On a Problem Raised by Gabriel and Beurling

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The aim of this paper is to study Gabriel's problem from the point of view of potential theory; that is, to study under what conditions on γ (and Ω)

$$\int_\gamma u(z)\,|dz| \le K \int_{\partial\Omega} u(z)\,|dz|,$$

where $\gamma \subset \Omega$ and u belongs to some set of functions (subharmonic or analytic). More specifically (but highly relevant), we examine under what conditions

$$\int_\gamma u(z)\,|dz| \le K \int_{\partial\Omega} u(z)V_\gamma(z)\,|dz|,$$

where $V_\gamma(z)$ is the normalized total angle subtended by γ at the point z.

The main idea in solving this problem is to exchange the roles of (a) the total angle with which the curve γ is seen from a point on the boundary and (b) the Poisson kernel of the domain Ω. We shall show that solving Gabriel's problem is equivalent to estimating the constant coming from this exchange; we shall call such constant *the functional K associated to the domain* Ω.

Several authors have studied Gabriel's problem for particular curves and domains. Our purpose in this paper is to survey this area, to unify proofs and presentation, and to describe the solution to some of the problems just mentioned. We will also describe a few problems that remain open.

The plan of the paper is as follows. In Section 2 we survey the history of the problem and present the known results; Section 3 is dedicated to studying the functional K introduced in this work. In Section 4 and Section 5, Theorems 1 and 3 unify and extend previous results for subharmonic and holomorphic functions and give the exact constant for Gabriel's problem. Also in Section 4 we pay special attention to the case when γ is a circle. Gabriel conjectured that, for holomorphic functions, the constant K is at most 2; in Section 5 we prove this conjecture false. Section 6 considers a classical problem (in geometric function theory) of Hayman and Wu under this viewpoint; we give a necessary geometric condition on the domain Ω for the constant K to be finite. In Section 7 we generalize Gabriel's problem to n dimensions for positive subharmonic functions and Γ convex. We finish in Section 8 by stating some questions that remain open.

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1. Introduction and Notation

Subharmonic functions u satisfy the submean value property as follows. Let C_s denote the circle $C_s = \{ z : |z| = s \}$, and let $0 < r < R$; then

$$\int_{C_r} u(z)\,|dz| \leq \frac{r}{R} \int_{C_R} u(z)\,|dz|, \tag{1}$$

where $|dz|$ denotes arclength.

The quotient r/R in (1) has a geometric meaning. For a point z exterior to a Jordan curve γ we define $V_\gamma(z)$, the (normalized absolute) *total angle* subtended by γ at z, as

$$V_\gamma(z) = \frac{1}{2\pi} \int_\gamma |\partial_\gamma \arg(z-w)|\,|dw|, \tag{2}$$

where $\partial_\gamma \arg(z-w)$ means the directional derivative of the argument in the direction γ'. (Here and hereafter, by the *exterior* of a Jordan curve γ we mean the unbounded component of $\hat{\mathbb{C}} \setminus \gamma$; similarly, by the *interior* of a Jordan curve γ we mean the bounded component of $\hat{\mathbb{C}} \setminus \gamma$.)

If we define ψ to be the angle between the segment from z to w and the tangent to the curve γ at the point w, then (2) can be written as

$$V_\gamma(z) = \frac{1}{2\pi} \int_\gamma \frac{|\sin\psi|}{|z-w|}\,|dw|$$

(see [B, p. 455; C; T, p. 340]). Observe that V_γ can be also defined as

$$V_\gamma(z) = \frac{1}{2\pi} \int_0^{2\pi} g_\gamma(z, e^{\imath\theta})\,d\theta,$$

where the function $g_\gamma(z, e^{\imath\theta})$ counts how many times the ray emerging from the point z in the direction θ intersects the curve γ.

If γ is a *closed* convex curve then, for every z exterior to the curve γ, $V_\gamma < 1$. In fact, this property characterizes convex curves: If $V_\gamma(z) < 1$ for every point z in the exterior of the Jordan curve γ, then γ is convex. Observe that if γ is C_r, Γ is C_R, and $z \in C_R$, then the quotient $r/R = \sin\big(\frac{\pi}{2} V_{C_r}(z)\big)$ and thus (1) can be written as

$$\int_{C_r} u(z)\,|dz| \leq \int_{C_R} u(z) \sin\left(\frac{\pi}{2} V_{C_r}(z)\right) |dz|. \tag{3}$$

It seems natural to ask to what extent result (3) can be generalized, a question which attracted a lot of attention in the 1930s. Gabriel was the first one to pose and study the following problem:

Given a pair of Jordan rectifiable curves Γ and γ, with γ contained in Ω, the interior of Γ, and given any positive number λ, find the best constant K such that, for all positive subharmonic functions u,

$$\int_\gamma u(z)^\lambda\,|dz| \le K \int_\Gamma u(z)^\lambda V_\gamma(z)\,|dz| \tag{G}$$

We shall refer to this question as *Gabriel's problem*.

Before finishing this section we will introduce some useful notation. Throughout this paper γ and Γ will be rectifiable curves, sometimes with additional properties (such as convexity) whenever specified; Γ is a Jordan curve and γ is contained in Ω, the interior of Γ. Also, z will always denote a point in γ and w a point in Γ. We shall denote their Euclidean distance by r; that is, $r = |z-w|$. We use $P_z^\Omega(w)$ to denote the normalized Poisson kernel of Ω at $z \in \gamma$ evaluated at $w \in \Gamma$ (i.e., the density of harmonic measure with respect to arclength).

In applications, the curves Γ or γ will often be convex; the results then are more precise and require an extra bit of notation. In the special case of Γ being convex: $\mathbb{H}_w$ will be the half-plane tangent to Γ at w that contains z; P_w will be its normalized Poisson kernel; and p will be the Euclidean distance from z to $\partial\mathbb{H}_w$. If γ is convex (or simply a segment): $\mathbb{H}_z$ will be the half-plane tangent to γ at z that contains w; Q_z will be its normalized Poisson kernel; and q will be the Euclidean distance from w to $\partial\mathbb{H}_z$. (If w happens to lie on the tangent line to γ at z, then $\mathbb{H}_z$ is either half-plane.)

Note that the subindex in $\mathbb{H}_z, Q_z, \ldots$ always stands for the point that is at the boundary.

2. Background

As mentioned in Section 1, in the 1930s Gabriel proposed the problem (G). Since then, different authors have studied it and obtained results for special curves or for special exponents. In this section we will describe the known results.

We shall divide this section into four parts: in Sections 2.1 and 2.2, the functions considered will be positive and subharmonic (restricting the exponent to 1 in 2.1). In 2.3 they will be holomorphic functions ($\lambda = \infty$); finally, in 2.4, we consider the closely related Hayman–Wu problem.

2.1. u Positive Subharmonic and $\lambda = 1$

In general, for positive subharmonic functions u, one has

$$\int_\gamma u(z)\,|dz| \le \int_\Gamma u(w)\left[\int_\gamma P_z^\Omega(w)\,|dz|\right]|dw|$$

(simply by comparing u with its minimal harmonic majorant).

In the particular case of γ being a circle C_r, say $C_r = \{\,z : |z| = r\,\}$, "separation" of variables simplifies the problem because

$$\int_{C_r} u(z)\,|dz| \le \int_\Gamma u(w)\left[\int_{C_r} P_z^\Omega(w)\,|dz|\right]|dw|$$
$$= \int_\Gamma u(w) P_0^\Omega(w)\,|dw|.$$

In this case, for Γ (with the property that, through each point p of Γ, one can draw a circle C_t of radius t independent of the point p such that its interior is exterior to Γ) Verblunsky [V] and Reuter [R] for $t = \infty$ (i.e., convex curves) showed that

$$\int_{C_r} u(z)\,|dz| \le \left(2 + \frac{r}{t}\right)\int_\Gamma u(w) V_{C_r}(w)\,|dw|.$$

2.2. *u Positive Subharmonic and* $\lambda \ge 1$

Gabriel [G3] studied the problem (G) for Γ convex and showed that, for $\lambda > 2$, the constant K is finite. He conjectured that this should be true for $\lambda \ge 1$, but Fenton [F] proved this false: K explodes at $\lambda = 2$ for the simple case when γ is a segment perpendicular to Γ.

We have stated that, for γ a circle, the constant K was always finite for $\lambda \ge 1$ (more precisely, $K \le 2 + r/t$ for r, t as before); this suggests that λ depends on the geometry of the curve γ. It would be nice to characterize those curves γ for which the constant K is finite for $\lambda \ge 1$.

In Section 4 we shall establish an analog of Gabriel's estimate, giving the exact constant not only for convex curves but for any Γ.

2.3. *u the Modulus of a Holomorphic Function*

The case of $\lambda = \infty$ for u being subharmonic and positive will be of special interest, because it is equivalent to Gabriel's problem (G) for u being the modulus of a holomorphic function and for exponent 1 (see Lemma 4 in Section 5.1).

Gabriel [G3] showed that K is finite if Γ is convex. As a matter of fact, he showed more: K is finite not only for exponent 1 for the modulus of a holomorphic function but for *any* positive exponent. Gabriel conjectured that $K \le 2$. We shall give a counterexample to this conjecture in Section 5.2.

Carlson [C] proved that, for any rectifiable curve γ and any positive number λ, $K \le 2$ if Γ a circle. As a corollary one obtains the result proved by Féjer and Riesz (see e.g. [D, p. 46]): for Γ the unit disc, γ the segment $(-1, 1)$, and $\lambda > 0$,

$$\int_{-1}^{1} |f(x)|^\lambda\,dx \le \frac{1}{2}\int_{\partial\mathbb{D}} |f(e^{\imath\theta})|^\lambda\,d\theta, \tag{4}$$

where the integral on the left is "counted only once" (i.e., we do *not* consider the curve $\gamma = [-1, 1]$ to be the closed curve that starts at -1, goes to 1, and then returns to -1 again). To see (4), simply observe that $V_{(-1,1)}(e^{\imath\theta}) = \frac{1}{4}$ for each $e^{\imath\theta} \in \partial\mathbb{D}$.

2.4. *A Related Problem*

We will also consider the problem of estimating the best constant $\tilde{K}$ such that, for special functions and exponents,

$$\int_\gamma |u(z)|^\lambda\,|dz| \le \tilde{K} \int_\Gamma |u(z)|^\lambda\,|dz| \tag{5}$$

(observe that no total angle is involved).

Gabriel obtained some results on (5) for very particular cases as follows. Given a circle Γ, a convex curve γ, and any exponent $\lambda > 0$, the constant $\tilde{K} \le 2$ for holomorphic functions u (see [G2]). Given Γ convex, γ a circle, and exponent $\lambda = 1$, the constant $\tilde{K} \le 2$ for positive subharmonic functions u (see [G1]). Also, Beurling [B, p. 457] considered the problem of estimating the best constant $\tilde{K}$ for some special curves. For more on this, see Section 3.3.

Observe that if the curve γ is convex then the total angle $V_\gamma < 1$; hence, in this case, the constant $\tilde{K}$ is at most the constant in Gabriel's problem (G).

2.5. *Application to Conformal Mappings*

One reason for the interest in these results is their application to the study of conformal mappings between simply connected (Jordan) domains. We offer two examples. First, consider g a conformal mapping from Ω onto the unit disc $\mathbb{D}$. Consider (5) when the exponent $\lambda = 1$, the function $u = g'$, and the curve γ is a segment. Then the problem of finding the best $\tilde{K}$ in (5) is the classical Hayman–Wu problem (see Section 6).

Likewise, consider g a conformal mapping from the unit disc $\mathbb{D}$ onto a simply connected (Jordan) domain Ω, and apply (4) to $f = g'$ with $\lambda = 1$. Then we obtain that the Euclidean length of hyperbolic geodesics in Ω is at most half the length of the boundary of Ω. For more on these applications see Section 6.

3. The Functional *K* Associated to a Domain Ω

We shall now describe a certain collection of functionals defined on the sphere bundle of Ω. We introduce the following notation. Given any direction **v** and any point $z \in \mathbb{R}^2$, let $L_{\mathbf{v},z}$ be a straight line containing z whose tangent is **v** (which divides the complex plane into two half-planes); let w be any point in $\mathbb{R}^2$. Denote by $Q_z(w)$ the normalized Poisson kernel of the half-plane with boundary $L_{\mathbf{v},z}$ that contains the point w. (If w lies on $L_{\mathbf{v},z}$ then we can take either half-plane; note that this use of $Q_z(w)$ is a slight abuse of notation.)

Fix the domain Ω and an exponent λ ($1 \le \lambda \le \infty$), and define the functional K_Γ^λ for every point $z \in \Omega$ and unit vector **v** (observe that $(z, \mathbf{v})$ is a point in the sphere bundle of Ω) as follows:

$$K_\Gamma^1(z,\mathbf{v}) = \sup_{w\in\Gamma} \frac{P_z^\Omega(w)}{Q_z(w)}, \quad \lambda = 1,$$

$$K_\Gamma^\lambda(z,\mathbf{v}) = \left\{ \int_\Gamma \left(\frac{P_z^\Omega(w)}{Q_z(w)}\right)^\beta P_z^\Omega(w)\,|dw| \right\}^{\lambda-1}, \quad 1 < \lambda < \infty,$$

$$K_\Gamma^\infty(z,\mathbf{v}) = \exp \int_\Gamma \log\left(\frac{P_z^\Omega(w)}{Q_z(w)}\right) P_z^\Omega(w)\,|dw|, \quad \lambda = \infty,$$

where $\beta = 1/(\lambda - 1)$.

3.1

Let us give a couple of relevant examples.

EXAMPLE 1. If Γ is a convex rectifiable Jordan curve, it is easy to estimate $K_\Gamma^\lambda(z,\mathbf{v})$ and to see that it is bounded if $\lambda > 2$. Toward this end, recall first the notation $Q_z(w)$, $P_w(z)$, and $\mathbb{H}_w$ (see Section 1). Since $\Omega \subset \mathbb{H}_w$, the integral of the functional K_Γ^λ can be bounded by

$$\int_\Gamma \left(\frac{P_z^\Omega(w)}{Q_z(w)}\right)^\beta P_z^\Omega(w)\,|dw| \le \int_\Gamma \left(\frac{P_w(z)}{Q_z(w)}\right)^\beta P_w(z)\,|dw|,$$

where again $\beta = 1/(\lambda-1)$. (Note that $w \in \Gamma$ is the variable of integration.)

If ϕ is the angle between the segment passing through z and w and the perpendicular line to $\partial\mathbb{H}_z$ containing z, and if ds is the arc element subtended at z, then

$$\begin{aligned}\int_\Gamma \left(\frac{P_w(z)}{Q_z(w)}\right)^\beta P_w(z)\,|dw| &\le \frac{1}{\pi}\int_\Gamma \frac{1}{|\cos\phi|^\beta}\,ds\\ &\le \frac{1}{\pi}\int_0^\pi \frac{1}{|\cos\phi|^\beta} 2\,d\phi\\ &= \frac{4}{\pi 2^{\beta+1}} B\left(\frac{1-\beta}{2}, \frac{1-\beta}{2}\right) < \infty,\end{aligned}$$

where the first inequality follows from the fact that

$$P_w(z) \le \frac{1}{|z-w|^2} \quad \text{and} \quad Q_z(w) = \frac{|\cos\phi|}{|z-w|^2}.$$

The last inequality comes from $\beta = 1/(\lambda-1)$. Here B denotes the Beta function (see [GR, p. 957]).

EXAMPLE 2. If Γ is the disc then $K_\Gamma^\infty(z,\mathbf{v}) = 0$. This will be obvious after the forthcoming Lemma 1.

Note that both $K_\Gamma^\lambda(z,\mathbf{v})$ and $K_\Gamma^\infty(z,\mathbf{v})$ are really defined on the sphere bundle: they depend not only on the point $z \in \Omega$ but also in the direction $\mathbf{v}$ of the tangent to γ at z (recall that $Q_z(w)$ depends on $\mathbf{v}$).

3.2

The following integral expression in the case $\lambda = \infty$ will be useful later.

LEMMA 1.

$$K_\Gamma^\infty(z,\mathbf{v}) = \exp \int_\Gamma \log \frac{1}{|2\sin\psi|}\,d\omega_z^\Omega,$$

where ψ is the angle between the segment from z to w and $L_{\mathbf{v},z}$ and where $d\omega_z^\Omega$ denotes harmonic measure in Ω.

Proof.

$$\int_\Gamma \log\left(\frac{P_z^\Omega(w)}{Q_z(w)}\right) P_z^\Omega(w)\,|dw| = \int_\Gamma P_z^\Omega(w)\log P_z^\Omega(w)\,|dw| - \int_\Gamma P_z^\Omega(w)\log(Q_z(w))\,|dw| = I + II.$$

Let us calculate I: If g denotes the Green's function for Ω with pole at z, then

$$I = \int_\Gamma \log\left(\frac{1}{2\pi}\frac{\partial g}{\partial \mathbf{n}}\right) P_z^\Omega(w)\,|dw| = \log\left(\frac{1}{2\pi}\right)1 + \int_\Gamma P_z^\Omega(w)\log\frac{\partial g}{\partial \mathbf{n}}\,|dw|.$$

Yet g is identically zero on Γ and so $\partial g/\partial\mathbf{n} = |\nabla g|$. This, together with the previous calculation, gives

$$I = \log\left(\frac{1}{2\pi}\right) + \int_\Gamma P_z^\Omega(w)\log|\nabla g|\,|dw|.$$

Let $\mathbb{D}$ be the unit disc, let $H\colon \mathbb{D}\to\Omega$ be conformal with $H(0) = z$, and let G be the Green's function on $\mathbb{D}$ with pole at zero. Then, by changing variables and observing that $g\circ H = G$, we have

$$\begin{aligned} I &= \log\frac{1}{2\pi} + \int_{\partial\mathbb{D}}\log|\nabla G|\,\frac{d\theta}{2\pi} - \int_{\partial\mathbb{D}}\log|H'|\,\frac{d\theta}{2\pi} \\ &= \log\frac{1}{2\pi} - \log|H'(0)| \\ &= \log\frac{1}{2\pi|H'(0)|}, \end{aligned}$$

where we have used that $|\nabla G| = 1$ on $\partial\mathbb{D}$.

We now calculate II. After the change of variables given by H, the integral II becomes

$$\begin{aligned} II &= \int_{\partial\mathbb{D}}\log\frac{\pi|H(\xi)-H(0)|^2}{q(H\xi)}\,\frac{|d\xi|}{2\pi} \\ &= \log\pi + \int_{\partial\mathbb{D}}\log\frac{|H(\xi)-H(0)|}{|\xi|}\,\frac{|d\xi|}{2\pi} + \int_{\partial\mathbb{D}}\log\frac{|H(\xi)-H(0)|}{q(H\xi)}\,\frac{|d\xi|}{2\pi} \\ &= \log(\pi|H'(0)|) + \int_{\partial\mathbb{D}}\log\frac{|H(\xi)-H(0)|}{q(H\xi)}\,\frac{|d\xi|}{2\pi}. \end{aligned}$$

Finally, adding I and II and doing the same (conformal) change of variables yields

$$
\begin{aligned}
\log K_\Gamma^\infty(z,\mathbf{v}) &= \log\frac{1}{2} + \int_{\partial\mathbb{D}} \log\frac{|H(\xi)-H(0)|}{q(H\xi)}\,\frac{|d\xi|}{2\pi}\\
&= \int_{\partial\mathbb{D}} \log\frac{1}{|2\sin\psi(\xi)|}\,\frac{|d\xi|}{2\pi}\\
&= \int_\Gamma \log\frac{1}{|2\sin\psi|}\,d\omega_z^\Omega,
\end{aligned}
$$

where the angle ψ and $d\omega_z^\Omega$ are as in Lemma 1. This finishes the proof. □

3.3

The expression

$$\int_\Gamma \log\frac{1}{|2\sin\psi|}\,d\omega_z^\Omega \tag{6}$$

appears in Beurling's work related to this type of problems [B, p. 457]. Beurling has implicitly conjectured that

$$2\exp\left(\int_\Gamma \log\frac{1}{|2\sin\psi|}\,d\omega_z^\Omega\right) \le 2$$

for every Γ convex, every point z in Ω, and every direction $\mathbf{v}$. Beurling's conjecture is still unresolved.

It is easily seen that

$$\sup_{\Gamma\text{ convex},z,\mathbf{v}} 2\exp\left(\int_\Gamma \log\frac{1}{|2\sin\psi|}\,d\omega_z^\Omega\right) \le 4$$

by observing

$$
\begin{aligned}
\int_\Gamma \log\frac{1}{|2\sin\psi|}\,d\omega_z^\Omega &= \int_0^\pi \log\frac{1}{2\sin\psi}\,d\theta(\psi)\\
&\le \log\frac{1}{2} + \int_0^\pi \log\frac{1}{\sin\psi}\,\frac{2}{\pi}\,d\psi\\
&= \log\frac{1}{2} + 2\log 2 = \log 2,
\end{aligned}
$$

where $\theta(\psi)$ denotes the harmonic measure in Ω of the arc I (which we now describe). Let v_1 and v_2 be counterclockwise and clockwise (respectively) rotations of angles ψ of the direction $\mathbf{v}$. Let r_1, $\mathbf{r}$, and r_2 be the rays emerging from the point z with directions v_1, $\mathbf{v}$, and v_2, respectively. Then we define the arc I on the curve Γ as the arc whose final points are the intersection of r_1 and r_2 with Γ and which contains the intersection point of $\mathbf{r}$ with Γ.

Beurling proved by a nice symmetry argument [B, p. 455] the nontrivial estimate

$$\sup_{\Gamma\text{ convex},z,\mathbf{v}} 2\exp\left(\int_\Gamma \log\frac{1}{|2\sin\psi|}\,d\omega_z^\Omega\right) \le 3.7.$$

Clearly, this supremum over convex curves Γ should be at least 2. For this, take Γ to be the unit circle; then

$$\int_\Gamma \log \frac{1}{|2\sin\psi|}\, d\omega_z^\Omega = 0$$

and Carlson's result in Section 2.3 follows.

The relevance of sharp bounds on the functional will become clear in Sections 4 and 5.

3.4

Before finishing this section we make a useful observation about $K_\Gamma^\infty(z, \mathbf{v})$. Denote

$$K(\Omega, z, \mathbf{v}) = \int_\Gamma \log \frac{1}{|2\sin\psi|}\, d\omega_z^\Omega.$$

First, note that obtaining a bound for $K(\Omega, z, \mathbf{v})$ valid for all triples $(\Omega, z, \mathbf{v})$ is equivalent to obtaining a bound for $K(\Omega, 0, \mathbf{v})$ valid for all pairs $(\Omega, \mathbf{v})$ with $0 \in \Omega$. We now have the following result, which will be crucial in Section 5 for showing that Gabriel's conjecture is not true.

Lemma 2. *For every domain Ω and point $z \in \Omega$,*

$$K(\Omega, z, \mathbf{v}) \le 0 \text{ for all } \mathbf{v} \iff K(\Omega, z, \mathbf{v}) = 0 \text{ for all } \mathbf{v}.$$

Proof. Integrating over all directions $\mathbf{v}$ the expression for $K(\Omega, z, \mathbf{v})$,

$$\int_{\partial\mathbb{D}} K(\Omega, z, \mathbf{v})\, \frac{d\mathbf{v}}{2\pi} = \int_{\partial\Omega}\int_{\partial\mathbb{D}} \log \frac{1}{|2\sin\psi|}\, \frac{d\mathbf{v}}{2\pi}\, d\omega_z^\Omega = 0,$$

where the last equality follows from Jensen's identity (see e.g. [Ru, p. 307]).

Thus, if $K(\Omega, z, \mathbf{v}) \le 0$ then $K(\Omega, z, \mathbf{v}) = 0$ for all $\mathbf{v}$. □

4. Subharmonic Functions

4.1

The main result of this section is the following theorem.

Theorem 1. *Let Γ, γ be rectifiable Jordan curves with γ contained in Ω, u positive and subharmonic on Ω, and $\lambda > 2$. Then*

$$\int_\gamma u(z)^\lambda\, |dz| \le K(\lambda, \Gamma) \int_\Gamma u(w)^\lambda V_\gamma(w)\, |dw|, \tag{7}$$

where

$$K(\lambda, \Gamma) = 2 \sup_{z\in\Omega,\, \mathbf{v}\in\partial\mathbb{D}} K_\Gamma^\lambda(z, \mathbf{v}). \tag{8}$$

The constant is sharp.

Observe that the constant K_Γ^λ does not depend explicitly on γ.

For Γ convex, one obtains Gabriel's result ([G3]). Recall that in this case and for exponent $\lambda > 2$, it follows that $K(\lambda, \Gamma) < \infty$ (see Example 1 in Section 3.1). Although the proof works for all $\lambda \ge 1$, given a convex curve Γ and an exponent

$\lambda \le 2$, $K(\lambda, \Gamma) = \infty$. Fenton [F] gave an example for $\Gamma = \partial\mathbb{H}$. (Actually, more is true: there is a pair $(z, \mathbf{v})$ such that $K_\Gamma^\lambda(z, \mathbf{v}) = \infty$.)

The condition that u be positive is necessary for Theorem 1 to hold, as the example $U(z) = \log|z|$ in the unit disc reveals (see [F]). For then, if γ is not a point, the integral on the right-hand side of (7) is zero while the left-hand side is positive.

Proof. Let $\gamma(t)$ $(0 \le t \le 1)$ be a parameterization of the rectifiable curve γ. For each $n \in \mathbb{N}$, consider a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ such that the points $\gamma(t_0), \dots, \gamma(t_n) \in \gamma$ divide the curve γ into n pieces of the same length. Approximate the curve γ by a polygonal P^γ with vertices $\gamma(t_0), \dots, \gamma(t_n) \in \gamma$ and sides $L_1, \dots, L_n$ (i.e., each side L_j is a Euclidean segment that starts at the point $\gamma(t_{j-1})$ and finishes at the point $\gamma(t_j)$).

Observe that

$$\operatorname{length} L_j = |\gamma(t_j) - \gamma(t_{j-1})| \le \frac{\operatorname{length}\gamma}{n} < \infty.$$

For any u, we have

$$\left| \int_\gamma u(z)\,|dz| - \sum_{i=1}^n \int_{L_j} u(z)\,|dz| \right| \to 0 \quad \text{as } n \to \infty,$$

$$|V_\gamma - V_{P^\gamma}| \to 0 \quad \text{as } n \to \infty.$$

It will be enough to prove (7) for the polygonal P^γ. Moreover, since both V_{P^γ} and the integral on the left-hand side of (7) are additive on the sides L_j of P^γ, it is enough to prove (7) for a segment, say L.

Recall the notation $Q_z(w)$. We want to show that

$$\int_L u(z)^\lambda\,|dz| \le K(\lambda, \Gamma) \int_\Gamma u(w)^\lambda V_L(w)\,|dw|,$$

which holds if and only if

$$u(z)^\lambda \le \frac{K(\lambda, \Gamma)}{2} \int_\Gamma u(w)^\lambda Q_z(w)\,|dw|. \tag{9}$$

To see this, simply observe that for any direction $\mathbf{v}$ with segment $L = [z, z + t\mathbf{v}]$ and for any $w \in \Gamma$, since $Q_z(w)$ depends only on $\mathbf{v}$ we have

$$\frac{V_L(w)}{|L|} \to \frac{1}{2}\frac{q}{\pi|w - z|^2} = \frac{Q_z(w)}{2}$$

as $t \to 0$.

We shall prove (9). Suppose first that u is harmonic; then, by Hölder's inequality,

$$\begin{aligned} u(z)^\lambda &= \left(\int_\Gamma u(w) P_z^\Omega(w)\,|dw| \right)^\lambda \\ &\le \left(\int_\Gamma u(w)^\lambda Q_z(w)\,|dw| \right) \left(\int_\Gamma \left(\frac{P_z^\Omega(w)}{Q_z(w)} \right)^\beta P_z^\Omega(w)\,|dw| \right)^{1/\beta}, \end{aligned} \tag{H}$$

where $\beta = 1/(\lambda - 1)$. Comparing this inequality with (9), for harmonic functions we obtain

$$K(\lambda,\Gamma) = 2 \sup_{z\in\Omega,\mathbf{v}\in\partial\mathbb{D}} \left(\int_\Gamma \left(\frac{P_z^\Omega(w)}{Q_z(w)} \right)^\beta P_z^\Omega(w)\,|dw| \right)^{1/\beta}$$
$$= 2 \sup_{z\in\Omega,\mathbf{v}\in\partial\mathbb{D}} K_\Gamma^\lambda(z,\mathbf{v}).$$

For u subharmonic, consider its harmonic majorant U. The result follows from applying (9) to U.

Note that the constant $K(\lambda,\Gamma)$ is the best possible; for $z_0 \in \Omega$ and direction $\mathbf{v}$, take u to be the harmonic function on Ω with boundary values $u(w) = (P_{z_0}^\Omega(w)/Q_{z_0}(w))^\beta$. For such u, equality in (H) is attained; that is,

$$u(z_0)^\lambda = K_\Gamma^\lambda(z_0,\mathbf{v}) \int_\Gamma u(w)^\lambda Q_{z_0}(w)\,|dw|.$$

Moreover, if γ_ε is the curve $\gamma_\varepsilon = z_0 + t\mathbf{v}$ for $t \in [0,\varepsilon]$ with $\varepsilon > 0$, then

$$\lim_{\varepsilon\to 0} \frac{\int_{\gamma_\varepsilon} u(z)^\lambda\,|dz|}{\int_\Gamma u(w)^\lambda V_{\gamma_\varepsilon}(w)\,|dw|} = 2K_\Gamma^\lambda(z_0,\mathbf{v}).$$

This concludes the proof of Theorem 1. □

There is a more useful estimate for the constant $K(\lambda,\Gamma)$ which involves both curves more explicitly but which is not sharp. For the sake of completeness we shall give it here.

LEMMA 3. *Let $\lambda \geq 1$, let λ' be such that $1/\lambda + 1/\lambda' = 1$, and let $\beta = \lambda'/\lambda$. Then, for $\gamma, \Gamma, \Omega, u$ as in Theorem 1,*

$$\int_\gamma u(z)^\lambda\,|dz| \leq C(\gamma,\Gamma) \int_\Gamma u(w)^\lambda V_\gamma(w)\,|dw|,$$

where

$$C(\gamma,\Gamma)^\beta = \sup_\Gamma \int_\Gamma \frac{1}{V_\gamma^\beta} \left(\int_\gamma (P_z^\Omega(w))^\lambda\,|dz| \right)^\beta.$$

Proof. It is just Minkowski's inequality. □

4.2. γ a Circle

As one can easily imagine, the case of the curve γ being a circle C_r (say, centered at the origin and of radius r) is particularly simple. Separation of variables simplifies the problem because

$$\int_{C_r} u(z)\,|dz| \leq \int_\Gamma u(w) \left[\int_{C_r} P_z^\Omega(w)\,|dz| \right] |dw|$$
$$= \int_\Gamma u(w) P_0^\Omega(w)\,|dw|.$$

In this case, Gabriel's inequality (G) holds for more general curves Γ and for all exponents $\lambda \geq 1$.

Before stating the principal result of this section, we will introduce the notion of exterior curvature, which will be a basic tool for handling the geometry of the curve Γ. Let Γ be a rectifiable Jordan curve. For almost every point $p \in \Gamma$ we consider the family of circles C_t^p of radii t tangent to Γ at p whose interiors do not intersect Γ and which are contained in the exterior of Γ. Let

$$t_p = \sup\{\text{radii of } C_t^p\}$$

($t_p = \infty$ at all points p if and only if Γ convex). We define the *exterior curvature* k_p^{ext} of Γ at p as

$$k_p^{\text{ext}} = \frac{1}{t_p}.$$

We can now state the main result of this section.

THEOREM 2. *For any circle C_r, any positive subharmonic function u, and any $\lambda \geq 1$,*

$$\int_{C_r} u(z)^\lambda\,|dz| \leq \int_\Gamma u(w)^\lambda \left[2\sin\left(\frac{\pi}{2}V_{C_r}(w)\right) + rk_w^{\text{ext}}\right]|dw|; \tag{10}$$

if $\lambda = 1$ it is sharp.

Observe that Theorem 2 implies the results obtained by Verblunsky and Reuter (their estimate uses the upper bound 1 instead of the factor $\sin\left(\frac{\pi}{2}V_{C_r}(w)\right)$; see Section 2.1).

Proof. We may assume that $\lambda = 1$, since u^λ is subharmonic whenever u is positive and subharmonic; we may also assume that u is harmonic, since u has a harmonic majorant v such that $u = v$ on Γ and $u \leq v$ on C_r. Thus,

$$\begin{aligned}\int_{C_r} u(z)\,|dz| &\leq \int_{C_r} v(z)\,|dz| = 2\pi r v(0)\\ &= 2\pi r\int_\Gamma v(w)P_0^\Omega(w)\,|dw| = 2\pi r\int_\Gamma u(w)P_0^\Omega(w)\,|dw|.\end{aligned} \tag{11}$$

For almost every $w \in \Omega$, consider the radius t_w of exterior curvature of Γ at the point w. To simplify notation, let

$$\begin{aligned} t &= t_w,\\ C_t &= \text{circle tangent to } \Gamma \text{ at } w \text{ of radius } t_w \text{ contained in } \Omega^c,\\ E_t &= \text{exterior of } C_t.\end{aligned}$$

Since $\Omega \subset E_t$ we have

$$P_0^\Omega(w) \leq P_0^{E_t}(w), \tag{12}$$

where $P_0^{E_t}(w)$ is the Poisson kernel of the domain E_t evaluated at the point 0.

Recall that if z_0 is the center of the circle C_t then

$$P_0^{E_t}(w) = \frac{1}{2\pi t}\frac{|z_0|^2 - t^2}{|z_0 + w|^2}.$$

To estimate this, let θ be the angle between the inward normal to Γ at w and the segment joining w and 0. A straightforward calculation shows

$$P_0^{E_t}(w) \le \frac{1}{2\pi}\left(\frac{2\cos\theta}{|w|} + \frac{1}{t}\right).$$

The normalized absolute total angle arises naturally, and since C_t is a circle it is easy to see that

$$V_{C_t}(w) = 4\frac{1}{2\pi}\arcsin\frac{t}{|w|},$$

so that

$$P_0^{E_t}(w) \le \frac{1}{2\pi}\left[\frac{2\cos\theta}{r}\sin\left(\frac{\pi}{2}V_{C_t}\right) + \frac{1}{t}\right]. \tag{13}$$

With estimates (12) and (13), and recalling the definition of exterior curvature k_w^{ext}, our initial inequality (11) becomes

$$\begin{aligned} 2\pi r\int_\Gamma u(w)P_0^{\Omega}(w)\,|dw| &\le \int_\Gamma 2\pi r u(w)P_0^{E_t}(w) \\ &\le \int_\Gamma u(w)\left[2\sin\left(\frac{\pi}{2}V_{C_t}\right) + rk_w^{\text{ext}}\right], \end{aligned}$$

and the theorem follows. $\square$

As claimed in the theorem, for $\lambda = 1$ the result is sharp. To see this, set Ω to be the upper half-plane and set γ to be the circle centered at $z = \imath L$ and with radius $r = 1$, where $L \gg 1$. Then

$$\int_{\mathbb{R}} u(\omega)\left[2\sin\left(\frac{\pi}{2}V_{C_r}\right) + rk_w^{\text{ext}}\right]|dw| \cong 2\int_{\mathbb{R}} u(w)\,|dw|,$$

where the number 2 is the constant estimated by Verblunsky and Reuter ($t_w = \infty$ for all $w \in \partial\mathbb{H}$).

5. Holomorphic Functions

It is of special interest to consider (G) when u is the modulus of a holomorphic function f. In this section we will first calculate the exact constant for this case and then give an example yielding $K > 2$ and so showing Gabriel's conjecture false (see Section 5.2).

5.1. The Exact Constant

We shall need the following observation of Hayman [H], which shows that studying Gabriel's problem (G) for u subharmonic positive and $\lambda = \infty$ is equivalent to studying (G) for $u = |f|$ with f holomorphic and exponent 1.

LEMMA 4. *For any pair of rectifiable Jordan curves Γ and γ, the following two conditions are equivalent.*

(A) *There exists a constant C such that, for every harmonic and positive function u,*

$$\int_\gamma e^{u(z)}\,|dz| \le C\int_\Gamma e^{u(w)}V_\gamma(w)\,|dw|.$$

(B) *There exists a constant C such that, for any polynomial P,*

$$\int_\gamma |P(z)|\,|dz| \le C\int_\Gamma |P(w)|V_\gamma(w)\,|dw|.$$

The constants in (A) *and* (B) *are the same.*

Hayman proved this without the weight V_γ and for γ a circle, but Lemma 4 follows in the very same way. Note also that, since (B) holds for any polynomial, it also holds for functions holomorphic on the closure of Ω.

Now we are ready to give the equivalent result to Theorem 1 in this context.

THEOREM 3. *Let Γ, γ be rectifiable Jordan curves with γ contained in Ω, and let f be holomorphic on Ω. Then*

$$\int_\gamma |f(z)|\,|dz| \le K(\infty,\Gamma)\int_\Gamma |f(w)|V_\gamma(w)\,|dw|, \tag{14}$$

where

$$K(\infty,\Gamma) = 2\sup_{z\in\Omega,\mathbf{v}\in\partial\mathbb{D}} K_\Gamma^\infty(z,\mathbf{v}). \tag{15}$$

The constant is sharp.

If Γ is convex then Gabriel's result follows. In this case, recall that $K(\infty,\Gamma) < \infty$ (see Sections 3.2 and 3.3). Note also that K_Γ^λ does not depend explicitly on the curve γ.

Proof. By Lemma 4 it is enough to show (14) for functions of the form $e^{u(z)}$ for u positive harmonic. As in the proof of Theorem 1, (14) will hold if and only if

$$e^{u(z)} \le \frac{K(\infty,\Gamma)}{2}\int_\Gamma e^{u(w)}Q_z(w)\,|dw|, \tag{16}$$

and we shall prove this.

Recall that u is harmonic and positive. Let v be a harmonic conjugate and consider the holomorphic function $f = e^{u+\imath v}$. Since $u = \log|f|$ is harmonic,

$$\begin{aligned} u(z) &= \int_\Gamma u(w)P_z^\Omega(w)\,|dw| \\ &= \int_\Gamma \left(u(w) + \log\left|\frac{Q_z(w)}{P_z^\Omega(w)}\right|\right)P_z^\Omega(w)\,|dw| + \int_\Gamma \log\left|\frac{P_z^\Omega(w)}{Q_z(w)}\right|P_z^\Omega(w)\,|dw|. \end{aligned}$$

Taking the exponential first and then applying Jensen's inequality,

$$e^{u(z)} = \exp\left\{\int_\Gamma \left(u(w) + \log\left|\frac{Q_z(w)}{P_z^\Omega(w)}\right|\right) P_z^\Omega(w)\,|dw|\right\}$$
$$\cdot \exp\left\{\int_\Gamma \log\left|\frac{P_z^\Omega(w)}{Q_z(w)}\right| P_z^\Omega(w)\,|dw|\right\}$$
$$\leq \int_\Gamma \exp\{u(w)\}Q_z(w)\,|dw| \exp\left\{\int_\Gamma \log\left|\frac{P_z^\Omega(w)}{Q_z(w)}\right| P_z^\Omega(w)\,|dw|\right\}. \quad \text{(J)}$$

Hence, for functions of the form $e^{u(z)}$ we have

$$\frac{K(\infty, \Gamma)}{2} = \sup_{z\in\Omega, \mathbf{v}\in\partial\mathbb{D}} \exp \int_\Gamma \log\left(\frac{P_z^\Omega(w)}{Q_z(w)}\right) P_z^\Omega(w)\,|dw|,$$

which proves the theorem. □

The constant $K(\infty, \Gamma)$ is sharp. To see this, consider $\mathbb{H}_{z_0}$ for each $z_0 \in \Omega$ and each direction $\mathbf{v}$. Define the set A_{z_0} as

$$A_{z_0}(\mathbf{v}) = \{\xi \in \Gamma : |\xi - a| < 10^{-10} \text{ for all } a \in \Gamma \cap \partial\mathbb{H}_{z_0}\},$$

and consider the functions

$$h(w) = \chi_{\Gamma\setminus A_z} \log\left(\frac{P_{z_0}^\Omega(w)}{Q_{z_0}(w)}\right),$$

$u(z) = P^\Omega[h]$ (i.e., u is the harmonic function on Ω with boundary values h),

$\tilde{u}(z) =$ a harmonic conjugate of u.

Then, equality in (J) is attained for $f(z) = e^{u+i\tilde{u}}$; that is,

$$|f(z_0)| = K_\Gamma^\infty(z_0, \mathbf{v}) \int_\Gamma |f(w)| Q_{z_0}(w)\,|dw|.$$

Moreover, if γ_ε is the curve $\gamma_\varepsilon = z_0 + t\mathbf{v}$ for $t \in [0, \varepsilon]$ with $\varepsilon > 0$, then

$$\lim_{\varepsilon\to 0} \frac{\int_{\gamma_\varepsilon} |f(z)|\,|dz|}{\int_\Gamma |f(w)| V_{\gamma_\varepsilon}(w)\,|dw|} = 2K_\Gamma^\infty(z_0, \mathbf{v}).$$

5.2. Counterexample to Gabriel's Conjecture

Gabriel [G3] proved that, for any f holomorphic and Γ convex and for any $\mu > 0$,

$$\int_\gamma |f(z)|^\mu\,|dz| \leq K \int_\Gamma |f(z)|^\mu V_\gamma(z)\,|dz|;$$

he conjectured that the constant K could be replaced by 2. We shall show this to be false and shall do so in the case of interest to u: when $\mu = 1$.

Recall that, by Lemma 1, the best possible constant K can be written as

$$K_\Gamma^\infty(z, \mathbf{v}) = \exp K(\Gamma, z, \mathbf{v})$$

with

$$K(\Gamma, z, \mathbf{v}) = \int_\Gamma \log \frac{1}{|2\sin\psi|}\, d\omega_z^\Omega;$$

here ψ is the angle between the segment from z to w and the tangent to γ at z, and $d\omega_z^\Omega$ denotes the harmonic measure in Ω. As we have already observed, the constant K_Γ^∞ does not depend explicitly on the curve γ. This is the key factor in the counterexample.

Consider the domain Ω to be the band $B = \{\, z \in \mathbb{C} : |\Re(z)| < 2 \,\}$, the direction $\mathbf{v} = (1, 0)$, and the sets

$$A = \{\, z \in \partial B : z = x + \imath y,\ |y| < 2/\sqrt{3} \,\},$$
$$D = \partial B \setminus A$$

(i.e., A is the set of points where $\log(1/|2\sin\psi|)$ is positive and D where it is negative). It is apparent that, from the point $z = 0$, A has more harmonic measure than D. This is why $K(B, 0, \mathbf{v})$ should be positive (note that $\mathbf{v}$ points in the x direction); thus $K_\Gamma^\infty(z, \mathbf{v}) > 1$ and thus $K(\infty, \Gamma) > 2$. We proceed to make this precise.

With Ω being the band B just described, take γ to be the segment $[-1, 1]$ and let $z = 0$. Since by symmetry

$$\begin{aligned} K(\partial B, 0, (1,0)) &= 4\int_0^\infty P_0^B(2 + \imath y) \log P_0^B(2 + \imath y)\, dy \\ &\quad + 4\int_0^\infty P_0^B(2 + \imath y) \log \frac{1}{Q_z(w)}\, dy \\ &= 4A + 4B, \end{aligned}$$

it will be enough to see that $A + B > 0$ to finish the counterexample.

We shall first estimate A. As shown in the proof of Lemma 1,

$$4A = \log \frac{1}{2\pi |H'(0)|}$$

where H is the conformal map $H : \mathbb{D} \to B$ with $H(0) = 0$. That is,

$$H(z) = \frac{4}{\pi} \imath \log \frac{1+z}{1-z}$$

and so

$$A = \log \frac{1}{2}.$$

To estimate B, observe that

$$P_0^B(2 + \imath y) = \frac{1}{8} \frac{1}{\cosh(y\pi/4)}.$$

We then have

$$\begin{aligned} B &= \int_0^\infty \frac{1}{8} \frac{1}{\cosh(y\pi/4)} \log \frac{\pi(y^2+4)}{y}\, dy \\ &= \log\left(\left[\frac{8}{\pi}\right]^{1/4} \left[\frac{\Gamma(1/4)}{\Gamma(3/4)}\right]^{1/2} \right), \end{aligned}$$

here $\Gamma(z)$ is the Gamma function (see [GR, p. 942]). Adding up A and B, we obtain

$$A + B = \log\left(\left[\frac{1}{2\pi}\right]^{1/4}\left[\frac{\Gamma(1/4)}{\Gamma(3/4)}\right]^{1/2}\right) \sim 0.083 > 0,$$

which shows that $K(\infty, \Gamma) > 2$.

It is important to observe that the foregoing is not a counterexample to Beurling's conjecture (see Section 3.3). The weight V_γ plays an important role.

Alternatively we could have proceeded as follows. Recall Lemma 2 and note that

$$K(\Omega, z, \mathbf{v}) = \int_\Gamma \log\frac{1}{|2\sin\psi|}\, d\omega_z^\Omega \le 0$$

for all z and v if and only if

$$\int_\Gamma \log\frac{1}{|2\sin\psi|}\, d\omega_z^\Omega = 0.$$

As one could expect (and as is quite easy to show), $K(\partial B, z, \mathbf{v}) < 0$ if $\mathbf{v}$ is the direction $\mathbf{v} = (0, 1)$ (i.e., if $\mathbf{v}$ points in the y direction). Thus, by Lemma 2, $K(\partial B, z, \mathbf{v})$ must be positive for some other directions $\mathbf{v}$.

6. Conformal Mapping: The Hayman and Wu Problem

The following question was raised by A. Weitsman.

> *Given a simply connected domain Ω bounded by a rectifiable Jordan curve Γ, find the best constant* $\mathrm{HW}(\Gamma)$ *such that, for every straight line L (or circle) and for every conformal mapping f from Ω onto the unit disc $\mathbb{D}$,*

$$\int_{L\cap\Omega} |f'(z)|\,|dz| \le \mathrm{HW}(\Gamma)\int_\Gamma |f'(z)|\,|dz|. \tag{17}$$

In other words, $\mathrm{length}(f(L\cap\Omega)) \le \mathrm{HW}(\Gamma)\,\mathrm{length}(\partial\mathbb{D})$. Observe that since the problem is Möbius invariant, circles and straight lines play the same role. Note also that there is no total angle in (17).

Hayman and Wu [HW] were the first to show that there exists a universal finite bound HW for *all* curves Γ (see also [GGJ]). The best result known today is that $\pi/2 \le \mathrm{HW} \le 2$, with both estimates due to Øyma (see [Ø1] and [Ø2], respectively). For the rest of this section, Ω, Γ, and L will remain as before.

Brown-Flinn [B-F] showed that if $L \subset \Omega$ then the sharp estimate is $\mathrm{HW} \le \pi/2$. The sharp constant for the general case is still unknown. The conjecture is, again, $\mathrm{HW} \le \pi/2$. If the simply connected domain Ω is convex, we showed that $\mathrm{HW} \le 1$ and this is sharp [FG]; moreover, $\mathrm{length}(f(L\cap\Omega)) < \mathrm{length}(\partial\mathbb{D})$.

Some authors have considered (17) for more general curves than straight lines and for more general functions than conformal ones. Bishop and Jones [BJ] characterized as *Ahlfors-regular* those curves for which (17) holds with a finite constant for any domain Ω and any conformal map f. Since arclength is a Carleson measure, it is straightforward to show that

$$\int_{L\cap\Omega} |g(z)|\,|dz| \le C \int_\Gamma |g(z)|\,|dz|$$

for all g holomorphic in $\bar{\Omega}$. Thus, (17) holds also for holomorhic functions (not necessarily the derivative of a conformal one).

It seems natural to ask whether (17) holds if one introduces the total angle V_L subtended at $w \in \Gamma$, and also if in this case it can be generalized by considering more general curves and functions. Hayman and Hall [HH] gave a nice estimate of the constant in (17) for general holomorphic functions and certain domains Ω. In fact, they generalized Brown's result (just recalled). Observe that there is *no* weight on the integral on the right-hand side of (17).

THEOREM HH. *Let Γ be a rectifiable Jordan curve whose interior Ω contains a circle C. Then, for any holomorphic function f in Ω,*

$$\int_C |f(z)|\,|dz| \le 4\pi \int_\Gamma |f(z)|\,|dz|.$$

Except for the constant, this result is a well-known corollary of the Hayman–Wu theorem and the characterization of Carleson measures in terms of the Möbius group. See [G, p. 239] and the proof that follows here. The emphasis now is in the best constant. Our proof is much simpler than the one in [HH], but the constant is worse.

Proof. Let ψ be any conformal map from Ω onto $\mathbb{D}$. Then, by Brown's result and conformal invariance,

$$\text{length}(\psi(C)) \le \pi^2;$$

that is,

$$\int_C |\psi'(z)|\,|dz| \le \frac{\pi}{2} \int_\Gamma |\psi'(z)|\,|dz|.$$

Let T be any Möbius transformation on $\mathbb{D}$. Applying the preceding inequality to the function $T \circ \psi$, we deduce that

$$\int_{\psi(C)} |T'(z)|\,|dz| \le \frac{\pi}{2} \int_{\partial\mathbb{D}} |T'(z)|\,|dz|.$$

Once there is an estimate as above on the constant for Möbius transformations, a nice result due to Treil and Volberg [TV] gives an estimate for holomorphic functions increasing the constant by a factor of 8. One obtains

$$\int_{\psi(C)} |f(z)|\,|dz| \le \frac{8\pi}{2} \int_{\partial\mathbb{D}} |f(z)|\,|dz|$$

for all functions f holomorphic on the closure of $\mathbb{D}$. The result follows. □

In the convex case (with C a segment), the result of [FG] and the foregoing proof together yield the constant 8. This is not the best constant: Beurling [B, p. 457] proved that the best constant in this case is 1 (and it is sharp). It would be nice to obtain the sharp bound in Theorem HH, which is a generalization of the classical Féjer-Riesz theorem.

We shall now consider the following problem:

For which rectifiable Jordan curves Γ *and* γ (*with* γ *in the interior of* Γ) *does there exist a constant* M *such that, for all conformal mappings* f *from* Ω *onto* $\mathbb{D}$,

$$\int_\gamma |f'(z)|\,|dz| \le M \int_\Gamma |f'(z)| V_\gamma(z)\,|dz|\,? \tag{P}$$

It is important to remark that this problem is not Möbius invariant (in particular, circles and lines are not the same for this problem). We now proceed to answer (P) depending on the geometries first of γ and then of Γ; we will do so in three different subsections.

6.1: γ a Straight Line. We shall show with an example that in this situation, unlike the case of the Hayman–Wu theorem (where there is no total angle), there is no universal bound for the problem (P). That is, there does not exist a finite constant M not depending on the curve Γ. In the rest of the section we shall denote γ by L.

The idea is to construct an increasing family of domains containing L so that, in the limit domain Ω, the total angle $V_L(w)$ is zero for almost all $w \in \partial\Omega$. Toward this end, take R_n to be the ray emerging from 0 with direction $2\pi/n$, and take Γ_n to be $\Gamma_n = R_n \cup R_{-n}$. Take L to be the segment $(-\infty, 0)$, and let f_n be a conformal map from $\Omega_n = \{ z \in \mathbb{C} : |\arg z| > 2\pi/n \}$ onto the unit disc $\mathbb{D}$ with $f_n(L) = (-1, 1)$. Then

$$\text{length}(f_n(L)) = \int_L |f_n'(z)|\,|dz| = 2 \quad \text{for all } n.$$

However, since $f_n' \in L^1(\Gamma_n)$ and $V_{(-\infty,0)}(w) \to 0$ as $n \to \infty$ for all $w \in \Gamma_n$,

$$\int_\Gamma |f_n'(z)| V_{(-\infty,0)}(z)\,|dz| \to 0 \quad \text{for all } n$$

and hence there is no finite universal constant M satisfying (P).

6.2: Conformal Mapping and Unpokeable Domains. As we have just observed in 6.1, there is no finite constant M for (P) in general (even for γ being a segment!). The reason for this lies in the geometry of the curve Γ. In order to have a finite constant M in (P), we need to place conditions on the geometry of Γ to guarantee that V_L cannot be too small for many points $w \in \Gamma$.

In this section we describe a class of domains for which we shall be able to obtain a positive result. We shall say these domains are "unpokeable." The following notation will be convenient to describe the domains of interest. For fixed $R > 1$ and $0 < \eta < \pi$ and for any $z \in \Omega$ and $e^{\imath\theta} \in \partial\mathbb{D}$, we define the *sector*

$$S(z, e^{\imath\theta}, R, \eta) = \{ w \in \mathbb{C} : \delta_\Omega(z) \le |w - z| < R\delta_\Omega(z), |\arg(w - z)e^{-\imath\theta}| < \eta/2 \},$$

where $\delta_\Omega(z)$ denotes the Euclidean distance from the point z to the boundary of the domain Ω. We define a *keyhole* to be the region

$$K(z, e^{\imath\theta}, R, \eta) = S(z, e^{\imath\theta}, R, \eta) \cup \mathbb{D}(z, \delta_\Omega(z)).$$

We say that the domain Ω *is unpokeable* if there exists $R > 1$ such that, for every $z \in \Omega$ and every $\xi \in \partial\Omega$ with $|z - \xi| = \delta_\Omega(z)$, we have

$$\{\, w \in \partial\Omega : \delta_\Omega(z) \le |z - w| \le R\delta_\Omega(z) \,\} \setminus S(z, e^{\imath\theta}, R, 1/R) \neq \emptyset,$$

where $\xi = z + e^{\imath\theta}\delta_\Omega(z)$. This condition simply says that there is a point of $\partial\Omega$ in the annulus centered at z of radii $\delta_\Omega(z)$ and $R\delta_\Omega(z)$ which is away from the ray emerging from z in the direction of ξ.

We characterize the unpokeable domains in terms of harmonic measure. We need the next lemma (which follows immediately from Beurling's projection theorem).

LEMMA 5. *If $z \in \Omega$ and $w \in \partial\Omega$ (say $w = z + te^{\imath\alpha}$, $|w - z| \le R\delta_\Omega(z)$), then*

$$\omega(z, \partial\Omega \cap S(z, e^{\imath\alpha}, 2R, \eta), K(z, e^{\imath\alpha}, 2R, \eta) \cap \Omega) \ge \Psi(R, \eta),$$

where Ψ is a certain universal continuous positive function defined for $R > 1$ and $0 < \eta < \pi$.

Now we will give the characterization in terms of harmonic measure.

LEMMA 6. *With the notation just described, the following two conditions are equivalent.*

(a) *Ω is unpokeable.*

(b) *There exists $R > 1$, an angle $\eta(R)$ $(0 < \eta < \pi)$, and a constant $c_0 = c_0(\eta, R)$ such that, for every $z \in \Omega$, there exists a direction $e^{\imath\alpha}$ such that*

$$\omega(z, S(z, e^{\imath\alpha}, R, \eta) \cap \partial\Omega, \Omega) \ge c_0$$

and there is a closest boundary point outside the sector $S(z, e^{\imath\alpha}, R, \eta)$. That is, there exists $\xi \in \partial\Omega$ with $\xi = z + e^{\imath\theta}$, where $|\theta - \alpha| > \eta/2$.

Observe that $c_0 = c_0(R, \eta)$, but it does not depend on the point z.

Proof.

(a) $\Rightarrow$ (b) Consider $R > 1$ given by the definition of unpokeable. For each $z \in \Omega$, consider a point $\xi = z + e^{\imath\theta}\delta_\Omega(z)$. By definition, there exist (i) a boundary point $w \in \partial\Omega$ in the annulus of radii $\delta_\Omega(z)$ and (ii) $R\delta_\Omega(z)$ which is not in the sector $S(z, e^{\imath\theta}, R, 1/R)$. Take the direction $e^{\imath\alpha}$ so that $\arg w = \alpha$. Then, using Lemma 5 together with the fact that $K \cap \Omega \subset \Omega$, we get (b) for $\eta = 1/R$.

(b) $\Rightarrow$ (a) This is immediate. For this let $\tilde{R} = \max(1/\eta, R)$. If

$$\omega(z, S(z, e^{\imath\alpha}, R, \eta) \cap \partial\Omega, \Omega) \ge c_0,$$

then it is straightforward to see that

$$\{\, w \in \partial\Omega : \delta_\Omega(z) \le |z - w| \le \tilde{R}\delta_\Omega(z)\} \setminus S(z, e^{\imath\theta}, \tilde{R}, 1/\tilde{R}) \neq \emptyset,$$

where ξ is a closest boundary point that can be written as $\xi = z + e^{\imath\theta}\delta_\Omega(z)$. Thus (a) holds for $\tilde{R}$. $\square$

If a simply connected domain is linearly connected [P, p. 103] or linearly locally connected [Ge], or if its complement satisfies the so-called corkscrew condition [JK, p. 93], then the domain is unpokeable. But there are unpokeable domains that are neither linearly (locally) connected nor whose complement satisfies the corkscrew condition, as the next example shows. Take directions θ_n in the unit disc to be $\theta_n = \pi/n$, and around each direction θ_n take a region

$$R_n = \left\{ z \in \mathbb{D} : |z| > \frac{1}{2},\ |\arg z - \theta_n| < \frac{\theta_n - \theta_{n+1}}{4} \right\};$$

now take the domain Ω to be $\Omega = \mathbb{D} \setminus \overline{\bigcup_{n\geq 1} R_n}$. That is, we are removing from the unit disc a countable collection of sectors, keeping the height fixed while shrinking the base. This domain Ω is unpokeable, but it is neither linearly connected nor locally linearly connected (observe that the two conditions needed for a set to have this property fail to be true), and its complement $\mathbb{C} \setminus \bar{\Omega}$ is not corkscrew.

6.3: Theorem. Now we state the theorem that gives a partial answer to (P).

THEOREM 4. *Let Ω be a Jordan unpokeable domain, $\Gamma = \partial\Omega$ rectifiable, and let γ be a rectifiable Jordan curve contained in Ω. Then, for any conformal map f from Ω onto the unit disc $\mathbb{D}$,*

$$\operatorname{length}(f(\gamma)) = \int_\gamma |f'(z)|\,|dz| \leq M \int_\Gamma |f'(z)| V_\gamma(z)\,|dz|, \tag{18}$$

where M depends only on the constant R.

It is important to recall that the Hayman–Wu theorem (see (17)) is not true for general Ω and rectifiable γ.

Proof. Recall the notation $Q_z(w)$ given in Section 1.7. Just as in the proof of Theorem 1, we need only show that

$$|f'(z)| \leq \frac{M}{2} \int_\Gamma |f'(w)| Q_z(w)\,|dw|. \tag{19}$$

Although the direction $\mathbf{v}$ does not appear explicitly, it plays an important role in (19) since $Q_z(w)$ depends on $\mathbf{v}$.

Without loss of generality, let the point z be $z = 0$, the Euclidean distance from 0 to $\partial\Omega$ be 1, and the point $\xi \in \partial\Omega$ that is closest to 0 be $\xi = 1$.

Since the domain Ω is unpokeable, by Lemma 6 there exist η $(0 < \eta < \pi)$ and $R > 1$ and also a direction $e^{\imath\alpha}$ $(|\alpha| > \eta/2)$ and a constant $c_0 > 0$ such that $\omega(z, \Gamma \cap S(0, e^{\imath\alpha}, R, \eta), \Omega) \geq c_0$.

We introduce the following notation:

$$S(\alpha) = S(0, e^{\imath\alpha}, R, \eta),$$
$$\Gamma^\alpha = \Gamma \cap S(\alpha), \qquad \Gamma^0 = \Gamma \cap S(0),$$
$$\Gamma^{\alpha,0} = \text{either } \Gamma^\alpha \text{ or } \Gamma^0$$

(i.e., if $\Gamma^{\alpha,0}$ appears in an inequality then it holds for either one of them).

We shall find it convenient to express some of the estimates in terms of the Poincaré geometry of Ω. Recall that the density λ_Ω of the Poincaré metric of Ω is given by

$$\lambda_\Omega(z) = \lambda_{\mathbb{D}}(gz)|g'(z)|,$$

where g is a conformal map from Ω onto $\mathbb{D}$. Recall also that $\lambda_{\mathbb{D}}(w) = 2/(1-|w|^2)$. The *Poincaré distance* between points p, q in Ω is denoted by $d_\Omega(p, q)$.

We shall divide the rest of the proof into two steps: first, we transform the problem into a conformally invariant one by estimating $Q_0(w)$; second, we rewrite the problem in terms of conformally invariant quantities.

Step I. We reduce the problem to show $|f'(0)| \leq C_1 \int_{\Gamma^{\alpha,0}} |f'(w)|\,|dw|$. Observe that

$$\int_\Gamma |f'(w)| Q_0(w)\,|dw| \geq \int_{\Gamma^\alpha \cup \Gamma^0} |f'(w)| Q_0(w)\,|dw|.$$

It therefore suffices to bound $Q_0(w)$ from below. The control on $Q_0(w)$ depends on the relative position of $\mathbf{v}$ and the segment $[0, 1]$.

We must consider two cases for $\mathbf{v}$ as follows.

(i) $|\mathbf{v} - \alpha| \leq \eta/4$. In this case the direction $\mathbf{v}$ can be close to $\arg w$ for $w \in \Gamma^\alpha$ and so $Q_0(w)$ can be arbitrarily small on Γ^α. Yet the points $w \in \Gamma^0$ have arguments far from the direction $\mathbf{v}$ and thus it is easily seen that

$$\begin{aligned}\int_{\Gamma^\alpha \cup \Gamma^0} |f'(w)| Q_0(w)\,|dw| &\geq \int_{\Gamma^0} |f'(w)| Q_0(w)\,|dw| \\ &\geq c(\eta, R) \int_{\Gamma^0} |f'(w)|\,|dw|.\end{aligned}$$

(ii) $|\mathbf{v} - \alpha| > \eta/4$. Just as in (i), the points $w \in \Gamma^\alpha$ have arguments far from the direction $\mathbf{v}$ and so

$$\begin{aligned}\int_{\Gamma^\alpha \cup \Gamma^0} |f'(w)| Q_0(w)\,|dw| &\geq \int_{\Gamma^\alpha} |f'(w)| Q_0(w)\,|dw| \\ &\geq c(\eta, R) \int_{\Gamma^\alpha} |f'(w)|\,|dw|.\end{aligned}$$

Thus,

$$\int_\Gamma |f'(w)| Q_0(w)\,|dw| \geq C \int_{\Gamma^{\alpha,0}} |f'(w)|\,|dw|$$

and it is enough to show that

$$|f'(0)| \leq C_1 \int_{\Gamma^{\alpha,0}} |f'(w)|\,|dw|. \tag{20}$$

Step II. We shall rewrite (20) in terms of conformally invariant quantities. Observe that the right-hand side of (20) can be written as

$$\int_{\Gamma^{\alpha,0}} |f'(w)|\,|dw| = \text{length}(f(\Gamma^{\alpha,0}))$$
$$= 2\pi\omega(0, f(\Gamma^{\alpha,0}), \mathbb{D}) = 2\pi\omega(f^{-1}(0), \Gamma^{\alpha,0}, \Omega).$$

When no confusion is possible we will write $\omega(z, A, \Omega)$ as $\omega(z, A)$. Let us denote $f^{-1}(0) = a$ for the rest of the proof.

Observe now that the left-hand side of (20) is

$$|f'(0)| = \frac{\lambda_{\mathbb{D}}(f(0))}{\lambda_\Omega(0)}.$$

In order to finish the proof, we want to relate $\lambda_{\mathbb{D}}(f(0))/\lambda_\Omega(0)$ and $\omega(a, \Gamma^{\alpha,0})$; to do so we must consider two cases depending on the hyperbolic distance between the points a and 0.

Case 1: $d_\Omega(0, a) \leq 1$. On one hand, by Harnack's inequality,

$$\omega(a, \Gamma^{\alpha,0}) \geq \omega(0, \Gamma^{\alpha,0})e^{-d_\Omega(0,a)} \geq c_0 e^{-1}.$$

On the other hand, by the $\frac{1}{4}$-Koebe theorem we have

$$|f'(0)| = \frac{\lambda_{\mathbb{D}}(f(0))}{\lambda_\Omega(0)} \leq 4d(0, \partial\Omega)\lambda_{\mathbb{D}}(f(0))$$
$$= \frac{8}{1 - |f(0)|^2} \leq 8e,$$

where the last inequality follows from the fact that

$$\frac{1 + |f(0)|}{1 - |f(0)|} = e^{d_{\mathbb{D}}(f(0),0)} = e^{d_\Omega(0,a)} \leq e.$$

Hence

$$|f'(0)| \leq C_1 \int_{\Gamma^{\alpha,0}} |f'(w)|\,|dw|$$

and the theorem follows.

Case 2: $d_\Omega(0, a) \geq 1$. Observe first that since $\delta_\Omega(0) = 1$,

$$|f'(0)| \leq C g_\Omega(0, a),$$

where $g_\Omega(z, a)$ is the Green's function of Ω with pole at a. In this case we thus want to relate $g_\Omega(0, a)$ to $\omega(a, \Gamma^{\alpha,0})$. The next elementary lemma will do the job.

LEMMA 7. *Let u be a harmonic and positive function in Ω. Then, for all $q, p \in \Omega$ such that $d_\Omega(p, q) \geq 1$,*

$$u(q) \geq c g_\Omega(p, q) u(p).$$

This lemma follows immediately from Harnack's inequality and harmonic majorant in the annulus $\{\xi \in \Omega : d_\Omega(\xi, p) > 1\}$.

Now take the function u to be $u(z) = \omega(z, \Gamma^{\alpha,0})$. Recall that $u(0) \geq c_0$ and apply the lemma with $p = a$ and $q = 0$ to obtain

$$
\begin{aligned}
|f'(0)| &\le Cg(0,a) \le C\frac{u(a)}{u(0)} \\
&\le Cu(a) \le C\int_{\Gamma^{\alpha,0}} |f'(w)|\,|dw|.
\end{aligned}
$$

This completes the proof of Theorem 4. □

7. An Extension to *n* Dimensions

Gabriel [G3] found an upper bound for the constant in the case $n = 2$. We will follow his ideas to estimate the constant for any $n \ge 3$. Throughout this section, Γ and γ will be n-dimensional hypersurfaces. The notation used here is the same as we have used all through the paper (with the obvious changes).

Let P and Q have the same meaning as in Section 1.7 (but in $\mathbb{R}^{n+1}_+ = \{(x,y) : x \in \mathbb{R}^n,\ y \in \mathbb{R}\}$); that is,

$$
Q = P_n \frac{q}{r^{n+1}} \quad \text{and} \quad P = P_n \frac{p}{r^{n+1}}, \quad \text{where } P_n = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi^{n+1}}}.
$$

Let S_n be the surface area of the unit ball in $\mathbb{R}^n$, that is,

$$
S_n = \frac{2\sqrt{\pi^{n+1}}}{\Gamma(\frac{n+1}{2})}.
$$

The main result of this section is the following theorem.

Theorem 5. *Let Γ and γ be smooth Jordan n-dimensional hypersurfaces, with u positive and subharmonic on the component of $\mathbb{R}^{n+1} \setminus \Gamma$ that contains γ, and let $\lambda > 2$. Then*

$$
\int_\gamma u(z)^\lambda\,|dz| \le K_n(\lambda,\Gamma)\int_\Gamma u(z)^\lambda V_\gamma(z)\,|dz| \tag{21}
$$

where $1/\lambda + 1/\lambda' = 1$, $\beta = \lambda'/\lambda$, $V_\gamma(z)$ is the normalized absolute total angle subtended by γ at z on Γ, and

$$
K_{n,\lambda} := K_n(\lambda,\Gamma) = S_n P_n \sup_z \left(\int_\Gamma \left(\frac{P_z^\Omega(w)}{Q}\right)^\beta P_z^\Omega(w)\,|dw|\right)^{1/\beta}. \tag{22}
$$

In the particular case of Γ being convex, there is an upper bound for $K_{n,\lambda}$:

$$
K_{n,\lambda} \le S_n(P_n)^\lambda \frac{2\sqrt{\pi}}{\Gamma(n/2)}\left[\frac{n-2}{1-\beta}B\left(\frac{n-2}{2},\frac{3-\beta}{2}\right)\right]^{1/\beta}. \tag{23}
$$

Here Γ and B denote the special Gamma and Beta functions, respectively (see e.g. [GR, pp. 942, 957]). Note that, for each fixed $2 < \lambda < \infty$, the right-hand side on (23) is $O(n^{1/2})$ as $n \to \infty$.

Proof. The proof follows the steps of that given in the 1-dimensional case. □

The condition of smoothness in Γ and γ could probably be weakened to one of rectifiability, in the appropriate sense.

8. Some Open Questions

QUESTION 1. Let u be a positive subharmonic function, Γ a convex (Jordan) curve, and $K(\lambda, \Gamma)$ as in Theorem 1. As mentioned in Section 4.1, if γ is a segment perpendicular to the curve Γ then the constant $K(\lambda, \Gamma)$ is finite if and only if $\lambda > 2$.

In Section 4.2 we saw that if γ is a circle then the constant $K(\lambda, \Gamma)$ is finite for any $\lambda \geq 1$. Hence, the best exponent that gives a finite constant is $\lambda = 1$. (As a matter of fact, if we let Γ be the upper half plane then, for $\gamma(t) = (t, |t|^\alpha)$ with $-1 \leq t \leq 1$, and $1 < \alpha \leq 2$, the best exponent is already $\lambda = 1$.) Consider λ as a function of the curve γ, $\lambda = \lambda(\gamma)$. It would be nice to have a better understanding of this limit exponent for a given curve γ contained, say, in the upper half-plane.

QUESTION 2. Let Ω be a Jordan domain with $\Gamma = \partial\Omega$, and consider the following two situations.

(A) Let γ be any rectifiable curve contained in Ω; then there exists a finite constant K such that, for any conformal mapping ψ from Ω onto $\mathbb{D}$,

$$\int_\gamma |\psi'(z)|\,|dz| \leq K(\Gamma) \int_\Gamma |\psi'(z)| V_\gamma(z)\,|dz|. \tag{24}$$

(B) Let γ be any rectifiable curve contained in Ω; then there exists a finite constant $\tilde{K}$ such that, for any holomorphic function f,

$$\int_\gamma |f(z)|\,|dz| \leq \tilde{K}(\Gamma) \int_\Gamma |f(z)| V_\gamma(z)\,|dz|. \tag{25}$$

If Ω is unpokeable, Theorem 4 establishes that (24) holds. Nevertheless, (25) will be false in general, as we now show.

Let $\mathbb{D}(2, 1)$ be the circle centered at 2 of radius 1, and let R be the region

$$R = \{ z \in \mathbb{C} : 2 \leq \Re z < \infty,\ -e^{-\Re z} < \Im z < e^{-\Re z} \}$$

(that is, R is the region between the graphs of $y = -e^{-x}$ and $y = e^{-x}$ for $x \geq 2$). Let Ω be the domain $\Omega = \hat{\mathbb{C}} \setminus (\overline{R \cup \mathbb{D}})$ (note that Ω is unpokeable). Take Γ to be the boundary of Ω. Recall (see the proof of Theorem 1) that finding the constant $\tilde{K}$ in (25) is equivalent to finding a constant C such that

$$|f(z)| \leq \frac{C}{2} \int_\Gamma |f(z)| Q_z(w)\,|dz|$$

for all points $z \in \gamma$ and direction $\mathbf{v}$.

Let $z = 0 \in \gamma$ and $\mathbf{v} = (1, 0)$ (that is, $\mathbf{v}$ is the direction of the real axis). Fix L large. Let h be the function on Ω with value 0 on $(\partial\mathbb{D}(2, 1) \cap \Gamma) \cup (\{ \xi \in \partial R : 2 < \Re\xi < L \} \cap \Gamma)$ and value L on $R_L = \{ \xi \in \partial R : \Re\xi \geq L \} \cap \Gamma$. Take the harmonic function u to be the Poisson integral of h and take the holomorphic function f to have modulus $|f(z)| = \exp u(z)$. Then:

(a) $|f(0)| = e^{L\omega(0, S_L, \Omega)}$;

(b) $\displaystyle\int_\Gamma |f(z)| Q_z(w)\, |dz| \le \left(1 + e^{-L} + \frac{e^{-L}\sqrt{1+e^{-2L}}}{L^2}\right) c.$

Since $\omega(0, S_L, \Omega) \sim 1/\sqrt{L}$ for L large,

$$\frac{|f(0)|}{\int_\Gamma |f(w)| Q\, |dw|} \to \infty \quad \text{as } L \to \infty;$$

the constant C can be as large as we want, so there does not exist a finite constant $\tilde{K}$ in (25).

Even though situations (A) and (B) are different problems, without the total angle V they would be the same property (since the arclength on γ is a Carleson measure). For the particular case of $\Omega = \mathbb{D}$, (25) holds by Carlson (see [C] or Section 2.3). It would be interesting to characterize those domains Ω for which (25) holds (or give a reasonable sufficient condition for (25) to hold).

QUESTION 3. It would be nice to consider the following generalization on Gabriel's problem in n dimensions:

> *Given a rectifiable Jordan n-dimensional hypersurface Γ and a rectifiable Jordan m-dimensional hypersurface γ with $m < n$ and γ contained in the interior of Γ, and given any positive number λ, find the best constant K_n such that, for all positive harmonic functions u,*

$$\int_\gamma u(z)^\lambda\, |dz| \le K_n \int_\Gamma u(z)^\lambda V_\Gamma(z)\, |dz|.$$

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