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Boundary Values and Mapping Degree

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Introduction

This note is an addendum to the paper of Alexander and Wermer [2], in which the authors relate the theory of linking numbers to the question of finding an analytic variety bounded by a given real, odd-dimensional submanifold of $\mathbb{C}^N$.

We give a characterization of the boundary values of holomorphic functions on certain domains in $\mathbb{C}^N$ in similar terms. In fact, the work of Alexander and Wermer contains such a characterization *in the case of functions of class* $\mathscr{C}^1$. It seems that the methods used in [2] require this degree of smoothness, but we have found that it is possible to obtain a result that characterizes the *continuous* functions that are boundary values of holomorphic functions that is entirely in the spirit of [2]. Specifically, we shall prove the following result.

Main Theorem. *Let Ω be a bounded domain in $\mathbb{C}^N$ with boundary of class $\mathscr{C}^2$, and assume that $\bar{\Omega}$ has a Stein neighborhood basis. A continuous function f on $b\Omega$ is of the form $F\big|_{b\Omega}$ for a function $F \in \mathscr{C}(\bar{\Omega})$ that is holomorphic on Ω if and only if the following condition is met.*

(∗) *With Γ_f the graph $\{(z, f(z)) : z \in b\Omega\}$, a compact subset of $\mathbb{C}^{N+1}$, if Q is a $\mathbb{C}^N$-valued holomorphic map defined on a neighborhood of $\bar{\Omega} \times \mathbb{C}$ with $Q^{-1}(0) \cap \Gamma_f = \emptyset$, then the degree of the map $b\Omega \to \mathbb{C}^N \setminus \{0\}$ given by $z \mapsto Q(z, f(z))$ is nonnegative.*

Recall that a closed set E in $\mathbb{C}^N$ is said to have a Stein neighborhood basis if it is the intersection of a sequence of domains of holomorphy in $\mathbb{C}^N$. If E is the closure of a strictly pseudoconvex domain or a polydisc in $\mathbb{C}^N$, then it has a Stein neighborhood basis.

The case of the main theorem in which f is of class $\mathscr{C}^1$ is contained in [2] as a very special case of the main results of that paper.

The main theorem seems to be new, even in the setting of classical function theory, where a version of the result is the following. Let $\mathbb{U}$ denote the open unit disc in the complex plane.

Corollary. *A continuous function f on $b\mathbb{U}$ extends holomorphically through $\mathbb{U}$ if and only if, for each polynomial $p(z) = p(z_1, z_2)$ in two complex variables*

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such that, with $\psi_{f,p}(\zeta) = p(\zeta, f(\zeta))$, if $\psi_{f,p}$ is zero-free on $b\mathbb{U}$ then the change in argument around $b\mathbb{U}$ of the function $\psi_{f,p}$ is nonnegative.

In particular, if the function f is smooth then the condition is that (with p and $\psi_{f,p}$ as in the corollary) the integral $(1/2\pi i)\int_{b\mathbb{U}}(d\psi_{f,p}/\psi_{f,p})$ be nonnegative.

The proof of the Main Theorem proceeds by induction on dimension. The case of planar domains is based on some results on polynomial convexity; the higher-dimensional case depends on a suitable slicing argument. The arguments make systematic use of the Bochner–Martinelli kernels.

We preface the proof with a section that assembles some information on degree theory.

1. Degree Theory

Our discussion draws essentially on the theory of the degree of mappings. We will restrict attention to the theory of the degree of mappings from sets in manifolds to Euclidean spaces.

Fix an oriented smooth manifold $\mathscr{N}$ of dimension N. To each triple (f, Ω, y) consisting of a relatively compact open set Ω in $\mathscr{N}$, a continuous map f from $\bar{\Omega}$ into $\mathbb{R}^N$, and a point $y \in \mathbb{R}^N \setminus f(b\Omega)$, there is assigned an integer $d(f, \Omega, y)$, *the degree of f*. The function d has the following properties:

(d1) $d(\mathrm{id}, \Omega, y) = 1$ if $y \in \Omega$;

(d2) $d(f, \Omega, y) = d(f, \Omega_1, y) + d(f, \Omega_2, y)$ whenever Ω_1 and Ω_2 are disjoint open subsets of Ω such that $y \notin f(\bar{\Omega} \setminus (\Omega_1 \cup \Omega_2))$;

(d3) $d(h(t, \cdot), \Omega, y(t))$ is independent of $t \in [0, 1]$ whenever $h\colon [0, 1] \times \bar{\Omega} \to \mathbb{R}^N$ is continuous, and $y\colon [0, 1] \to \mathbb{R}^N$ is continuous and satisfies $y(t) \notin h(t, b\Omega)$ for all $t \in [0, 1]$;

(d4) given (f, Ω, y) and (g, Ω, y) as before, if $f\big|_{b\Omega} = g\big|_{b\Omega}$ then $d(f, \Omega, y) = d(g, \Omega, y)$;

(d5) if $\Omega_1 \subset \Omega$ (Ω_1 open), then $d(f, \Omega, y) = d(f, \Omega_1, y)$ if $y \notin f(\bar{\Omega} \setminus \Omega_1)$.

Property (d4) implies that one can assign a degree to (f, Ω, y) when the continuous map f to $\mathbb{R}^N$ is defined only on $b\Omega$ and satisfies the condition that $y \notin f(b\Omega)$.

Degree theory in the form that we shall need is developed in [3] and [17]. An axiomatic development (for maps from $\mathbb{R}^N$ to itself) is given in [4].

If Ω is a domain in the plane with boundary a finite collection of mutually disjoint simple closed curves and if $f \in \mathscr{C}(\bar{\Omega})$, then

$$d(f, \Omega, 0) = \frac{1}{2\pi i}\Delta_{b\Omega} \log f = \frac{1}{2\pi}\Delta_{b\Omega} \operatorname{Arg} f.$$

Given a bounded domain Ω in $\mathscr{N}$ with smooth boundary, if $f\colon b\Omega \to \mathbb{R}^N \setminus \{0\}$ is a smooth map then the degree is given by an integral formula,

$$d(f, \Omega, 0) = \frac{1}{S_{N-1}} \int_{b\Omega} f^*\tau,$$

with $S_{N-1} = 2\pi^{N/2}/\Gamma(N/2)$ (the area of the unit sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$) and with τ the form given by

$$\tau = \frac{1}{|x|^N} \sum_{j=1}^{N} (-1)^{N-1} x_j dx_1 \wedge \cdots \wedge [j] \wedge \cdots \wedge dx_N,$$

where the expression $[j]$ indicates the omission of the jth term; see [17]. The $(N-1)$th de Rham cohomology group with complex coefficients of $\mathbb{R}^N \setminus \{0\}$ is isomorphic to $\mathbb{C}$ and so, if ϑ is any closed smooth $(N-1)$-form on $\mathbb{R}^N \setminus \{0\}$ that is not exact, then

$$d(f, \Omega, 0) = c(\vartheta) \int_{b\Omega} f^* \vartheta$$

for a constant $c(\vartheta)$ that depends only on the form ϑ, not on the domain Ω or on the map f. The constant $c(\vartheta)$ is determined by the condition that the form $\tau - c(\vartheta)\vartheta$ be exact. Given ϑ, we can determine $c(\vartheta)$ by taking Ω to be the unit ball in $\mathbb{R}^N$ and f the identity map. It follows that $c(\vartheta)$ is determined by the equation $c(\vartheta)^{-1} = \int_{\mathbb{S}^{N-1}} \vartheta$.

It will be convenient to use the following notation: If $g = (g_1, \dots, g_r)$ is a vector of complex-valued functions on $\mathbb{C}^N$, then $\omega(g) = \omega(g_1, \dots, g_r) = dg_1 \wedge \cdots \wedge dg_r$ and

$$\omega'(g) = \omega'(g_1, \dots, g_r) = \sum_{j=1}^{r} (-1)^{j-1} g_j dg_1 \wedge \cdots \wedge [j] \wedge \cdots \wedge dg_r.$$

In particular, with $g_j = z_j$ this yields $\omega(z) = dz_1 \wedge \cdots \wedge dz_r$. Similarly,

$$\omega'(\bar{z}) = \sum_{j=1}^{r} (-1)^{j-1} \bar{z}_j d\bar{z}_1 \wedge \cdots \wedge [j] \wedge \cdots \wedge d\bar{z}_r.$$

Recall that the Bochner–Martinelli kernel β_N is the $(N, N-1)$-form on $\mathbb{C}^N$ defined by

$$\begin{aligned} \beta_N &= \left(\frac{1}{2\pi i}\right)^N |z|^{-2N} \bar{\partial}|z|^2 \wedge (\bar{\partial}\partial|z|^2)^{N-1} \\ &= c_N |z|^{-2N} \omega'(\bar{z}_1, \dots, \bar{z}_N) \wedge \omega(z_1, \dots, z_N), \end{aligned}$$

where c_N denotes the constant $(-1)^{(1/2)N(N-1)}(N-1)!/(2\pi i)^N$. This form is closed and $\bar{\partial}$-closed on $\mathbb{C}^N \setminus \{0\}$. It is not exact. The Bochner–Martinelli kernel gives an integral formula: If Ω is a bounded domain in $\mathbb{C}^N$ with smooth boundary, and if g is a function holomorphic on a neighborhood of $\bar{\Omega}$, then

$$\int_{b\Omega} g\beta_N = \begin{cases} g(0) & \text{if } 0 \in \Omega, \\ 0 & \text{if } 0 \notin \bar{\Omega}. \end{cases}$$

As follows from the preceding remarks, the Bochner–Martinelli kernel can be used to compute the degree of certain maps in $\mathbb{C}^N$. Given a bounded domain Ω in $\mathbb{C}^N$ with smooth boundary and given a smooth map $f : b\Omega \to \mathbb{C}^N \setminus \{0\}$, the degree of f is given by

$$d(f, \Omega, 0) = \int_{b\Omega} f^* \beta_N.$$

2. A Theorem in the Plane

For X a compact subset of $\mathbb{C}$, we shall use $\mathscr{R}(X)$ to denote the uniform closure in the space $\mathscr{C}(X)$ of complex-valued, continuous functions on X of the space of rational functions on $\mathbb{C}$ that have no poles on X. For a given set X, this space may or may not coincide with the space $A(X)$ of functions that are continuous on X and holomorphic on its interior. (For a systematic discussion of the question of the equality of $\mathscr{R}(X)$ and $A(X)$ for compacta X in the plane, see [7] and [18].) We shall also use the notation that, for a closed set $E \subset \mathbb{C}^N$, $\mathscr{Q}_E$ is the space of functions defined and holomorphic on a neighborhood of $E \times \mathbb{C}$ in $\mathbb{C}^{N+1}$. The neighborhood depends on the particular function; it is not assumed to be a product set.

The result to be established in this section is the following.

THEOREM. *Let Ω be a bounded connected open set in $\mathbb{C}$. Assume that each point of $b\Omega$ is a peak point for the algebra $\mathscr{R}(\bar{\Omega})$. If $f \in \mathscr{C}(b\Omega)$, then there is a function F that is continuous on $\bar{\Omega}$ and holomorphic on Ω with $F\big|_{b\Omega} = f$ if and only if the following condition holds.*

(†) *For every $p \in \mathscr{Q}_{\bar{\Omega}}$ such that for no $\zeta \in b\Omega$ does the quantity $\psi_{f,p}(\zeta) = p(\zeta, f(\zeta))$ vanish, the degree of the map $\zeta \mapsto \psi_{f,p}(\zeta)$ from $b\Omega$ to $\mathbb{C} \setminus \{0\}$ is nonnegative.*

The theorem applies in particular to all domains Ω for which $b\Omega$ consists of a finite number of mutually disjoint simple closed curves. (No regularity is required in this case beyond that imposed by the condition of being a simple closed curve; in particular, each of the curves might have locally finite area.) The condition is satisfied also by certain infinitely connected domains—for example, one obtained by excising from the open unit disc the union of countably many mutually disjoint closed subdiscs whose centers cluster only on the unit circle. Other (more exotic) examples can be found in [7] and [18].

Proof. We assume, as we can without loss of generality, that the origin lies in the domain Ω.

It is evident that, in the geometric situation of the theorem, if $f \in \mathscr{C}(\bar{\Omega})$ is holomorphic in Ω and if $p \in \mathscr{Q}_{\bar{\Omega}}$ has the property that $\psi_{f,p}$ does not vanish on $b\Omega$, then the degree of $\zeta \mapsto \psi_{f,p}(\zeta)$ from $b\Omega$ to $\mathbb{C} \setminus \{0\}$ is nonnegative. Let $\Omega_o \Subset \Omega$ be a domain with smooth boundary and with $\psi_{f,p}$ zero-free on $\Omega \setminus \Omega_o$. Then $d(\psi_{f,p}, \Omega, 0) = d(\psi_{f,p}, \Omega_o, 0)$ by property (d5) of the mapping degree. However, since $\psi_{f,p}$ is holomorphic in Ω, it follows that $d(\psi_{f,p}, \Omega_o, 0)$ is the nonnegative integer $(1/2\pi i)\Delta_{b\Omega_o} \operatorname{Arg} \psi_{f,p}$.

Denote by Γ_f the graph $\Gamma_f = \{(z, f(z)) : z \in b\Omega\}$. By $\mathscr{Q}_{\bar{\Omega}}(\Gamma_f)$ we denote the uniform closure in $\mathscr{C}(\Gamma_f)$ of the (restrictions to Γ_f of the elements of the) algebra $\mathscr{Q}_{\bar{\Omega}}$. This is a *uniform algebra* on the set Γ_f.

We denote the hull of the set Γ_f with respect to the algebra $\mathscr{Q}_{\bar{\Omega}}$ by $\mathscr{Q}_{\bar{\Omega}} - \operatorname{hull} \Gamma_f$. By definition, a point $z \in \bar{\Omega} \times \mathbb{C}$ is *not* in $\mathscr{Q}_{\bar{\Omega}} - \operatorname{hull} \Gamma_f$ if and only if there exists a $q \in \mathscr{Q}_{\bar\Omega}$ with $q(z) = 1 > \operatorname{supp}_{\Gamma_f}|q|$. The set $\mathscr{Q}_{\bar\Omega} - \operatorname{hull}\Gamma_f$ is a compact set; it is the spectrum (or maximal ideal space) of the algebra $\mathscr{Q}_{\bar\Omega}(\Gamma_f)$.

Define $\pi_1, \pi_2 \colon \mathbb{C}^2 \to \mathbb{C}$ to be the projections given by $\pi_1(z_1, z_2) = z_1$ and $\pi_2(z_1, z_2) = z_2$, respectively. Then $b\Omega \subset \pi_1(\mathscr{Q}_{\bar\Omega} - \operatorname{hull}\Gamma_f) \subset \bar\Omega$.

We begin by showing that the set $\mathscr{Q}_{\bar\Omega} - \operatorname{hull}\Gamma_f$ is not contained in $\pi^{-1}(b\Omega)$. Suppose it is. Let $\delta_o = \inf\{1/|z| : z \in b\Omega\}$, and let $h \in \mathscr{C}(b\Omega)$ be the function given by $h(z_1, z_2) = 1/z_1$. Because $\mathscr{Q}_{\bar\Omega} - \operatorname{hull}\Gamma_f$ is the spectrum of the algebra $\mathscr{Q}_{\bar\Omega}(\Gamma_f)$, we have that $\pi_1(\mathscr{Q}_{\bar\Omega} - \operatorname{hull}\Gamma_f)$ is the spectrum of the element $\pi_1\big|_{\Gamma_f}$ of the algebra $\mathscr{Q}_{\bar\Omega}(\Gamma_f)$. Consequently, if $g \in \mathscr{O}(\pi_1(\mathscr{Q}_{\bar\Omega} - \operatorname{hull}\Gamma_f))$, then $g \circ \pi_1 \in \mathscr{Q}_{\bar\Omega}(\Gamma_f)$. As each point of $b\Omega$ is a peak point for $\mathscr{R}(b\Omega)$, it follows that $\mathscr{R}(b\Omega) = \mathscr{C}(b\Omega)$ (see [7, p. 543]), whence there is a function $q \in \mathscr{Q}_{\bar\Omega}$ with $|q(z) - h(z)| < \frac{1}{2}\delta_o$ for all $z \in \Gamma_f$. As maps from Γ_f to $\mathbb{C} \setminus \{0\}$, the functions q and h are homotopic. Consequently, the degrees of the maps $\zeta \mapsto q(\zeta, f(\zeta))$ and $\zeta \mapsto h(\zeta, f(\zeta))$ from $b\Omega$ to $\mathbb{C} \setminus \{0\}$ are the same. However, the former degree is nonnegative (by hypothesis), and the latter is -1, a contradiction. Thus, $\pi_1(\mathscr{Q}_{\bar\Omega} - \operatorname{hull}\Gamma_f)$ must meet the connected open set Ω. The maximum principle then implies that $\pi_1(\mathscr{Q} - \operatorname{hull}\Gamma_f)$ contains Ω.

Next, the set $(\mathscr{Q}_{\bar\Omega} - \operatorname{hull}\Gamma_f) \cap \pi_1^{-1}(b\Omega)$ coincides with the set Γ_f. To see this, suppose z^o to be a point in $((\mathscr{Q}_{\bar\Omega} - \operatorname{hull}\Gamma_f) \cap \pi_1^{-1}(b\Omega)) \setminus \Gamma_f$. There is then a regular Borel probability measure μ on Γ_f with $\int_{\Gamma_f} q\, d\mu = q(z^o)$ for all $q \in \mathscr{Q}$. (The measure is a *representing measure* for the point z^o with respect to the algebra $\mathscr{Q}_{\bar\Omega}(\Gamma_f)$.) Because each point of $b\Omega$ is a peak point for $\mathscr{R}(\bar\Omega)$, the measure μ is supported on the set $(\{z_1^o\} \times \mathbb{C}) \cap \Gamma_f$. This set is a singleton, for Γ_f is a graph. It follows that μ is a point mass, whence z^o must lie in Γ_f—a contradiction.

We shall show that π_1 carries $\mathscr{Q}_{\bar\Omega} - \operatorname{hull}\Gamma_f$ injectively onto $\bar\Omega$. In order to do this, we use the notion of *maximum-modulus algebra*. (See [1] and [13].) Set $\mathscr{A} = \{f \mid \mathscr{Q}_{\bar\Omega} - \operatorname{hull}\Gamma_f \setminus \Gamma_f : f \in \mathscr{Q}_{\bar\Omega}(\Gamma_f)\}$, an algebra of continuous, complex-valued functions on the locally compact space $X = \mathscr{Q} - \operatorname{hull}\Gamma_f \setminus \Gamma_f$. Let $p = \pi_1\big|_X$. Then $(\mathscr{A}, X, \Omega, p)$ is a maximum-modulus algebra on X over Ω with projection p in the sense of [1].

Introduce the function $\delta \colon \Omega \to [0, \infty)$ by

$$\delta(\zeta) = \operatorname{diameter} \pi_2(\pi_1^{-1}(\zeta) \cap \mathscr{Q}_{\bar\Omega} - \operatorname{hull}\Gamma_f).$$

The compactness of $\mathscr{Q}_{\bar\Omega} - \operatorname{hull}\Gamma_f$ implies the boundedness of the function δ. The function δ tends to 0 at $b\Omega$. Otherwise, there is a sequence $\{\zeta_k\}_{k=1}^{\infty}$ of points in Ω with $\zeta \to b\Omega$ as $k \to \infty$ and with $\operatorname{diameter} \pi_2\pi_1^{-1}(\zeta) > \eta$ for some positive η and all k. If $\zeta_k \to \zeta_o \in b\Omega$, then π_1 carries two distinct points of Γ_f onto the point ζ_o. However, the map π_1 is injective over Γ_f.

According to [1, Thm. 11.7], the function $\log \delta$ is subharmonic on Ω. As it tends to $-\infty$ at $b\Omega$, it must be identically $-\infty$ on Ω. That is, π_1 is injective on $\mathscr{Q}_{\bar\Omega} - \operatorname{hull}\Gamma_f$. Thus, there is a continuous function $F \colon \bar\Omega \to \mathbb{C}$ whose graph is $\mathscr{Q}_{\bar\Omega} - \operatorname{hull}\Gamma_f$; F agrees with f on $b\Omega$.

The theory of maximum-modulus algebras implies that F is holomorphic on Ω. To see this, we can invoke the general theory of maximum-modulus algebras as given in [1]. Alternatively, we can appeal to Rudin's treatment [13] of local maximum-modulus algebras. If we set $\mathscr{B} = \{\varphi \in \mathscr{C}(\Omega) : \varphi(z) = f(z, F(z))$ for some $f \in \mathscr{A}$ and all $z \in \Omega\}$, then $\mathscr{B}$ is a *local maximum-modulus algebra* in the sense of [13]. It contains the identity map—namely, the map $z \mapsto \pi_1(z, F(z))$—so each element of $\mathscr{B}$ is holomorphic on Ω. In particular, since $F(z) = \pi_2(z, F(z))$, we find that F is holomorphic, as we wished to show. The theorem is proved. $\square$

REMARK. By construction, the function F is holomorphic on Ω and continuous on $\bar{\Omega}$. Nothing we have done implies that $F \in \mathscr{R}(\bar{\Omega})$.

REMARK. The hypothesis in the preceding theorem that each point of $b\Omega$ be a peak point for the algebra $\mathscr{R}(\bar{\Omega})$ cannot be completely abandoned. Let I be the closed interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$ and let $\Omega = \mathbb{U} \setminus I$; thus, $b\Omega = b\mathbb{U} \cup I$. Let $f : b\mathbb{U} \to \mathbb{C}$ be given by

$$f(z) = \begin{cases} z & \text{if } z \in b\mathbb{U}, \\ 0 & \text{if } z \in I. \end{cases}$$

The function f is not the boundary value of any function holomorphic on Ω, continuous on $\bar{\Omega}$. However, it does satisfy the condition that, if $p(z_1, z_2)$ is a polynomial in two complex variables such that the function $\psi_{f,p}(z) = p(z, f(z))$ has no zero on $b\Omega$, then the degree of the map $\psi_{f,p}$ on $b\Omega$ is nonnegative. The points of the interval I are not peak points for the algebra $\mathscr{R}(\bar{\Omega})$.

3. The Induction Step

We have proved the Main Theorem, and somewhat more, in the 1-dimensional case; we now show that the N-dimensional case is a consequence of the $(N-1)$-dimensional case. Thus, we assume that the Main Theorem has been established in the case of domains in $\mathbb{C}^{N-1}$ and continuous functions on their boundary.

Let Ω be a bounded domain in $\mathbb{C}^N$ with boundary of class $\mathscr{C}^2$, and let f be a continuous function on $b\Omega$ that satisfies the hypotheses of the Main Theorem. As in the statement of that theorem, let Γ_f denote the graph of the function f, a subset of $\mathbb{C}^{N+1}$.

Let Π be a complex affine hyperplane in $\mathbb{C}^N$ that meets $b\Omega$ transversely. We will show that the restriction $f\big|_{(b\Omega\cap\Pi)}$ satisfies the hypotheses of the theorem (in the $(N-1)$-dimensional case) and so extends holomorphically into the slice $\Omega \cap \Pi$. For notational convenience, we assume that Π is $\mathbb{C}^{N-1} = \mathbb{C}^{N-1} \times \{0\} = \{z_N = 0\} \subset \mathbb{C}^N$.

Let $Q_1, \dots, Q_{N-1}$ be functions defined and holomorphic on a neighborhood W' of $(\mathbb{C}^{N-1} \cap \bar{\Omega}) \times \mathbb{C}$ such that their set of common zeros is disjoint from the graph

$$\Gamma'_f = \{(z_1, \dots, z_{N-1}, 0, f(z_1, \dots, z_{N-1}, 0)) : (z_1, \dots, z_{N-1}, 0) \in b\Omega\}.$$

Since $\bar{\Omega}$ has a Stein neighborhood basis, there is a Stein neighborhood W of $\bar{\Omega} \times \mathbb{C}$ such that $W \cap (\mathbb{C}^{N-1} \times \mathbb{C}) \subset W'$. Each of the functions $Q_1, \dots, Q_{N-1}$ extends to be holomorphic on W; we denote these extensions also by $Q_1, \dots, Q_{N-1}$. Denote by Q' the map $Q' = (Q_1, \dots, Q_{N-1})$ from W' to $\mathbb{C}^{N-1}$. We must show that, if $\Psi'\colon \mathbb{C}^{N-1} \to \mathbb{C}^{N-1}$ is given by $\Psi'(z_1, \dots, z_{N-1}) = Q'(z_1, \dots, z_{N-1}, f(z_1, \dots, z_{N-1}, 0))$, then the degree of Ψ' as a map from $b\Omega \cap \Pi$ to $\mathbb{C}^{N-1}$ is nonnegative.

Let Q be the $\mathbb{C}^N$-valued map defined for (z, ζ) in a neighborhood of $\bar{\Omega} \times \mathbb{C}$ by $Q(z, \zeta) = (Q'(z_1, \dots, z_{N-1}, \zeta), z_N)$. The set $Q^{-1}(0)$ is disjoint from Γ_f. By hypothesis, the map $\Psi\colon b\Omega \to \mathbb{C}^N \setminus \{0\}$ given by $\Psi(z_1, \dots, z_N) = Q(z, f(z))$ has nonnegative degree.

The point is that *the degree of the map Ψ' is the same as the degree of the map* Ψ. This fact is contained in [2, Lemma 1.1]. There is a derivation of the fact, based on the Bochner–Martinelli integral, as follows.

Because the function f is only continuous, a preliminary step is required. Let ϱ be a defining function of class $\mathscr{C}^2$ for the domain Ω so that ϱ is defined on a neighborhood of $\bar{\Omega}$, $d\varrho \neq 0$ on $b\Omega$, and $\Omega = \{\varrho < 0\}$. We suppose that the function f has been extended to a continuous function defined on $\bar{\Omega}$ and smooth in Ω. For example, f might be harmonic in Ω with the given values on $b\Omega$. If $\delta > 0$ is small then, with $\Omega_\delta = \{\varrho < -\delta\}$, the set $b\Omega_\delta$ is again a manifold of class $\mathscr{C}^2$, as is $b\Omega_\delta \cap \mathbb{C}^{N-1}$. Moreover, given that δ is small enough, the function f will not assume the value 0 in the set $\Omega \setminus \bar{\Omega}_\delta$. Then $d(\Psi, \Omega, 0) = d(\Psi, \Omega_\delta, 0)$ and $d(\Psi', \Omega \cap \mathbb{C}^{N-1}, 0) = d(\Psi', \Omega_\delta \cap \mathbb{C}^{N-1}, 0)$. The upshot of this is that, for the purpose of verifying that the degree of Ψ is the same as the degree of Ψ', we can assume the function f to be of class $\mathscr{C}^2$. Thus, we must prove that $\int_{b\Omega} \Psi^* \beta_N = \int_{b\Omega\cap\Pi} \Psi'^* \beta_{N-1}$. This is equivalent to proving that $\int_{\Gamma_f} Q^* \beta_N = \int_{\Gamma_f'} Q'^* \beta_{N-1}$. Note that the last two formulas cannot be written without the assumption that f be smooth.

We need a fact about the Bochner–Martinelli kernel β_N on $\mathbb{C}^N$: It is exact in $\mathbb{C}^N \setminus \mathbb{C}^{N-1}$. This is well known and due originally to Martinelli [12]. It is a consequence of a simple, direct calculation. If Ξ is the $(N, N-2)$-form on $C^N \setminus \{0\}$ given by

$$\Xi = |z|^{-2(N-1)} \omega'(\bar{z}_1, \dots, \bar{z}_{N-1}) \wedge \omega(z_1, \dots, z_N)$$

then $d\Xi = \bar{\partial}\Xi = (-1)^{N-1}(N-1) z_N |z|^{-2N} \omega'(\bar{z}_1, \dots, \bar{z}_N) \wedge \omega(z_1, \dots, z_N)$. From this it follows that, on the set in $\mathbb{C}^N$ where $z_N \neq 0$, if γ_N denotes the constant $(-1)^{N-1}(-1)^{(1/2)N(N-1)}(N-2)!/(2\pi i)^N$, then

$$\beta_N = d\left(\gamma_N \frac{1}{z_N} \Xi\right).$$

If we pull this formula back to $\mathbb{C}^{N+1}$ by way of the map Q, we obtain

$$Q^*\beta_N = d\left(\gamma_{N-1} \frac{1}{(2\pi i)^{N-1}} \frac{1}{z_N} Q^* \Xi\right).$$

Since

$$Q^*\Xi = \frac{\omega'(\bar{Q}_1, \dots, \bar{Q}_{N-1}) \wedge \omega(Q_1, \dots, Q_{N-1}) \wedge dz_N}{\{|Q_1|^2 + \cdots + |Q_{N-1}|^2 + |z_N|^2\}^{N-1}},$$

it follows that

$$Q^*\beta_N = d\Big(\gamma_{N-1}\omega'(\bar{Q}_1 \wedge \cdots \wedge \bar{Q}_{N-1}) \wedge \omega(Q_1, \dots, Q_{N-1})\{|Q_1|^2 + \cdots + |Q_{N-1}|^2 + |z_N|^2\}^{-(N-1)} \wedge \frac{dz_N}{2\pi i z_N}\Big).$$

Now apply Stokes's theorem and Fubini's theorem (in the form given, e.g., in [17]):

$$\int_{\tilde\Gamma} Q^*\beta_N = \lim_{\varepsilon\to 0^+} \int_{\tilde\Gamma\cap\{|z_N|>\varepsilon\}} Q^*\beta_N = \lim_{\varepsilon\to 0^+} -\gamma_{N-1} \int_{\{|z_N|=\varepsilon\}} \left(\int_{\tilde\Gamma\cap\{z\in\mathbb{C}^N:\zeta=z_N\}} \frac{\omega'(\bar{Q}_1, \dots, \bar{Q}_{N-1}) \wedge \omega(Q_1, \dots, Q_{N-1})}{\{|Q_1|^2 + \cdots + |Q_{N-2}|^2 + |z_N|^2\}^{N-1}} \right) \frac{dz_N}{2\pi i z_N}.$$

The negative sign in the last equation arises from considerations of orientation. As $\varepsilon \to 0^+$, the last quantity tends to $\int_{\tilde\Gamma'} Q'^*\beta_{N-2}$. Thus, we have the desired equality of the two degrees in question.

The inductive hypothesis now implies that the function f continues holomorphically into the domain $\Pi \cap \Omega$. That is, we have shown that *if* Π *is a complex-affine hyperplane in* $\mathbb{C}^N$ *that meets* $b\Omega$ *transversely, then* $f\big|_{(\Pi\cap b\Omega)}$ *extends holomorphically into the slice* $\Pi \cap \Omega$.

If $N = 2$, we are in the situation of the 1-dimensional extension property considered in [15] (cf. [8]). The result of [15] implies that, as desired, f extends holomorphically through Ω. If $N > 2$, a simpler argument is possible. Given a point $w \in \Omega$, let Π be a complex-affine hyperplane in $\mathbb{C}^N$ through w that meets $b\Omega$ transversely. Define $F_\Pi(w)$ to be the value of the holomorphic extension F_Π of $f\big|_{(\Pi\cap\Omega)}$ through $\Pi \cap \Omega$. Since $N \geq 3$, the value of $F_\Pi(w)$ does not depend on the choice of Π; denote this value by $F(w)$. This gives a well-defined function F on Ω that is holomorphic and assumes the values f on $b\Omega$. The Main Theorem is proved.

4. Extensions

We shall now consider certain extensions of the work just given.

A. The Convex Case

It is more or less evident that an analog of the Main Theorem can be established in the setting where D is a bounded convex domain in $\mathbb{C}^N$ and $f \in \mathscr{C}(bD)$ is a continuous function that satisfies condition $(*)$ of the Main Theorem. (We consider *arbitrary* convex domains, not only those with smooth boundaries; thus, we admit polydiscs or convex polyhedra among other examples.)

The basis for this assertion is as follows. The compact set $\bar{D}$ does have a neighborhood basis that consists of Stein domains. The slices of D by complex lines are convex 1-dimensional domains, so the result of Section 2 applies to them. The necessary 1-dimensional extension result, tailored to the setting of convex domains, is contained in [8, Thm. 3.2.1]. Thus, the only point that needs to be verified in this case is a fact about degrees.

Precisely put, what has to be established is the following. Let Δ be a bounded convex domain in $\mathbb{C}^N$. If $f \in \mathscr{C}(b\Delta)$ satisfies condition $(*)$ and if Π is a complex hyperplane that meets Δ, then f_Π, the restriction of f to $b\Delta \cap \Pi$, has the corresponding property.

This implication can be established as follows. Without loss of generality, we suppose Π to be the coordinate axis $\{z \in \mathbb{C}^N : z_N = 0\}$. Let Δ_N denote the intersection $\Delta \cap \Pi$. Fix a holomorphic mapping Q' from a neighborhood of $\bar{\Delta}_N \times \mathbb{C}$ into $\mathbb{C}^{N-1}$ such that the zero locus of the map Ψ' given by $\Psi' = (Q'(z_1, \ldots, z_{N-1}, 0, f(z_1, \ldots, z_{N-1}, 0))$ is disjoint from the graph of f_Π. We are to prove that the degree of the map Ψ' from $b\Delta \cap \Pi$ to $\mathbb{C}^{N-1}$ is nonnegative.

The map Q' can be extended to a holomorphic $\mathbb{C}^{N-1}$-valued map (still denoted by Q') defined on a neighborhood of $\bar{\Delta} \times \mathbb{C}$. Let Q be the $\mathbb{C}^N$-valued map defined on a neighborhood of $\bar{\Delta} \times \mathbb{C}$ by $(z, \zeta) \mapsto ((Q'(z_1, \ldots, z_{N-1}, 0, \zeta), z_N)$. Then the $\mathbb{C}^N$-valued map Ψ defined on $b\Delta$ by $\Psi(z) = Q(z, f(z))$ is zero-free; its degree as a map to $\mathbb{C}^N \setminus \{0\}$ is nonnegative. Note that we can extend the map f through Δ as a smooth function.

Consider a smoothly bounded convex domain D that is a relatively compact subset of Δ and that is large enough for the quantity $Q(z, f(z))$ to be zero-free on the compact set $\bar{\Delta} \setminus D$. Then the degree of the map Ψ from $b\Delta$ to $\mathbb{C}^N \setminus \{0\}$ is the same as the degree of the map $\Psi_D \colon bD \to \mathbb{C}^N \setminus \{0\}$. Also, the degree of the map Ψ' from $b\Delta \cap \Pi$ to $\mathbb{C}^{N-1} \setminus \{0\}$ is the same as the degree of the map Ψ'_D from $bD \cap \Pi$ to $\mathbb{C}^{N-1} \setminus \{0\}$.

The analysis using the Bochner–Martinelli integral invoked in Section 3 shows the degree of Ψ_D to be the same as the degree of Ψ'_D. Thus, a result analogous to the Main Theorem is established in the case of arbitrary bounded convex domains.

B. The Manifold Case

The work carried out in Sections 2 and 3 is set in the context of domains in $\mathbb{C}^N$. But little effort is required to extend it to certain manifold situations.

Toward this end, if $\mathcal{R}$ is an open Riemann surface and if X is a compact subset of $\mathcal{R}$, then we define $\mathscr{R}(X)$ to be the closure in $\mathscr{C}(X)$ of the (restrictions to X of) functions defined and holomorphic on various neighborhoods of X in $\mathcal{R}$. Runge's theorem implies that, when $\mathcal{R}$ is the complex plane, this notion of $\mathscr{R}(X)$ coincides with that used in Section 2.

If D is a relatively compact domain in $\mathcal{R}$ for which each point of bD is a peak point for the algebra $\mathscr{R}(\bar{D})$, then the continuous functions f on bD that extend holomorphically through D admit a characterization precisely parallel to that given in the Theorem of Section 2: *They are those functions satisfying the condition* $(\dagger_{\mathcal{R}})$ *that, if p is a function holomorphic on a neighborhood in $\mathcal{R} \times \mathbb{C}$ of the set $\bar{D} \times \mathbb{C}$ such that $p^{-1}(0)$ is disjoint from the graph $\Gamma_f = \{(z, f(z)) : z \in bD\}$, then the degree of the map $z \mapsto \varphi_p(z) = p(z, f(z))$ from bD into $\mathbb{C}$ is nonnegative.* The argument of Section 2 applies, mutatis mutandis, to deal with the present situation.

Once we have this, we can formulate and prove a result on domains in manifolds as follows. Fix a complex manifold $\mathscr{M}$ of dimension M and in it a relatively compact domain D with bD of class $\mathscr{C}^2$. We assume that $\bar{D}$ has a Stein neighborhood basis. It then entails no loss of generality to suppose that $\mathscr{M}$ is itself a Stein manifold and so, by the embedding theorem, that $\mathscr{M}$ is a closed submanifold of $\mathbb{C}^N$ for some suitably large N. In this setting, we have the following result. *The continuous function f on bD extends holomorphically through D if and only if it satisfies the condition $(*_{\mathscr{M}})$: For every holomorphic $\mathbb{C}^M$-valued map Q defined on a neighborhood of $\bar{D}\times\mathbb{C}$ in $\mathscr{M}\times\mathbb{C}$ such that $Q^{-1}(0)$ is disjoint from the graph $\Gamma_f = \{(z, f(z)) : z \in bD\}$, the map $z \mapsto Q(z, f(z))$ from bD into $\mathbb{C}^M \setminus \{0\}$ has nonnegative degree.*

Unlike our proof of the Main Theorem, the proof we give for this is not based on induction on dimension. Rather, we shall apply some information about the Bochner–Martinelli kernel to show that it follows from the hypotheses that, for every nonsingular 1-dimensional complex submanifold Σ of a neighborhood of $\bar{D}$ in $\mathscr{M}$ that meets bD transversely, the restriction $f\big|_{(\Sigma\cap bD)}$ extends holomorphically into the slice $\Sigma \cap D$. Having established this point, by [16] we can conclude that f extends holomorphically through D. We establish the stated 1-dimensional extension property first in the particular case that the curve Σ is a complete intersection, so that there exists a map $\varphi = (\varphi_1, \dots, \varphi_{M-1})$ from a neighborhood of $\bar{D}$ in $\mathscr{M}$ to $\mathbb{C}^{M-1}$ whose zero locus is Σ and whose differential has maximal rank at each point of Σ.

Consider a function G defined and holomorphic on a neighborhood of $\bar{D}\times\mathbb{C}$ that satisfies the condition that the zero locus of G be disjoint from the partial graph $\{(z, f(z)) : z \in (\Sigma \cap bD)\}$. We want to show that the map $z \mapsto G(z, f(z))$ from $\Sigma \cap bD$ to $\mathbb{C}\setminus\{0\}$ has nonnegative degree. Having fixed G, there is no loss in assuming that f is smooth on bD. (This follows from arguments we have already used.) Hence, we must prove that the winding number $(1/2\pi i)\int_{\Sigma\cap bD}(dG/G)$ is nonnegative.

By construction, the holomorphic map $\Phi\colon \bar{D}\times\mathbb{C} \to \mathbb{C}^M$ given by

$$\Phi(z,\zeta) = (\varphi(z), G(z,\zeta)) = (\varphi_1(z), \dots, \varphi_{M-1}(z), G(z,\zeta))$$

does not vanish at any point of the graph Γ_f of f (on the whole of bD). Accordingly, the integral $\int_{\Gamma_f}\Phi^*\beta_M$ is nonnegative.

The map Φ is constructed so that it carries $\bar{D}\setminus\Sigma$ into $\mathbb{C}^M\setminus\lambda$, where λ is the complex line $\{z \in \mathbb{C}^M : z_1 = \cdots = z_{M-1} = 0\}$. In Section 6, a primitive for the Bochner–Martinelli kernel is constructed in the domain $\mathbb{C}^M\setminus\lambda$; if $\Theta \in \mathcal{E}^{2M-2}(\mathbb{C}^M\setminus\lambda)$ is the form given in Section 6 (with M here replacing the N of Section 6), then

$$\Phi^*\beta_M = d\Phi^*(c_M\Theta),$$

where (as in Section 1) c_M denotes the constant $(-1)^{(1/2)M(M-1)}(M-1)!/(2\pi i)^M$. We shall write $|\varphi|$ for the quantity $\sqrt{|\varphi_1|^2+\cdots+|\varphi_{M-1}|^2}$. Let $\varepsilon > 0$ be a small positive number. By Stokes's theorem,

$$\begin{aligned}
&\int_{\Gamma_f} \Phi^*\beta_M \\
&\quad = \lim_{\varepsilon\to 0^+} \int_{\Gamma_f\cap\{|\varphi(z)|>\varepsilon\}} \Phi^*\beta_M = \lim_{\varepsilon\to 0^+} \int_{b(\{\Gamma_f\cap\{|\varphi|>\varepsilon\})} c_M\Theta \\
&\quad = \lim_{\varepsilon\to 0^+} \int_{\Gamma_f\cap\{|\varphi|=\varepsilon\}} \biggl(\frac{c_M}{2(M-1)(|\varphi|^2+|G|^2)^{M-1}} \\
&\qquad\qquad \times \sum_{r=0}^{M-2} \binom{M-1}{r} |G|^{2M-2r-4}|\varphi|^{2r-2M+2} \biggr) \\
&\qquad \times [(-1)^M (Gd\bar{G}-\bar{G}dG)\wedge\omega'(\bar{\varphi})\wedge\omega(\varphi) + (-1)^{M-1}|G|^2\omega(\bar{\varphi})\wedge\omega(\varphi)].
\end{aligned}$$

Since $|\varphi| = \varepsilon$ on the path of integration, it follows from Stokes's theorem and Fubini's theorem for forms that

$$\begin{aligned}
\int_{\Gamma_f} \Phi^*\beta_M &= \lim_{\varepsilon\to 0^+} \int_{\{\zeta\in\mathbb{C}^{M-1}:|\zeta|=\varepsilon\}} \biggl(\int_{\Gamma_f\cap\varphi^{-1}(\zeta)} \frac{c_M}{2(M-1)(\varepsilon^2+|G|^2)^{M-1}} \\
&\quad \times \sum_{r=0}^{M-2} \binom{M-1}{r} G^{2M-2r-4}|\varepsilon|^{2r-2M+2}(-1)^M(Gd\bar{G}-\bar{G}dG)\biggr) \omega'(\bar{\zeta})\wedge\omega(\zeta).
\end{aligned}$$

Observe that the term in the primitive $\Phi^*\Theta$ for $\Phi^*\beta_M$ that contains $\omega(\bar{\varphi})\wedge\omega(\varphi)$ contributes nothing to the integral: It is a $(2M-2)$-form in $d\varphi$ and so induces the 0-form on the ζ-path of integration (on which $|\varphi| = \varepsilon$).

The ζ-path of integration is the $(2M-3)$-dimensional sphere of radius ε; its volume is $(2\pi^{M-1}/(M-2)!)\varepsilon^{2M-3}$, and the coefficient of each term in the form $\omega'(\bar{\varphi})$ is one of the coordinates of φ and hence is $O(\varepsilon)$. Consequently, the only term in the sum under the last integral that (in the limit) can make a nonzero contribution is the one corresponding to $r = 0$. Thus, by using the equality $c_M = ((-1)^{M-1}(M-1)/2\pi i)c_{M-1}$, we reach

$$\begin{aligned}
\int_{\Gamma_f} \Phi^*\beta_M &= \lim_{\varepsilon\to 0^+} \int_{\{\zeta\in\mathbb{C}^{M-1}:|\zeta|=\varepsilon\}} \biggl(\int_{\Gamma_f\cap\varphi^{-1}(\zeta)} \frac{c_M(-1)^M|G|^{2M-4}(Gd\bar{G}-\bar{G}dG)}{2(M-1)(\varepsilon^2+|G|^2)^{M-1}} \biggr) \\
&\qquad \times \varepsilon^{-2M+2}\omega'(\bar{\zeta})\wedge\omega(\zeta) \\
&= \frac{1}{4\pi i} \int_{\Gamma\cap\varphi^{-1}(0)} \left\{ \frac{dG}{G} - \frac{d\bar{G}}{\bar{G}} \right\}.
\end{aligned}$$

Observe that the quantity $(1/2\pi i)\int_{\varphi^{-1}(0)}(dG/G)$ is the degree d of the mapping $z \mapsto G(z, f(z))$ from $\Sigma\cap bD$ to $\mathbb{C}\setminus\{0\}$. The quantity $(1/2\pi i)\int_{\varphi^{-1}(0)}(d\bar{G}/\bar{G})$ is $-d$.

We have established that

$$\int_{\Gamma_f} \Phi^*\beta_M = \frac{1}{2\pi i}\int_{\Sigma\cap bD} \frac{dQ}{Q}.$$

By hypothesis, the former number is nonnegative. Consequently, the latter degree is, too. We may conclude that the restriction $f\big|_{(\Sigma\cap bD)}$ extends holomorphically into the slice $\Sigma\cap D$.

This has been done under the assumption that the curve Σ is a complete intersection. In dimension ≥ 3, this condition is not restrictive, as follows from work of Forster and Ramspott [5; 6]. That work is not simple, and we prefer to proceed without appeal to it. Moreover, it is not true that every nonsingular curve is a complete intersection in the 2-dimensional case: There are unsolvable Cousin II problems on certain 2-dimensional Stein manifolds.

However, it is true that, in every dimension ≥ 2, a nonsingular curve can be approximated by a collection of irreducible branches of nonsingular complete intersections. This device allows us to obtain the desired 1-dimensional extension property in the general case.

To do this, fix a nonsingular analytic curve Σ in a Stein neighborhood W of the compact set $\bar{D}$ in $\mathscr{M}$. The first observation is that, although Σ may not be a complete intersection in W, there is a neighborhood V of Σ $(V \subset W)$ in which Σ is a complete intersection. This follows because a neighborhood of Σ in W is biholomorphically equivalent to a neighborhood of the zero section of the normal bundle of the embedding $\Sigma \hookrightarrow W$. (See [10, p. 257].) This normal bundle is a holomorphic vector bundle over the noncompact Riemann surface Σ and so is trivial. (See [9, p. 303].) Thus, if V is a sufficiently small neighborhood of Σ in W, then Σ is a complete intersection in V. Let F be a holomorphic map from a neighborhood of $\bar{V}$ to $\mathbb{C}^{M-1}$ that defines Σ as an analytic set. (We may have to shrink V a little to have F be defined on $\bar{V}$.) Let $W_1 \Subset W$ be a Stein domain whose closure is $\mathscr{O}(W)$-convex. The set $\Sigma \cap \bar{W}_1$ is then $\mathscr{O}(W)$-convex. Consequently, if V is chosen to be sufficiently thin and to be a Stein domain, then there will exist a sequence $\{F_n\}_{n=1,2,\dots}$ of holomorphic maps from W to C^{M-1} that converges uniformly on $\bar{V} \cap \bar{W}_1$ to f. The maps F_n can be chosen such that (a) for each n, 0 is a regular value of F_n, and (b) $\Sigma_n = F_n^{-1}(0)$ is transverse to bD.

For each n, the restriction $f\big|_{(\Sigma_n \cap bD)}$ extends holomorphically into the slice $\Sigma_n \cap D$. We are to deduce from this that $f\big|_{(\Sigma \cap bD)}$ extends holomorphically through the slice $\Sigma \cap D$.

For this, since V is a Stein domain it will suffice to show that, for every holomorphic 1-form α on V, the integral $\int_{\Sigma \cap bD \cap V} f\alpha$ vanishes. (See [11].) We already know that, for each n, the integral $\int_{\Sigma_n \cap bD \cap V} f\alpha$ vanishes. Hence, we must prove that

$$\lim_{n\to\infty} \int_{F_n^{-1}(0)\cap bD\cap V} f\alpha = \lim_{n\to\infty} \int_{F^{-1}(0)\cap bD\cap V} f\alpha.$$

By a device already used several times, it entails no loss of generality for us to suppose that the function f is smooth and defined on W. Then, by Stokes's theorem,

$$\lim_{n\to\infty} \int_{F_n^{-1}(0)\cap bD\cap V} f\alpha = \lim_{n\to\infty} \int_{F_n^{-1}(0)\cap D} df \wedge \alpha$$

and also

$$\int_{F^{-1}(0)\cap bD\cap V} f\alpha = \int_{F^{-1}(0)\cap D} df \wedge \alpha.$$

We shall show that the right-hand sides of the last two equations are the same.

In this, we need the following remark. Let ω denote the fundamental area form on $\mathbb{C}^N$: $\omega = (-i/2)\sum_{j=1}^N dz_j \wedge d\bar{z}_j$. Thus, the area of a 1-dimensional variety E in an open subset of $\mathbb{C}^N$ is $\int_E \omega$.

Let T and T' be thin tubes in $\mathbb{C}^N$ over the system of curves $\Sigma \cap bD$ with $T' \Subset T$. Let χ be a nonnegative function of class $\mathscr{C}^\infty$ on $\mathbb{C}^N$ with $\chi = 1$ on a neighborhood of $\bar{T}'$ and with $\chi = 0$ on a neighborhood of $\mathbb{C}^N \setminus T'$. Then

$$\lim_{n\to\infty} \int_{T'\cap\Sigma_n} \chi\, df \wedge \alpha = \int_{T'\cap\Sigma} \chi\, df \wedge \alpha.$$

If T' is sufficiently small, then the latter quantity is small.

It follows that if T' is chosen so that the area of the part of $F^{-1}(0)$ in T' is small, then in

$$\begin{aligned}\int_{F_n^{-1}(0)\cap D} & df \wedge \alpha - \int_{F^{-1}(0)\cap D} df \wedge \alpha \\ &= \int_{F_n^{-1}(0)\cap D} \chi\, df \wedge \alpha - \int_{F^{-1}(0)\cap D} \chi\, df \wedge \alpha \\ &\quad + \int_{F_n^{-1}(0)\cap D} (1-\chi)\, df \wedge \alpha - \int_{F^{-1}(0)\cap D} (1-\chi)\, df \wedge \alpha\end{aligned}$$

we have that the first summand on the right is small uniformly in n for sufficiently large n while the second summand tends to zero as $n \to \infty$. It follows that, as desired,

$$\lim_{n\to\infty} \int_{F_n^{-1}(0)\cap D} df \wedge \alpha = \int_{F^{-1}(0)\cap D} df \wedge \alpha,$$

which completes the proof.

5. Two Open Questions

In this work we have repeatedly imposed the hypothesis that the closure of our domain D have a Stein neighborhood basis. It is not at all obvious that this hypothesis is necessary for the conclusion, and we ask whether it can be replaced by something weaker. In particular, might it suffice for the boundary of the domain to be connected? (The hypothesis that the closure of the domain has a Stein neighborhood basis implies that the boundary of the domain is connected.) The Stein neighborhood basis hypothesis is stable under passage to intersections by lower-dimensional hypersurfaces; the hypothesis of having connected boundary is not.

The second open question arises in connection with the classical description of the boundary values of holomorpic functions on the unit disc $\mathbb{U}$. From classical function theory, a continuous function f on $b\mathbb{U}$ extends holomorphically through $\mathbb{U}$ if and only if the Fourier coefficients $\hat{f}(n) = (1/2\pi)\int_{-\pi}^{\pi} f(e^{i\vartheta})e^{-in\vartheta}\, d\vartheta$ vanish for $n = -1, -2, \ldots$. The vanishing of these integrals for all $n = -1, -2, \ldots$ is equivalent to the vanishing of the integral

$$\frac{1}{2\pi i}\int_{|z|=1}\frac{f(z)}{z-w}\,dz$$

for all $w \in \mathbb{C}$ with $|w| > 1$. Accordingly, we ask for a simple, direct proof of the equivalence—in the case of functions on the unit circle—of these classical conditions to the positivity conditions given by the Main Theorem and its corollary.

6. Appendix: A Primitive for the Bochner–Martinelli Kernel

In this appendix we shall determine an explicit primitive for the Bochner–Martinelli kernel in the complement of a complex line through the origin in $\mathbb{C}^N$.

In order to avoid dealing with constants in some calculations, we shall not work initially with the Bochner–Martinelli kernel β_N itself but rather with the form

$$k = |z|^{-2N}\omega'(\bar{z}) \wedge \omega(z),$$

defined and of bidegree $(N, N-1)$ on $\mathbb{C}^N \setminus \{0\}$. The kernel k differs from β_N only by the constant factor $c_N = (-1)^{(1/2)N(N-1)}(N-1)!/(2\pi i)^N$.

Denote by λ the complex line

$$\lambda = \{z \in \mathbb{C}^N : z_1 = \cdots = z_{N-1} = 0\},$$

and define $H\colon (\mathbb{C}^N \setminus \lambda) \times [0,1] \to \mathbb{C}^N \setminus \lambda$ by

$$H(z,t) = H(z_1,\ldots,z_{N-1},z_N,t) = (z_1,\ldots,z_{N-1},tz_N).$$

The map H is a homotopy in $\mathbb{C}^N \setminus \lambda$ between the identity map on $\mathbb{C}^N \setminus \lambda$ and the projection $z \mapsto (z_1,\ldots,z_{N-1},0)$.

The form k is smooth on $\mathbb{C}^N \setminus \lambda$ and so (since k is a closed form) the homotopy formula for forms [14, p. 8-53] provides the formula $k = d\Theta$, where Θ denotes the $(2N-2)$-form obtained as follows. Write $H^*k = dt \wedge \vartheta + \eta$ with η a $(2N-1)$-form on $(\mathbb{C}^N \setminus \lambda) \times [0,1]$ that does not have a factor dt and with ϑ a $(2N-2)$-form on the same manifold that does not have a factor dt. (To be sure, the coefficients of ϑ and η will depend on t.) Then Θ is the form given by $\Theta = \int_0^1 \vartheta\, dt$. (In connection with the homotopy formula, note that the range of the map $z \mapsto H(z,0)$ is of complex dimension $N-1$, so that the form k induces on it the zero form.)

We determine Θ explicitly as follows. In this it will be convenient to write z' for $(z_1,\ldots,z_{N-1}) \in \mathbb{C}^{N-1}$ if $z = (z_1,\ldots,z_{N-1},z_N) \in \mathbb{C}^N$. We have

$$H^*k = (|z'|^2 + t^2|z_N|^2)^{-N}\omega'(\bar{z}_1,\ldots,\bar{z}_{N-1},t\bar{z}_N) \wedge \omega(z_1,\ldots,z_{N-1},tz_N).$$

Then

$$\omega(z_1,\ldots,z_{N-1},tz_N) = t\omega(z) + z_N\omega(z') \wedge dt$$

and

$$\begin{aligned}\omega'(\bar{z}_1,\ldots,\bar{z}_{N-1},t\bar{z}_N) = \sum_{j=1}^{N-1}&(-1)^{j-1}\bar{z}_j d\bar{z}_1 \wedge \cdots \wedge [j] \wedge \cdots \wedge d(t\bar{z}_N)\\ &+ (-1)^{N-1}t\bar{z}_N\omega(\bar{z}_1,\ldots,\bar{z}_{N-1}).\end{aligned}$$

From this it follows (after some computation) that

$$\Theta = \left\{ \frac{1}{2(N-1)(|z'|^2 + |z_N|^2)^{N-1}} \sum_{r=0}^{N-2} \binom{N-1}{r} |z_N|^{2N-2r-4} |z'|^{2r-2N+2} \right\}$$
$$\times [(-1)^N (z_N d\bar{z}_N - \bar{z}_N dz_N) \wedge \omega'(\bar{z}') \wedge \omega(z')$$
$$+ (-1)^{N-1} |z_N|^2 \omega(\bar{z}') \wedge \omega(z')].$$

This is the desired primitive—a d-primitive—for k on $\mathbb{C}^N \setminus \lambda$. We then have that

$$\beta_N = d[c_N \Theta]$$

on $\mathbb{C}^N \setminus \lambda$. (In this formula, c_N denotes the constant introduced in Section 1 in connection with β_N.)

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