Isometric Cusps in Hyperbolic 3-Manifolds

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1. Introduction

Given a cusped finite-volume hyperbolic 3-manifold M, each cusp has two natural invariants associated to it. First, there is the so-called cusp shape, which is the Euclidean metric on a torus cross-section of the cusp, up to scaling. It can be described by an equivalence class of parallelograms that correspond to fundamental domains for the action of the cusp subgroup on a horizontal plane in the upper half-plane model. The second natural invariant is the maximal cusp volume, which is obtained by expanding the cusp until it first touches itself and then computing the corresponding volume.

In [10] it was proved that the set of possible cusp shapes corresponding to cusps in hyperbolic 3-manifolds is dense in the set of possible Euclidean metrics on a torus. In some sense, one expects a panoply of nonisometric cusps in different manifolds.

However, in [8] and [9], the authors provided examples of manifolds (and orbifolds) with two cusps such that surgery on one cusp leaves the cusp shape of the remaining cusp invariant. In particular, this generates an infinite set of manifolds, each with a single cusp having the same cusp shape. In [3] it was demonstrated that these examples also have the same maximal cusp volume.

Define two cusps in two possibly distinct hyperbolic 3-manifolds to be *maximally isometric* if there is an isometry of the interior of one maximal cusp to the interior of the other. In particular, this occurs if and only if both the cusp shape and the maximal cusp volume are the same for the two cusps.

In this paper, a list of "generic cusps" is provided and defined up to maximal isometry such that one can choose one of these cusps and then—by removing three disjoint simple closed curves from any closed 3-manifold or two disjoint simple closed curves from any cusped hyperbolic 3-manifold—the resulting manifold is hyperbolic and one of the new cusps is maximally isometric to the chosen cusp. The set of generic cusps contains a large variety of cusps, including ones of maximal cusp volume 4, 6, $2\sqrt{3}$, $2(1+\sqrt{3})$, and $8\sqrt{3/5}$. Moreover, if one removes four disjoint simple closed curves from a closed hyperbolic 3-manifold, or three

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from a cusped hyperbolic 3-manifold, then one can realize cusps of maximal cusp volume 8, 10, $2\sqrt{7}$, $\sqrt{11}$, and $\sqrt{15}$, among others.

In essence, we are cutting open the original manifold and inserting a rigid submanifold, a so-called *walnut*, that contains the cusp in question. The shell of the walnut is made up of two twice-punctured disks that protect the interior from deformation.

Thus, for example, one sees that the labeled component of the hyperbolic link complement shown in Figure 1 has cusp of maximal isometry type completely determined by the fact that another component wraps around it. The rest of the link is essentially irrelevant. The maximal isometry type of this cusp would be preserved no matter where we inserted this link component and its wrapping partner in this or any other link complement. This maximal cusp will have volume 4, meridian of length 2, longitude of length 4, and meridian and longitude at right angles.

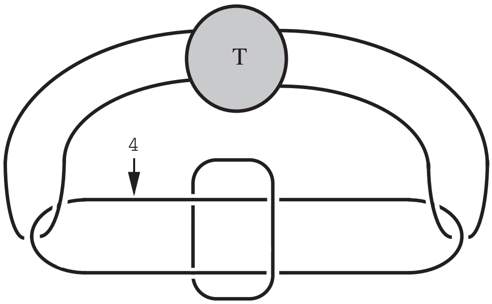


Figure 1 Maximal isometry type of cusp completely determined

The particular walnut just described seems to appear in hyperbolic 3-manifolds in a natural context. If M is a hyperbolic 3-manifold, it is known that the complement of the shortest geodesic is another hyperbolic 3-manifold. Utilizing the SNAPPEA hyperbolic structures program, Thurston [11] noticed the following fact. Starting with a finite-volume hyperbolic 3-manifold, if one repeatedly removes the shortest geodesic, replacing the previous manifold with a complete hyperbolic manifold at each stage, then empirically the shortest geodesic becomes, in a finite number of steps, one of complex length $2.122\ldots \pm 1.809\ldots i$ (where the real part is the length of the geodesic and the imaginary part is the twist angle). From that point on, all subsequent removals of the shortest geodesic leave the resulting shortest geodesic of this same complex length. Walnuts give a plausible explanation for this phenomenon, as follows.

Suppose that a sequence of shortest geodesics has been removed and, in the process, a walnut of the type described in Figure 1 occurs. That is to say, there is a single component bounding a twice-punctured disk and then another component that bounds a twice-punctured disk, with both punctures coming from the first component. In fact, this walnut is a Borromean rings complement that has been cut open along a twice-punctured disk. As such, it contains two shortest geodesics of length $2.122\ldots \pm 1.809\ldots i$. If these are the shortest geodesics in the overall manifold, then the removal of either one yields a manifold that is the same as the previous manifold—except that a Whitehead link complement that has been cut open along a twice-punctured disk has been glued in, sharing one twice-punctured disk with the previous walnut and creating a larger walnut. In particular, the shortest geodesic inside this new expanded walnut has the same complex length as before. The removal of the geodesic has had no impact on the manifold outside the walnut, since it is shielded by the two twice-punctured disks. As we repeat this process, we continue to glue in Whitehead link walnuts, leaving the length of the resulting shortest geodesic unchanged. This process continues on forever.

Although there is not yet a proof that this phenomenon in particular is occurring, one can use SNAPPEA to see that, in any given case, if enough removals of shortest geodesics have been performed to reach the length $2.122\ldots \pm 1.809\ldots i$ then each subsequent removal increases the volume of the manifold by $3.6638\ldots$, which is the volume of the Whitehead link complement.

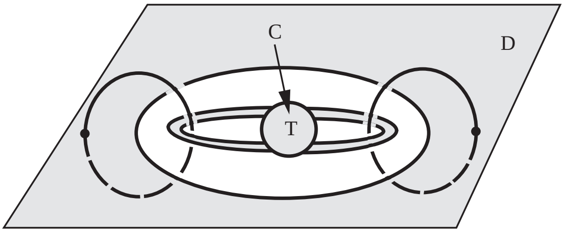
In Section 2, we list some of the generic cusps occurring in walnuts that are available. In Section 3, we prove that certain modifications to a cusped hyperbolic manifold will preserve its hyperbolicity. Section 4 is devoted to a proof of the main result that these generic cusps can be inserted by removing three or four simple closed curves. In Section 5, we generalize to remove a handlebody and two cusps from a closed hyperbolic 3-manifold or a handlebody and one cusp from a cusped hyperbolic 3-manifold, so that the resulting manifold has a totally geodesic boundary of specified isometry type. This generates a variety of new examples of infinite sets of manifolds, all with isometric totally geodesic boundaries. Previous examples appeared in [9] and then [6].

This paper could not have been completed without the aid of the hyperbolic structures program called SNAPPEA that was written by Jeffrey Weeks [12].

2. Generic Cusps

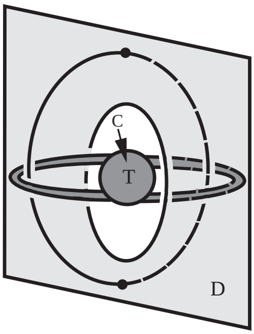
In this section we introduce a set of cusps, denoted $\boldsymbol{C}$, that will be our generic set. There will be two types of such cusps, modeled on the cusps appearing in two types of link complements. A Type I cusp will correspond to a cusp that appears somewhere in the tangle T in Figure 2a. The bands that enter and leave T to the left and right represent a set of parallel strands. A Type II cusp will correspond to a cusp that appears somewhere in the tangle T in Figure 2b, possibly including the strands that come out of T.

In both cases, our primary concern is that the particular cusp, when maximized, does not intersect the twice-punctured disk denoted D (which, in these figures, includes the point at ∞). This will allow us to cut the link complement open along D while preserving the maximal isometry type of the cusp C. In essence, the maximal cusp resides in a protective walnut once the manifold is cut open. Ultimately,



Type I cusp

a)



Type II cusp

b)

Figure 2 Types of generic cusps

we will cut another manifold open along another twice-punctured disk and then insert this walnut. Since a twice-punctured disk is totally geodesic in a hyperbolic 3-manifold (see e.g. [1]), one can identify the lift of the twice-punctured disk in the horoball diagram provided by the SNAPPEA computer program (see [4] and [12]) for any particular link and so determine whether or not the maximal cusp intersects it. With the exception of the volume 6, 4, 8, and 10 cusps in Figures 3b, 4a, 5a, and 5c (respectively), each of the maximal cusps labeled in Figures 3, 4 and 5 does not intersect the twice-punctured disk D. In the remaining four cases, the twice-punctured disk is tangent to the maximal cusp at a point of tangency of that cusp with itself. However, these maximal cusps touch themselves at (resp.) 4, 2, 8, and 5 points, only one tangency point of which occurs on the twice-punctured disk. Hence, when these link complements are cut open along D, the fact that the maximal cusps still touch themselves at other points will prevent the maximal isometry types of these cusps from changing.

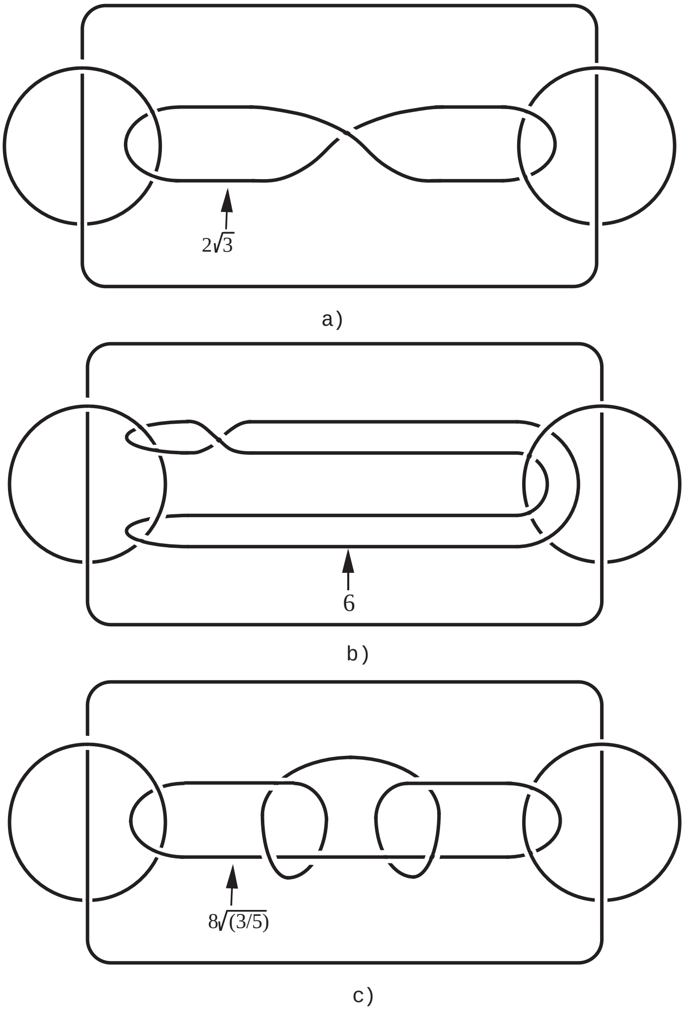


Figure 3 Generic cusps of Type I in the set $\boldsymbol{C}$

The labeled cusps in Figure 3 and 4 are generic cusps in the set $\boldsymbol{C}$. The labeled cusps in Figure 5 are generic cusps in a set denoted $\boldsymbol{C}'$.

In Figure 6 there is an example of a maximal cusp that does pass through the twice-punctured disk D. To get from one link complement to the other, one can cut the first complement open along what would be the disk D, rotate a half turn, and then reglue. Here, the maximal cusp corresponding to the particular labeled component in the first link passes through the twice-punctured disk, so when we cut open along the twice-punctured disk, twist a half-twist, and reglue, we change the cusp volume. There is no cocoon. In the second link complement, the maximal

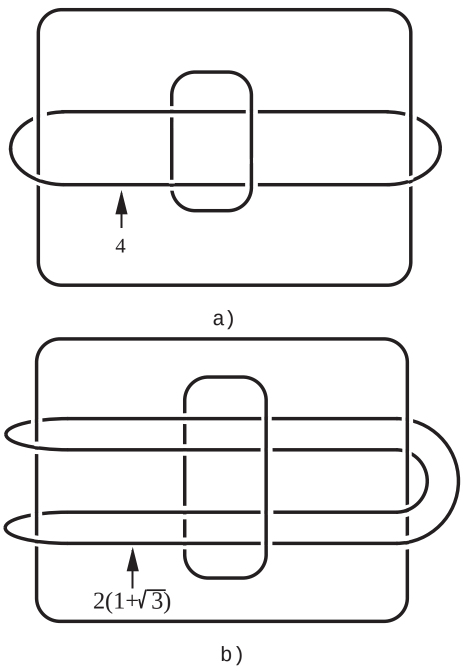


Figure 4 Generic cusps of Type II in the set ***C***

cusp just touches the twice-punctured disk but does not pass through it. Although that situation occurs for four of the cusps appearing in Figures 3, 4, and 5, those cusps had additional points of tangency. Here, there is only one point of tangency for the maximal cusp with itself, and it occurs on the twice-punctured disk. So when we cut open along that disk and glue the resulting manifold into another cut-open manifold, the maximal cusp volume of this cusp can change. This is the only example of a cusp of Type I or Type II where this phenomenon has been seen to occur. In all other cases, empirical evidence suggests that the maximal isometry type of the cusp C is preserved when the link complement is cut open along D.

3. Cutting and Pasting

In this section we prove the following theorem, which will be of use when handling cusps of Type II.

THEOREM 3.1. *Let M be a finite-volume hyperbolic* 3*-manifold containing a submanifold M^* bounded by a quadruply punctured sphere F such that M^* is homeomorphic to the complement in the ball bounded by F of the tangle in Figure 7a. Then if M^* is replaced in M by the submanifold that is the complement of the*

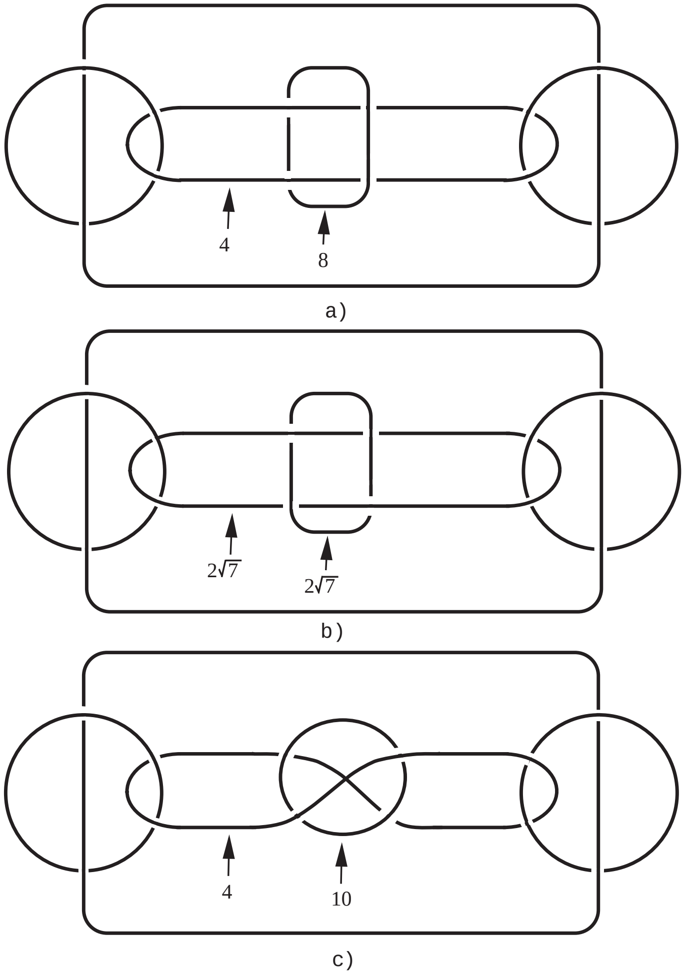


Figure 5 Generic cusps in the set $\boldsymbol{C}'$, when more curves can be removed from the manifold

tangle in Figure 7b, the resulting manifold Q is hyperbolic unless one of the following cases occurs:

(a) *there exists another cusp U in M as in Figure 8*; *or*
(b) *M is the complement of a link as in Figure 9.*

If case (a) does occur, then replacing the submanifold bounded by F as in Figure 8 with the complement of the tangle in Figure 7b results in a hyperbolic manifold—unless M is the complement of the Borromean rings or the complement of the sibling of the Borromean rings as appear in Figure 10.

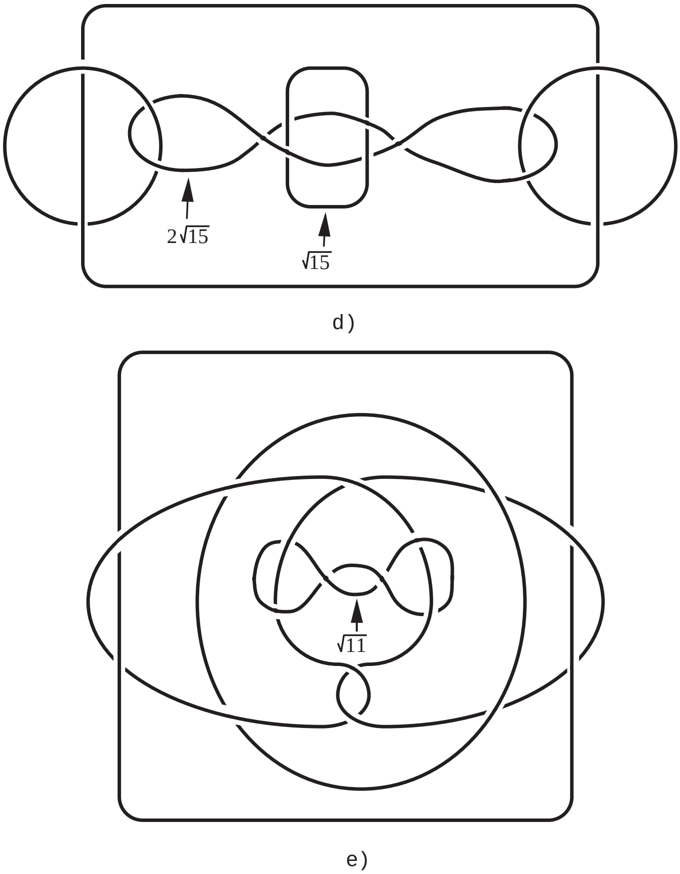
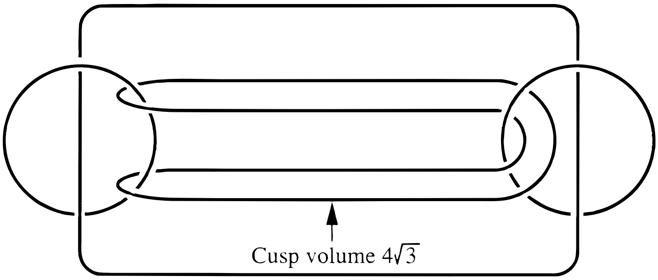
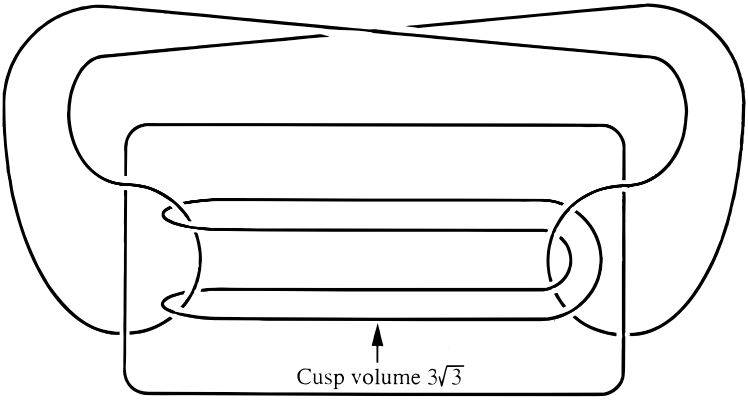


Figure 5 (cont.)

Proof. Suppose first that neither cases (a) nor (b) in the statement of the theorem occurs. Since a thrice-punctured sphere (pictured in Figure 7a as the twice-punctured disk D that is bounded by the trivial link component) is always totally geodesic, take the geodesic g that lies in the twice-punctured disk D, begins and ends on that link component, and separates the two punctures. Lemma 2.3 of [2] implies that Theorem 2.2 of [2] applies to g. Hence, there is a simple closed curve γ that can be removed from M so that the resulting manifold contains a twice-punctured disk D' with boundary a longitude of the removed curve and punctures corresponding to meridians of the cusp C. In fact, we would like the simple closed curve γ to correspond to traveling parallel to g on one side of D, then around a meridian of C, and then parallel to g on the other side of D and then around another meridian of C. In other words, we want it to be the case that the manifold

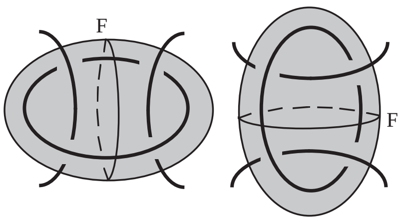


a)



b)

Figure 6 Cusps can intersect the twice-punctured disk *D*



a) b)

Figure 7 Replacing a) by b)

Q obtained from M by replacing M^* by the complement of the tangle in Figure 8 is hyperbolic. This requires us to extend the proof of Theorem 2.2 of [2] to show that the complement of γ in M is hyperbolic. That proof relies on Theorem 2.1 of [2]. Conditions 2 and 4 of the hypotheses of that theorem are satisfied by γ;

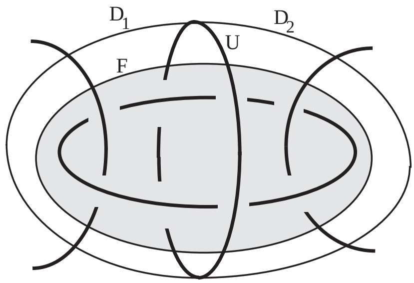


Figure 8 Cusp can be isotoped to lie in a regular neighborhood of F

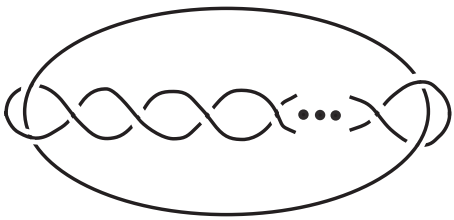


Figure 9 This case cannot occur

however, the curve γ might need to be modified in order to satisfy conditions 1 and 3 of those hypotheses.

Condition 1 states that γ is not contained in a neighborhood of an existing cusp in M. Suppose this were not satisfied. Such a neighborhood N cannot correspond to any of the cusps appearing in Figure 8. Let C' be the cusp with such a neighborhood. The boundary of the neighborhood is an essential torus T in M. Since the cusp does not intersect the twice-punctured disk D, T can be isotoped to avoid D. By the proof of Theorem 2.2 of [2], D' is essential in $M - \gamma$. Hence, T can be isotoped to intersect D' only in essential curves on T and D'. Since D must be avoided by these intersection curves, all such are parallel to γ on D'. There must be at least one such curve, since γ lies in the neighborhood N. However, then γ is isotopic to the outermost such curve in D' and is therefore isotopic to a (p, q)-curve on the boundary of the neighborhood N. But this implies that Condition (a) in the statement of this theorem is satisfied, a contradiction.

Condition 3 of Theorem 2.1 of [2] states that it cannot be the case that γ is a torus knot in a solid torus V such that ∂V is incompressible in $M - \text{int}(V)$. However, if this condition is not satisfied, the core of the solid torus V can be removed instead of γ. As occurs in the proof of Theorem 2.2, the resulting manifold is homeomorphic to the manifold obtained from M by replacing M^* by the complement of the tangle in Figure 8.

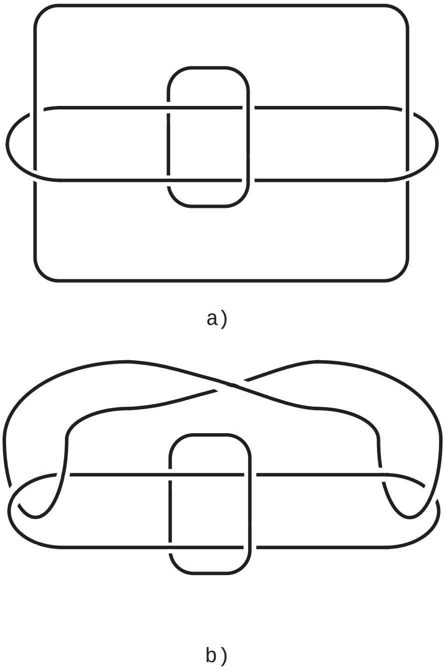


Figure 10 Borromean rings and sibling

Thus, we may now assume that the manifold obtained from M by drilling out the additional component γ is hyperbolic. From [1], we know that the two twice-punctured disks D_1 and D_2 appearing in Figure 8 are totally geodesic. If they are isotopic in M, then it must be the case that the complement in M of the tangle contained outside $D_1 \cup D_2$ is a product. This can only occur if M is a link as in Figure 9, which cannot occur because we are assuming we are not in case (b). Thus, the two totally geodesic twice-punctured disks are distinct. Since two totally geodesic surfaces can intersect only along essential curves and since these two do not so intersect, they are also disjoint as totally geodesic surfaces. Cutting $M - C'$ open along these two disks yields two components. By gluing together the two resulting twice-punctured disks on the boundary of each component—so that the result is orientable and the boundary of one disk is glued to the boundary of the other—we obtain two manifolds, one of which is the complement of the Borromean rings and one of which is the manifold obtained by replacing in M the submanifold M^* by the complement of the tangle in Figure 7b.

As for the addendum to the theorem, in the event that case (a) does hold, instead of drilling out an additional component we can simply utilize the component that is already there as our γ and then repeat the same argument. This shows that if that component and the rest of that part of the manifold contained in F are replaced by Figure 7b then the result is hyperbolic, unless the rest of the manifold outside F is a product. In this case it is a product and M is the complement of either the Borromean rings or its sibling link, as in Figure 10. □

4. The Main Theorem

THEOREM 4.1. *Let C be a choice of a cusp from the set in $\boldsymbol{C}$. Then for any closed* 3*-manifold there exist three disjoint simple closed curves—or, for any cusped finite volume hyperbolic* 3*-manifold there exist two disjoint simple closed curves—such that one of the new cusps in the complement of the curves will be maximally isometric to C.*

Proof. For any compact 3-manifold without boundary, it is known that there always exists a simple closed curve such that the complement is a finite-volume hyperbolic 3-manifold; see [7] (or [2] for an alternative proof). Thus, choosing any such curve to remove, one need then deal with only the second possibility in the statement of the theorem—namely, removing two disjoint simple closed curves from a cusped hyperbolic 3-manifold.

Let M be a finite-volume hyperbolic 3-manifold with one or more cusps. By Theorem 2.4 of [2], a simple closed curve can be removed from M such that the resulting manifold M' contains an essential embedded thrice-punctured sphere S, with one boundary curve from S on the boundary of the new cusp. Call this boundary curve of S a *longitude* of the new cusp.

Let M_L be the link complement corresponding to the particular cusp C from $\boldsymbol{C}$ that we wish to create. Suppose, first of all, that M_L is of Type I. Cut M' open along the thrice-punctured sphere S and call the result M'^-; cut M_L open along the twice-punctured disk D and call the result M_L^-. Glue each of the two copies of S in M'^- to one of the two copies of D in M_L^-, calling the result Q. By results of [1], the gluings can be done by isometries in such a way that the resulting manifold Q is orientable and hyperbolic with volume equal to the sum of the volumes of M' and M_L. Because the thrice-punctured spheres are glued together via isometries, there is no deformation of the hyperbolic structure on M' or M_L in the creation of Q. This immediately implies that the cusp corresponding to C in Q has the same modulus as C. Moreover, since the maximal cusp C either did not intersect the twice-punctured disk D in M_L, or (if it did touch D) did so in a single point of tangency with itself, and since there were additional points of tangency, C remains maximal inside the copy of M_L^- contained within Q. Hence there is an isometric copy of the maximal cusp C contained within Q. Note that Q is obtained from M by removing two cusps.

Now suppose that M_L is of Type II. We will show that Theorem 3.1 can be applied to M'. First note that case (a) in the statement of that theorem cannot occur, for if it did then there would be a disk in M with boundary on the cusp C, a contradiction to the hyperbolicity of M. Case (b) cannot occur for M' either; if it did then M would be the complement of the trivial knot and thus not hyperbolic.

Thus, we know that if we replace a neighborhood of the twice-punctured disk in M' that appears as the complement of the tangle in Figure 7a with the complement of the tangle in Figure 7b, the resulting manifold M'' is hyperbolic. We can now cut M'' open along the twice-punctured disk in Figure 7b to obtain M''^{-}. We now glue M''^{-} to M_L^{-}, and the argument repeats exactly as in the previous case. □

Note that this proof above immediately extends as follows.

COROLLARY 4.2. *Let C be a choice of a cusp from the set in $\mathbf{C}'$. Then for any closed hyperbolic* 3*-manifold there exist four disjoint simple closed curves—or, for any cusped finite volume hyperbolic* 3*-manifold there exist three disjoint simple closed curves—such that one of the new cusps in the complement of the curves will be isometric as a maximal cusp to C.*

5. Totally Geodesic Boundary

Let Q' be a manifold obtained as the complement in S^3 of the spatial graph G' shown in Figure 11a, including the graph G and the three trivial components L_B, L_C, L_D. We assume that the graph G has all vertices of valence at least 3 and has hyperbolic complement Q containing no essential annuli. An example of such a graph G and the corresponding graph G' appear in Figure 11b. The complement of that G was proved to be hyperbolic in [7] and to have no essential annuli.

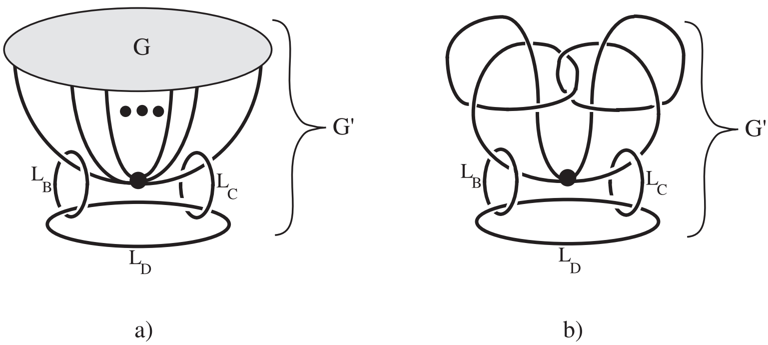


Figure 11 $Q = S^3 - G$, $Q' = S^3 - G'$

THEOREM 5.1. *Let G be as in Figure 11a with all vertices of valence at least* 3*. If $Q = S^3 - N(G)$ is hyperbolic and contains no essential annuli, then the same holds for $Q' = S^3 - N(G')$.*

Proof. We are assuming that Q is hyperbolic and contains no essential annuli. Thus, Q can be realized as a hyperbolic manifold with totally geodesic boundary.

We must prove that drilling out the three additional link components to obtain Q' preserves hyperbolicity and does not introduce any essential annuli. By work of Thurston [11], it is enough to show that Q' is irreducible, boundary-irreducible, and contains no essential tori or annuli. It is clear that the resulting manifold is irreducible and boundary-irreducible. Let B, C, D denote the obvious twice-punctured disks bounded by L_B, L_C, L_D, respectively.

Suppose there were an essential torus in Q'; it must then be compressible in Q. Let E be the disk bounded by an arc in L_D, an arc in B, an arc in C, and an arc passing through the visible vertex in G. Since the union of a neighborhood of $B \cup C \cup D \cup E$ with $N(G)$ does not change $N(G)$ topologically, the torus T must intersect at least one of these four disks. Assume that we have minimized the number of such intersection curves. If T intersects only E, then it must do so along an essential curve in T, meaning that T would be compressible in Q'. Hence, T must intersect B, C, or D. Because any simple closed curve in B, C, or D is isotopic to the boundary of Q', the existence of such a T forces the existence of an essential annulus in Q'. Thus, it is enough to show that there are no essential annuli in Q'.

Choose an annulus with as few intersection curves with B, C, and D as possible. Note that such an annulus must have no simple closed intersection curves. For if there were such a curve, one could cut along it and slide one of the new boundaries of the resulting annuli out along the disk to the boundary of the manifold, obtaining an essential annulus with fewer intersection curves.

Such an annulus A cannot have one boundary component on $\partial N(G)$ and one on $\partial N(L_D)$, as all curves on $\partial N(L_D)$ are trivial in $\pi_1(S^3 - N(G))$ but all nontrivial curves on $\partial N(G)$ are nontrivial in $\pi_1(S^3 - N(G))$. The annulus cannot have one boundary component on $\partial N(G)$ and one on $\partial N(L_B)$, since all nontrivial nonmeridianal curves on $\partial N(L_B)$ are nontrivial in $\pi_1(S^3 - N(L_D))$ while all curves on $\partial N(G)$ are trivial in $\pi_1(S^3 - N(L_D))$. If the boundary of A on $\partial N(L_B)$ were a meridian, we could then fill in L_B to obtain from A a disk in Q with boundary a nontrivial curve on $\partial N(G)$, a contradiction to the hyperbolicity of Q. The same holds if the second boundary component of A were on $\partial N(L_C)$.

The annulus cannot have one boundary component on $\partial N(L_B)$ and one on $\partial N(L_D)$ because any nontrivial nonmeridianal curve on $\partial N(L_B)$ is homotopic to a power of a meridian on an edge of G, which is nontrivial in $\pi_1(S^3 - N(G))$, whereas all curves on $\partial N(L_D)$ are trivial in $\pi_1(S^3 - N(G))$. If one boundary of A is a meridianal curve on $\partial N(L_B)$ then, once L_B is filled in, there is a disk with boundary a nontrivial curve on $\partial N(L_B)$ in the complement of $L_C \cup L_D$, a contradiction to the fact these two link components nontrivially link one another. The same holds if one boundary component of A is on $\partial N(L_C)$ and one on $\partial N(L_D)$.

Suppose that one boundary component of A is on $\partial N(L_B)$ and one on $\partial N(L_C)$. Then both boundary curves must be longitudes, since these are the only curves on the two boundaries that are equivalent as elements of $\pi_1(S^3 - N(L_B \cup L_C))$. In Q, one can cap off the annulus with the punctured disks B and C to obtain a properly embedded annulus A' in Q. It cannot be essential in Q and so it must boundary compress, implying that it is parallel to the boundary of $N(G)$. Hence, the edge J that punctures B must be the same edge that punctures C. Since L_D punctures A' through B and C and since L_D is unknotted, the annulus A' must be unknotted and hence the edge J of G that begins and ends on the visible vertex of G is unknotted. The link component L_D is parallel to that edge, and since L_D bounds a disk unpunctured by G, J bounds a disk unpunctured by G. But this contradicts the boundary irreducibility of Q.

Suppose that both boundary components of A are on $\partial N(G)$. If A intersects B or C, it must do so in an arc that begins and ends at the puncture in B or C caused by $N(G)$. The annulus is essential, so the arc must encircle the puncture caused by L_D. But then A intersects D. Since A has no boundary on L_D, this forces a simple closed curve intersection of A with D, a contradiction with our previous assumption that no such curves exist. So A does not intersect B, C, or D. Because A is inessential in Q, it follows that A and an annulus on $\partial N(G)$ together bound a solid torus V in Q that must contain at least one of L_B, L_C, L_D as an essential curve. However, L_D is trivial in Q, so if it were essential in the solid torus then this would imply that the boundary components of A were trivial in $\pi_1(Q)$, a contradiction. Both L_B and L_C bound once-punctured disks in Q, each punctured by G. Therefore each must be nontrivial in V, and contained within B and C are disks in $S^3 - \text{int}(V)$ punctured once by G with boundary in ∂V. Hence there is a sphere in S^3 punctured twice by G such that it bounds a ball containing L_B, L_C, and L_D but not the visible vertex of G. Then, since Q is acylindrical, G intersects this ball in an unknotted arc. This arc is a subarc of a single edge that begins and ends at the visible vertex and passes through B and C. Because the sphere separates this arc from the rest of G, the boundary of $N(G)$ is compressible in Q, a contradiction.

Suppose now that both boundary components are on $\partial N(L_B)$. If they are not meridians of L_B then there must be at least one arc of intersection of A with D. Since A does not boundary-compress, that arc must loop around the puncture corresponding to L_C. However, then A intersects C. Since A does not touch $\partial N(L_C)$, A must intersect C in at least one simple closed curve, a contradiction. If both boundary components of A are meridians on $\partial N(L_B)$, this contradicts the irreducibility of $Q - (L_C \cup L_D)$. The same holds if both boundary components are on $\partial N(L_C)$.

Suppose that both boundary components of A are on $\partial N(L_D)$. If they are both meridians then there would exist an essential sphere in $Q - (L_B \cup L_C)$, contradicting its irreducibility. If the two boundary curves are nonmeridians on $\partial N(L_D)$, then they must be parallel (p, q)-curves on $\partial N(L_D)$, with $q \neq 0$, and there exist intersection arcs of A with each of B and C. By the boundary incompressibility of A, each such arc must encircle the puncture made by G in each of these disks. By gluing a second annulus A' from $\partial N(L_D)$ to A along its boundary components, we have a torus T in Q'. Choosing innermost intersection arcs on each of B and C, they each form part of the boundary of meridianal disks for this torus that are each punctured once by G. Hence, G lies in a solid torus, with two meridianal disks that are each punctured once by G. Since the solid torus does not intersect itself, A can only intersect each of B and C in a single arc. Therefore, its boundary components each consist of a longitude and some number of meridians, and A is isotopic to A' in S^3. The two meridianal disks cut the solid torus into two balls, one of which contains the visible vertex of G. In order that there be no essential annulus in Q, we must have that the other ball contains only a trivial arc from G. However, this implies that there is a single edge from G that is puncturing both B and C and that D can be isotoped to that edge. In particular, that edge can be separated from the rest of G by a disk with boundary on a neighborhood of the visible vertex, contradicting the boundary irreducibility of Q. □

THEOREM 5.2. *Let S be the genus-2 surface with the particular hyperbolic structure of the totally geodesic boundary corresponding to the manifold Q' shown in Figure 5a, where Q is hyperbolic and contains no essential annuli. Then for any closed hyperbolic 3-manifold one can remove a handlebody and two disjoint simple closed curves—or, for any cusped finite volume hyperbolic 3-manifold one can remove a handlebody and a single disjoint simple closed curve—such that the totally geodesic boundary of the resulting hyperbolic manifold is isometric to S.*

Proof. We repeat the proof of Theorem 4.1 except that—instead of gluing in the link complement M_L cut open along a twice-punctured disk—we glue in Q' cut open along the twice-punctured disk D. That the structure of the twice-punctured disk is unique prevents any distortion of the hyperbolic metric on the totally geodesic boundary. The resulting manifold can be obtained by drilling out a handlebody and a simple closed curve from the original manifold. □

We note that there is a maximal embedded neighborhood for a totally geodesic boundary. In some cases, one could check that this neighborhood avoids the relevant twice-punctured disk and hence its volume is preserved in all resulting manifolds. However, we have not done that in this case.

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