|u(x)-u(x_0)-Du(x_0)\cdot(X^\alpha(x)-X^\alpha(x_0))|^q\,d\mu(x)\biggr)^{1/q} = 0. \quad (8)$$

Proof. Note first that, according to [7, Sec. 14.6], the Lebesgue differentiation theorem (2) holds under the foregoing assumptions on X. Let f_k $(k=1,2,\dots)$ be a sequence of locally Lipschitz functions converging to u both in $H^{1,p}(X,d,\mu)$ and pointwise μ-a.e. in X. Let $x_0\in U^\alpha$ be an $L^p(\mu)$-Lebesgue point of Du and a point of density of U^α such that $f_k(x_0)\to u(x_0)$ as $k\to\infty$.

Let $v_k(x)=f_k(x)-u(x_0)-Du(x_0)\cdot(X^\alpha(x)-X^\alpha(x_0))$; let $r>0$ and $B_j=B(x_0,2^{-j}r)$, $j=0,1,\dots$. Since $v_{k,B_j}=\fint_{B_j}v_k\,d\mu\to v_k(x_0)$ as $j\to\infty$, we obtain (as in the proof of Theorem 4.1) that

$$\frac{1}{r}\biggl(\fint_{B(x_0,r)}|v_k|^q\,d\mu\biggr)^{1/q}\le C\sup_{0<\rho\le r}\biggl(\fint_{B(x_0,\sigma\rho)}g_{v_k}^p\,d\mu\biggr)^{1/p}+\frac{|f_k(x_0)-u(x_0)|}{r}.$$

The first term on the right-hand side is estimated using (7), and since $Dv_k(x)=Df_k(x)-Du(x_0)$, letting $k\to\infty$ yields

$$\begin{aligned}\frac{1}{r}\biggl(\fint_{B(x_0,r)}|u(x)-u(x_0)-Du(x_0)\cdot(X^\alpha(x)-X^\alpha(x_0))|^q\,d\mu(x)\biggr)^{1/q}\\ \le C\sup_{0<\rho\le r}\biggl(\fint_{B(x_0,\sigma\rho)}|Du(x)-Du(x_0)|^p\,d\mu(x)\biggr)^{1/p},\end{aligned}$$

which tends to zero as $r\to 0$. □

Combining Theorem 4.5 and a general version of Theorem 3.1 (see [7, Thm. 5.1]), we derive the following analog of Theorem 1.5 in the setting of metric measure spaces.

COROLLARY 4.6. *Let $X=(X,d,\mu)$ be a metric measure space equipped with a doubling Borel regular measure μ. Assume that X admits a weak* $(1,p)$*-Poincaré inequality with* $1<p<\infty$ *and that μ satisfies the decay condition* (3). *Let $X^\alpha\colon X\to\mathbf{R}^{k(\alpha)}\subset\mathbf{R}^N$ be the "coordinate" functions provided by Theorem 4.4 and let $u\in H^{1,p}(X,d,\mu)$. Then the following are true.*

(i) *If $p<s$, then* (8) *holds for μ-a.e. $x_0\in U^\alpha$ and all $q<sp/(s-p)$. If $p=s$, then the same is true for all $q<\infty$.*
(ii) *If $p>s$, then u can be modified on a set of μ-measure zero so that* (6) *holds for u and μ-a.e. $x_0\in U^\alpha$.*

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