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Normal Embeddings of Semialgebraic Sets

Lev Birbrair & Tadeusz Mostowski

1. Introduction

This paper is devoted to some metric properties of semialgebraic sets with singularities. The metric theory of singularities considers sets as metric spaces. There are several classification problems in this theory; here we consider the problem of bi-Lipschitz classification.

Two metric spaces (X_1, d_1) and (X_2, d_2) are called *bi-Lipschitz equivalent* if there exist a homeomorphism $F: X_1 \to X_2$ and two positive constants K_1 and K_2 such that

$$K_1 d_1(x, y) \le d_2(F(x), F(y)) \le K_2 d_1(x, y)$$

for every $x, y \in X_1$. The homeomorphism F is called *a bi-Lipschitz map*. The bi-Lipschitz classification is stronger than topological and weaker than analytical classifications.

We can define two natural metrics on the same semialgebraic subset of $\mathbb{R}^n$: induced and length. The definition of the length metric came from differential geometry (see, for example, [G]). It is defined as the infimum of lengths of piecewise smooth curves connecting two given points. The Lipschitz classification in terms of the induced metric is more rigid: the equivalence in the induced metric implies the equivalence in the length metric, but not inversely. There exists a special type of sets—so-called normally embedded sets—such that these two classifications are equivalent. A set is *normally embedded* if the induced metric is equivalent to the length metric in the usual sense of metric spaces (see Definition 2.1). Every nonsingular compact semialgebraic subset is normally embedded, but the converse is not true.

The main result of the paper is the following normal embedding theorem: *Every compact semialgebraic set is bi-Lipschitz equivalent to some normally embedded semialgebraic set with respect to the length metric.* It is a metric analog of the normalization theorem [L] or of the desingularization theorem [H]. The proof is based on the so-called pancake decomposition (Section 2) created by Parusinski [P] and Kurdyka [K] (see also [KM] for details). Using a pancake decomposition we can define the pancake metric (Section 3), a semialgebraic metric equivalent

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to the length metric. Then by the tent-procedure (Section 4), which is some sort of blow-up, we obtain the result.

To relate this theorem to the problem of bi-Lipschitz classification, we formulate the following corollary: *Let X be a compact semialgebraic set. Let L_X be a set of all semialgebraic sets bi-Lipschitz equivalent to X with respect to the length metric. We define a semiorder relation on L_X in the following way: $X_2 \prec X_1$ if there exists a map $F\colon X_1 \to X_2$ bi-Lipschitz with respect to the length metric and Lipschitz with respect to the induced metric. Then L_X contains a unique (up to a bi-Lipschitz equivalence with respect to the induced metric) maximal element. This element is normally embedded.* The local two-dimensional version of this proposition is actually stated in [BS].

All theorems and propositions hold true for subanalytic sets. The proofs are the same.

2. Normal Embedding of Semialgebraic Subsets

Let X be a semialgebraic connected subset of $\mathbb{R}^n$. Since every connected semialgebraic set is arcwise connected, we can define the two following metrics on X. The first is the *induced metric* from $\mathbb{R}^n$, which we denote by d_{ind}. The second is the *length metric* (or *internal metric*), defined as follows. Let $x_1, x_2 \in X$, and let Γ be the set of all piecewise smooth curves γ connecting x_1 and x_2 (i.e., $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x_1$, $\gamma(1) = x_2$); define $d_l(x_1, x_2) = \inf_{\gamma\in\Gamma} l(\gamma)$, where $l(\gamma)$ means the length of γ.

DEFINITION 2.1. The set X is called *normally embedded in* $\mathbb{R}^n$ if the metrics d_{ind} and d_l are equivalent. It means that there exists a constant $C > 0$ such that, for each $x_1, x_2 \in X$, we have $d_l(x_1, x_2) \le C d_{\text{ind}}(x_1, x_2)$.

REMARK 2.1. In the same way, we can define the normal embedding in every stratified arcwise connected subset $Y \subset \mathbb{R}^n$. Observe that if X is normally embedded in Y and Y is normally embedded in Z, then X is normally embedded in Z.

DEFINITION 2.2. The set X is called *locally normally embedded at the point* $x_0 \in X$ if there exists a ball $B_{x_0,r}$ centered at x_0 and of radius r such that the set $B_{x_0,r} \cap X$ is normally embedded. In other words, we say that the germ of X at x_0 is normally embedded or the pair (X, x_0) is normally embedded.

DEFINITION 2.3. Let X be a semialgebraic set, and let $Y \subset X$. We say that Y is *relatively normally embedded in* X if there exists a constant $C > 0$ such that, for all $x \in X$ and $y \in Y$, we have $d_l(x, y) \le C d_{\text{ind}}(x, y)$.

REMARK 2.2. Let $X = \bigcup_{i=1}^k Y_i$, and let each Y_i be relatively normally embedded in X. Then X is normally embedded.

PROPOSITION 2.1. *Let X be a compact set locally normally embedded at each point $x \in X$. Then X is normally embedded.*

EXAMPLE 2.1. Every compact smooth submanifold X of $\mathbb{R}^n$ is normally embedded.

EXAMPLE 2.2. The standard cusp $X = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^3 = x_2^2\}$ is not normally embedded at 0. To see this, consider a sequence $t_i = 1/i$ and the points $x^i = (t_i, t_i^{3/2})$ and $y^i = (t_i, -t_i^{3/2})$. Then we have $d_l(x^i, y^i) = 2t_i + o(t_i)$, but $d_{\text{ind}}(x^i, y^i) = 2t_i^{3/2}$. This means that $d_l(x^i, y^i)$ cannot be estimated from above by $Cd_{\text{ind}}(x^i, y^i)$.

EXAMPLE 2.3. A standard β-horn ($\beta \geq 1$) (see [B])

$$H_\beta = \{(x_1, x_2, y) \in \mathbb{R}^3 \mid (x_1^2 + x_2^2)^q = y^{2p};\ y \geq 0;\ \beta = p/q;\ p, q \in \mathbb{N}\}$$

is normally embedded.

EXAMPLE 2.4. A standard β-Holder triangle ($\beta \geq 1$) (see [B])

$$T_\beta = \{(x, y) \in \mathbb{R}^2 \mid y \leq x^\beta;\ y \geq 0;\ 0 \leq x \leq 1\}$$

is normally embedded. (The proof of this fact is straightforward.) A double β-Holder triangle

$$DT_\beta = \{(x, y) \in \mathbb{R}^2 \mid |y| \leq x^\beta,\ 0 \leq x \leq 1\}$$

is also normally embedded.

EXAMPLE 2.5. Consider a subset X of $\mathbb{R}^3$ defined in the following way. Let H_β be a standard β-horn and let $P_{x_1}\colon H_\beta \to \mathbb{R}^2$ be a projection: $P_{x_1}(x_1, x_2, y) = (x_2, y)$. Clearly, $P_{x_1}(H_\beta) = DT_\beta$. Let $\beta_1 > \beta$; then $DT_{\beta_1} \subset DT_\beta$. Define $X = H_\beta - \text{Int}(P_{x_1}^{-1}(DT_{\beta_1}))$. By the same arguments as in Example 2.2, we obtain that X is not normally embedded.

PROPOSITION 2.2 (Pancake Decomposition) [K; KM; P]. *Let $X \subset \mathbb{R}^n$ be a closed semialgebraic set. Then there exists a finite set of subsets $\{X_i\}$ such that:*

(1) *all X_i are semialgebraic closed subsets of X;*
(2) $X = \bigcup_i X_i$;
(3) $\dim(X_i \cap X_j) < \min(\dim X_i, \dim X_j)$ *for every $i \neq j$; and*
(4) *X_i are normally embedded in $\mathbb{R}^n$.*

We shall call these sets X_i *pancakes* and a decomposition satisfying conditions (1)–(4) will be called *a pancake decomposition*.

REMARK 2.3. This result is due to A. Parusinski. Another proof was given in [K]; this proof is presented (for open X being the essential case) in [KM]. In general, L-regular sets from [P] have some additional properties. Namely, each L-regular set is normally embedded but not vice versa. Here we shall use only the properties (1)–(4) just described.

Our main result is the following.

THEOREM 2.1. *Let X be a compact connected semialgebraic subset of $\mathbb{R}^n$. Then, for every $\varepsilon > 0$, there exists a semialgebraic set $X^\varepsilon \subset \mathbb{R}^m$ such that*:

(1) *X^ε is semialgebraically bi-Lipschitz equivalent to X with respect to the length metric*;
(2) *X^ε is normally embedded in $\mathbb{R}^m$; and*
(3) *the Hausdorff distance between X and X^ε is less than ε.*

3. Pancake Metric

Let $X \subset \mathbb{R}^n$ be a closed connected semialgebraic set. Let $\{X_j\}_1^N$ be a pancake decomposition of X. Consider $x_1, x_2 \in X$, and let $\{y_1, \dots, y_k\}$ be a sequence of points satisfying the following conditions:

(1) $y_1 = x_1$ and $y_k = x_2$;
(2) every couple y_i, y_{i+1} lies in one pancake X_j; and
(3) if $y_i, y_{i+1} \in X_j$, then $y_s \notin X_j$ for $s \neq i$, $s \neq i+1$.

Denote by Y_{x_1,x_2} the set of all finite sequences satisfying conditions (1)–(3). For every sequence $y = \{y_1, \dots, y_k\} \in Y_{x_1,x_2}$, we set $l(y) = \sum_{i=2}^k d_{\text{ind}}(y_i, y_{i-1})$ and (finally) we put

$$d_p(x_1, x_2) = \inf_{y \in Y_{x_1,x_2}} l(y).$$

Note that X, being semialgebraic and connected, is actually arcwise connected; hence $d_p(x_1, x_2)$ is well-defined for any couple (x_1, x_2) (see the proof of Theorem 3.2 to follow). We call d_p the *pancake metric.*

THEOREM 3.1. *The function $d_p: X \times X \to \mathbb{R}$ is semialgebraic and defines a metric in X.*

This was actually proved in [KO]; we recall the argument for convenience. We shall need the following lemmas.

LEMMA 3.1. *There exists $K > 0$ such that, for any $x_1, x_2 \in X$, we have*

$$d_p(x_1, x_2) \geq K d_l(x_1, x_2).$$

Proof. Let $K = \min K_j$, where K_j is a constant corresponding to the pancake X_j (see Section 2). Thus, for every $y = \{y_1, \dots, y_k\}$ we have

$$d_{\text{ind}}(y_i, y_{i-1}) \geq K d_l(y_i, y_{i-1}).$$

Hence

$$d_p(x_1, x_2) \geq \sum_{i=2}^k K d_l(y_i, y_{i-1}) \geq K d_l(x_1, x_2),$$

and the lemma is proved. □

LEMMA 3.2. *For every $x_1, x_2 \in X$, there exists $y \in Y_{x_1,x_2}$ such that $d_p(x_1, x_2) = l(y)$.*

Proof. A subsequence of pancakes $\{X_i\} \subset \{X_j\}_1^N$ is called *admissible* (for x_1, x_2) if there exists $y = \{y_1, \dots, y_k\} \in Y_{x_1,x_2}$ such that $y_i \in X_i$. It is enough to prove that, for any admissible subsequence of pancakes $\{X_i\}$, there exists $\tilde{y} \in Y_{x_1,x_2}$ such that $\tilde{y}_i \in X_i$ and $l(\tilde{y}) = \min\{l(y) \mid y \in Y_{x_1,x_2},\ y_i \in X_i\}$. Consider a closed ball $B_{x_1, 2d_p(x_1,x_2)} = B$ centered at the point x_1 and of radius $2d_p(x_1, x_2)$. Let $\tilde{X}_i = X_i \cap X_{i+1} \cap B$ (here $i = 2, \dots, k-1$; $\{X_i\}$ is the fixed admissible subsequence of pancakes). We define the function $\tilde{l}\colon \tilde{X} \times \cdots \times \tilde{X}_{k-1} \to \mathbb{R}$ by $\tilde{l}(y_1, \dots, y_{k-1}) = l(y)$, where $y = \{x_1, y_2, \dots, y_{k-1}, x_2\}$. The function $\tilde{l}$ is continuous and defined on a compact set $\tilde{X}_2 \times \cdots \times \tilde{X}_{k-1}$. Thus, there exists $\tilde{y} \in Y_{x_1,x_2}$ such that $\tilde{y}_i \in \tilde{X}_i$ and $l(\tilde{y}) = \min\{l(y) \mid y \in Y_{x_1,x_2},\ y_i \in \tilde{X}_i\}$ $(i = 2, \dots, k-1)$. □

The sequence $y \in Y_{x_1,x_2}$ such that $d_p(x_1, x_2) = l(y)$ we call *a minimizing sequence corresponding to* x_1, x_2.

COROLLARY 3.1. *The pancake metric is a semialgebraic function defined on* $X \times X$.

This follows from the quantifier elimination theorem of Tarski and Seidenberg and the fact that the graph of l is semialgebraic.

Proof of Theorem 3.1. Let us prove now that d_p is a metric. The first two axioms of a metric follow immediately from the definition of d_p and Lemma 3.1.

To prove the triangle inequality, consider three points $x_1, x_2, x_3 \in X$. Let $y^1 = \{y_1^1, y_2^1, \dots, y_{k1}^1\} \in Y_{x_1,x_2}$ be a minimizing sequence corresponding to x_1, x_2, $y^2 = \{y_1^2, y_2^2, \dots, y_{k2}^2\} \in Y_{x_3,x_2}$ a minimizing sequence corresponding to x_3, x_2, and $y^3 = \{y_1^3, y_2^3, \dots, y_{k3}^3\} \in Y_{x_1,x_3}$ a minimizing sequence corresponding to x_1, x_3. Thus the sequence $z = \{y_1^3, y_2^3, \dots, y_{k3}^3, y_1^2, y_2^2, \dots, y_{k2}^2\} = \{z_1, z_2, \dots, z_{k3+k2}\}$ satisfies the conditions of the definition of a pancake metric $d_p(x_1, x_2)$, except possibly condition (3). To improve it we use the following simplification procedure. If two nonconsecutive points in the sequence y^3 belong to the same pancake X_j, then we skip all points lying in between. Repeating this simplification procedure a finite number of times, we obtain a sequence $z \in Y_{x_1,x_3}$. By the triangle inequality, for the induced distance we have

$$d_p(x_1, x_3) \le l(z) \le l(y^3) = l(y^1) + l(y^2) = d_p(x_1, x_2) + d_p(x_2, x_3).$$

Theorem 3.1 is proved. □

THEOREM 3.2. *The pancake metric is bi-Lipschitz equivalent to the length metric.*

Proof. Let $y \in Y_{x_1,x_2}$ be a minimizing sequence corresponding to $x_1, x_2 \in X$. Let $\{X_i\}_1^{k-1}$ be an admissible sequence of pancakes corresponding to y. Since each X_i is a pancake, there exists a piecewise smooth path $\eta_i\colon [0,1] \to X_i$ such that $\eta_i(0) = y_i$, $\eta_i(1) = y_{i+1}$, and $l(\eta_i) \le K(X_i) d_{\text{ind}}(y_i, y_{i+1})$, where $l(\eta_i)$ denotes the length of η_i. Let $\eta = \eta_1 \eta_2 \cdots \eta_{k-1}$ (the usual product of paths); then $l(\eta) = \sum_{i=1}^{k-1} l(\eta_i)$. Let $K = \max_i K(X_i)$. We thus have

$$d_l(x_1, x_2) \le l(\eta) \le K \sum_{i=1}^{k-1} d_{\text{ind}}(y_i, y_{i+1}) = K d_p(x_1, x_2).$$

Let $\gamma\colon[0,1]\to X$ be any piecewise smooth path connecting x_1 and x_2, and define the following sequence of points $y_i\in Y_{x_1,x_2}$. Put $y_1=x_1$. The point y_1 belongs to some pancake X_{i1}. Define $t_2=\sup\{t\mid\gamma(t)\in X_{i1}\}$. If $x_2\notin X_{i1}$ then $t_2\neq 1$. Let $y_2=\gamma(t_2)$, and suppose that $y_2\in X_{i1}\cap X_{i2}$. Define $t_3=\sup\{t\mid\gamma(t)\in X_{i2}\}$ and $y_3=\gamma(t_3)$. The number of pancakes is finite, so this process will stop for some $y_k=x_2$. Clearly, $y=\{y_1,y_2,\dots,y_k\}\in Y_{x_1,x_2}$ and $l(\gamma)\geq l(y)$. Since γ is an arbitrarily chosen curve, we are led to the inequality

$$d_l(x_1,x_2)\geq d_p(x_1,x_2).$$

Theorem 3.2 is proved. □

4. Tents and Tent Procedure

Let X be a compact connected semialgebraic set, and let $\{X_i\}_{i=1}^N$ be a pancake decomposition of X. Consider the function $\rho_i\colon X\to\mathbb{R}$ defined by $\rho_i(x)=d_p(x,X_i)$, where d_p is the pancake distance. Let $\Gamma_i\subset\mathbb{R}^{n+1}$ denote the graph of ρ_i, and put

$$\mu_i(x)=(x,\rho_i(x)).$$

PROPOSITION 4.1. *The mapping $\mu_i\colon X\to\Gamma_i$ has the following properties.*

(1) *μ_i is a bi-Lipschitz map with respect to the length metrics on X and on Γ_i.*
(2) *$\mu_i(X_j)$ is a pancake in $\mu_i(X)$; in other words, $\{\mu_i(X_j)\}_{j=1}^N$ is a pancake decomposition of $\mu_i(X)$.*
(3) *$\mu_i(X_i)$ is relatively normally embedded in $\mu_i(X)$.*

Proof. (1) Recall that, in any metric space, the distance to a fixed set is a Lipschitz function. So ρ_i is Lipschitz with respect to the pancake metric and, by Theorem 3.2, also with respect to the length metric. Thus, μ_i is Lipschitz with respect to the length metric. Note that $\mu_i^{-1}=\left(\pi\big|_{\Gamma_i}\right)$, where π is a projection on the first n coordinates. Hence, μ_i^{-1} is Lipschitz with respect to the length metric.

(2) There exists a constant B, depending only on n, such that

$$\max\{d_{\text{ind}}(x_1,x_2),|\rho_i(x_1)-\rho_i(x_2)|\}\leq Bd_{\text{ind}}(\mu_i(x_1),\mu_i(x_2))$$

for any $x_1,x_2\in X_j$. Since X_j is a pancake, we obtain

$$d_l(x_1,x_2)\leq Ld_{\text{ind}}(\mu_i(x_1),\mu_i(x_2))$$

for some $L>0$. By (1), the mapping μ_i is bi-Lipschitz; hence,

$$d_l(\mu_i(x_1),\mu_i(x_2))\leq Kd_{\text{ind}}(\mu_i(x_1),\mu_i(x_2))$$

for some $K>0$ and any $x_1,x_2\in X_j$.

(3) We shall prove that there exists a constant $K>0$ such that

$$d_l(\mu_i(x),\mu_i(y))\leq Kd_{\text{ind}}(\mu_i(x),\mu_i(y))$$

for any $x\in X_i$ and $y\in X$. Indeed, by (1), it is enough to find a $K_1>0$ such that

$$d_l(x,y)\leq K_1d_{\text{ind}}(\mu_i(x),\mu_i(y)).$$

Suppose that $\rho_i(y) \leq d_{\text{ind}}(x, y)$, and consider $\tilde{x} \in X_i$ such that $\rho_i(y) = d_p(\tilde{x}, y)$. By the definition of a pancake metric, we have

$$d_p(x, y) \leq d_{\text{ind}}(x, \tilde{x}) + d_p(y, \tilde{x}).$$

Since $d_p(y, \tilde{x}) = \rho_i(y) \leq d_{\text{ind}}(x, y)$ we obtain, by the triangle inequality,

$$d_{\text{ind}}(x, \tilde{x}) \leq 2d_{\text{ind}}(x, y)$$

and thus

$$d_p(x, y) \leq 3d_{\text{ind}}(x, y).$$

Hence, by Theorem 3.2,

$$d_l(x, y) \leq 3Cd_{\text{ind}}(\mu_i(x), \mu_i(y)),$$

where $C > 0$ is a constant satisfying $d_l \leq Cd_p$.

Suppose now that $\rho_i(y) > d_{\text{ind}}(x, y)$. As before, for $\tilde{x} \in X_i$ such that $\rho_i(y) = d_p(\tilde{x}, y)$ we have

$$d_p(x, y) \leq d_p(y, \tilde{x}) + d_{\text{ind}}(x, y) + d_{\text{ind}}(\tilde{x}, y) \leq 3d_p(\tilde{x}, y).$$

On the other hand,

$$\rho_i(y) \leq \max\{d_{\text{ind}}(x, y), \rho_i(y)\} \leq Bd_{\text{ind}}(\mu_i(x), \mu_i(y))$$

for some $B > 0$ depending only on n. Hence,

$$d_p(x, y) < 3Bd_{\text{ind}}(\mu_i(x), \mu_i(y)).$$

By the equivalence of length and pancake metrics we obtain, as before,

$$d_l(x, y) < Kd_{\text{ind}}(\mu_i(x), \mu_i(y))$$

for some $K_1 = 3C \max\{1, B\}$. □

The set $\mu_i(X)$ we call an *i-tent over* X, and the map μ_i we call *i-tent procedure.*

The following proposition is easy; the proof is the same as that of (2) in Proposition 4.1.

PROPOSITION 4.2. *Let $Y \subset X$ be relatively normally embedded in X. Then $\mu_i(Y)$ is relatively normally embedded in $\mu_i(X)$.*

Proof of the Main Theorem 2.1. Let us define the set $\tilde{X} \subset \mathbb{R}^{n+k}$ as follows.

(1) Fix some pancake decomposition $\{X_i\}_{i=1}^k$ of X.

(2) Apply 1-tent procedure to X, and put $\tilde{X}^0 = X$ and $\tilde{X}^1 = \mu_1(X)$.

(2′) Define $\tilde{X}_j^1 = \mu_1(X_j)$; then $\{\tilde{X}_j^1\}_{j=1}^k$ is a pancake decomposition if $\tilde{X}^1$ (by Proposition 4.1).

$(i+1)$ Define $\tilde{X}^i = \mu_i(\tilde{X}^{i-1})$.

$((i+1)')$ Define $\tilde{X}_j^i = \mu_i(\tilde{X}_j^{i-1})$.

Put $\tilde{X} = \tilde{X}^k$.

In the final step we obtain a decomposition $\tilde{X} = \bigcup_{i=1}^k \tilde{X}_j^k$. By Proposition 4.2, all $\tilde{X}_j^k$ are relatively normally embedded. Hence, by Remark 2.2, $\tilde{X}$ is normally embedded in $\mathbb{R}^{n+k}$. It follows from Proposition 4.1 that $\tilde{X}$ is semialgebraically bi-Lipschitz equivalent to X with respect to the length metric.

To prove that $\tilde{X}$ can be chosen close to X (in the sense of Hausdorff distance), it is enough to replace ρ_i (in the tent construction) by $\delta \cdot \rho_i$, with δ small. For example, $\delta = \varepsilon/(k \cdot \operatorname{diam}_{d_p} X)$, where k is the number of pancakes and $\operatorname{diam}_{d_p} X$ is the diameter of X with respect to the pancake metric. □

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References

[B] L. Birbrair, *Local bi-Lipschitz classification of 2-dimensional semi-algebraic sets,* Houston J. Math. 25 (1999), 453–471.

[BS] L. Birbrair and M. Sobolevsky, *Realization of Hölder complexes,* Ann. Fac. Sci. Toulouse Math. (6) 8 (1999), 35–44.

[G] M. Gromov, *Structures metriques pour les varietes riemanniennes,* Cedic, Paris, 1981.

[H] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero,* Ann. of Math. (2) 79 (1964), 109–326.

[K] K. Kurdyka, *On a subanalytic stratification satisfying a Whitney property with exponent 1,* Lecture Notes in Math., 1524, pp. 316–323, Springer-Verlag, Berlin, 1992.

[KM] K. Kurdyka and T. Mostowski, *The gradient conjecture of R. Thom,* preprint (1998).

[KO] K. Kurdyka and P. Orro, *Distance geodesique sur un sous-analytique,* Rev. Mat. Univ. Complut. Madrid 10 (1997), 173–182.

[L] H. B. Laufer, *Normal two-dimensional singularities,* Princeton Univ. Press, Princeton, 1971.

[P] A. Parusinski, *Lipschitz stratifications of subanalytic sets,* Ann. Sci. Ecole. Norm. Sup. (4) 27 (1994), 661–696.

L. Birbrair
Departamento de Matematica
Universidade Federal do Ceara
Campus do Pici, Bloco 914
CEP 60455-760, Fortaleza-Ce
Brasil

lev@mat.ufc.br

T. Mostowski
Uniwersytet Warszawski
Wydzial Matematyki
Informatyki i Mechaniki
Instytut Matematyki
ul. Banacha 2
02-097 Warszawa
Poland

tmostows@duch.mimuw.edu.pl