# Green's Functions for Irregular Quadratic Polynomial Automorphisms of $\mathbb{C}^3$

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## 1. Introduction

We first recall some basic facts about the dynamics of (holomorphic) polynomial automorphisms of $\mathbb{C}^2$. It was shown in [FM] that the ones with interesting dynamics are affinely conjugated to a finite composition of generalized Hénon maps—that is, maps of the form $(x, y) \to (P(x) - ay, x)$, where $P$ is a holomorphic polynomial in $\mathbb{C}$. The dynamics of such maps is studied in detail in a sequence of papers by Bedford and Smillie, starting with [BS], in [FS], and also in [H].

For simplicity, let us refer to the case of one generalized Hénon map, $h(x, y) = (P(x) - ay, x)$, where $P$ has degree $d \geq 2$. There are two posibilities: either the forward iterates $h^n$ of $h$ can escape to infinity at super-exponential rate ($\sim$ (const)$^{d^n}$) or they are locally uniformly bounded. The first situation occurs on an open set $U^+$ and the second on the complement $K^+ = \mathbb{C}^2 \setminus U^+$. Then the Fatou set of $h$, defined in the usual sense as the largest open set on which the iterates $\{h^n\}$ form a normal family, is given by $U^+ \cap \operatorname{int} K^+$, while the Julia set is $\partial K^+$. Similar statements hold for the inverse map $h^{-1}$, the corresponding sets being denoted by $U^-$ and $K^-$. Using these facts, one defines (pluricomplex) Green's functions which measure the (super-exponential) rate of escape to infinity in forward/backward time:

$$G^\pm(w) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ \|h^{\pm n}(w)\|,$$

where $w = (x, y) \in \mathbb{C}^2$. These functions are continuous plurisubharmonic on $\mathbb{C}^2$ and actually pluriharmonic on $U^+$ (resp. on $U^-$). Moreover, $K^\pm = \{G^\pm = 0\}$. The Green's functions are used to define the currents $\mu^\pm = dd^c G^\pm$, supported on $\partial K^\pm$, which satisfy $h^\star \mu^\pm = d^{\pm 1}\mu^\pm$ $\big($here $d^c = \frac{1}{2\pi i}(\partial - \bar\partial)\big)$. It follows that $\mu = \mu^+ \wedge \mu^-$ is a probability measure supported on $\partial K^+ \cap \partial K^-$, which is invariant under $h$. The currents $\mu^\pm$ and the invariant measure $\mu$ are important tools in understanding the dynamics of $h$.

It is an interesting problem to study the dynamics of polynomial automorphisms of $\mathbb{C}^N$ in dimensions higher than 2. To our knowledge, there are only a few attempts in this direction, which we briefly recall now. The theory of Hénon maps in $\mathbb{C}^2$ carries over to the special class of shift-like polynomial automorphisms of $\mathbb{C}^N$, which are introduced and studied in [BP].

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A polynomial automorphism $h$ of $\mathbb{C}^N$ and its inverse $h^{-1}$ can be regarded as meromorphic maps of $\mathbb{P}^N$, which are well-defined away from the indeterminacy set $I^+$ (resp. $I^-$). This approach is used by Sibony in [S]. He calls $h$ a regular automorphism if $I^+ \cap I^- = \emptyset$. Note that this is always the case for Hénon maps in $\mathbb{C}^2$. For such automorphisms he defines the Green's functions $G^\pm$ and the currents $\mu^\pm$ as in the 2-dimensional case. The closed sets $K^\pm = \{G^\pm = 0\}$ are still the sets of points with bounded forward (resp. backward) orbit; on the complements $U^\pm$ the iterates escape to infinity at super-exponential rate. Moreover, Sibony shows that there exists a positive integer $l$ such that the measure $(\mu^+)^l \wedge (\mu^-)^{N-l}$ is an invariant probability measure with interesting dynamical properties. Hence the dynamics of regular polynomial automorphisms is similar in many aspects to that of Hénon maps of $\mathbb{C}^2$. For the study of dynamics of birational maps of $\mathbb{P}^2$ we refer to [D].

In this paper we consider the dynamics of polynomial automorphisms of $\mathbb{C}^3$, which are not regular in the sense of Sibony. The polynomial automorphisms of degree 2 have been classified up to affine conjugacy into seven classes by Fornæss and Wu [FW]. For this reason, we restrict our attention to the quadratic case. However, when dealing with the inverse maps, we must consider quite often polynomial automorphisms of higher degree.

Of these seven classes, two consist of affine automorphisms and elementary automorphisms, which have simple dynamics. (See [FW], and also [FM], since they are in direct analogy with the 2-dimensional case.) It turns out that the remaining five classes have rich dynamics, exhibiting new interesting phenomena. These new dynamical behaviors are different in many aspects from the 2-dimensional case and from the higher-dimensional cases studied in [BP] and [S].

In Sections 2 through 6, we consider the five classes of polynomial automorphisms just mentioned. We use the same order and notations $H_1$ through $H_5$ as in [FW]. The main goal is to introduce the Green's functions $G^\pm$, to understand their properties, and to identify the new dynamical phenomena that occur in dimension 3. Our approach is as follows: We find first the open set $U^+$ where the forward iterates $H^n$ of the map $H$ under consideration escape to infinity at the super-exponential rate $(\text{const})^{2^n}$, which, of course, is the highest possible rate. We then prove that, on the complement $K^+ = \mathbb{C}^3 \setminus U^+$, the iterates either are bounded or they escape to infinity at a much slower rate (e.g., in some cases exponential $\sim (\text{const})^n$). The existence of this slower order of growth of the iterates is one of the new phenomena alluded to previously. Using these facts, we can follow the technique in [BS] to define the Green's function $G^+$. A similar study is done in each case for the inverse map $H^{-1}$, leading to $U^-$, $K^-$, and $G^-$. In Section 7 we summarize the main conclusions regarding new dynamical behaviors that occur in this setting.

The maps of the classes $H_1$ and $H_2$ (discussed in Sections 2 and 3) are semi-direct products, and the results we obtain in these cases are fairly complete. For these maps, the Green's functions $G^\pm$ are pluriharmonic on $U^\pm$, and exactly one of the sets $K^\pm$ (say, $K^+$) consists of both points with bounded orbit and points with orbit escaping to infinity at exponential rate. The other set, $K^-$, consists only of points with bounded (backward) orbit. Thus we obtain an invariant measure by taking the wedge product of $\mu^+ \wedge \mu^-$ with a third invariant $(1, 1)$ current arising from the set of points with bounded forward orbit inside $K^+$.

For the other classes it also happens in general that $G^+$ and $G^-$ are both pluriharmonic on $U^+$ and $U^-$, respectively. This is not the case in the situations considered in [BP] and [S], and it makes it impossible to construct invariant measures using only the currents $\mu^\pm$ (as it is done in [BP] and [S]). Indeed, we have $\mu^+ \wedge \mu^+ = \mu^- \wedge \mu^- = 0$. Moreover, sometimes it also occurs under these circumstances that $H^\star\mu^+ = 2\mu^+$, but $H^\star\mu^- = \frac{1}{3}\mu^-$ since the degree of $H^{-n}$ is $3^n$.

Throughout the paper we will use the following notation: $w = (x, y, z)$ for a point of $\mathbb{C}^3$, and $w_n = (x_n, y_n, z_n) = H^n(x, y, z)$, $n \geq 1$, where $H^n$ denotes the $n$th iterate of a self-map $H$ of $\mathbb{C}^3$. We also let $\| \cdot \|_+ = \max\{\| \cdot \|, 1\}$, where $\| \cdot \|$ denotes the Euclidean norm. Whenever we refer to $\mathbb{P}^3$ as a compactification of $\mathbb{C}^3$, we denote by $[x : y : z : t]$ the homogeneous coordinates on $\mathbb{P}^3$ and we identify $\mathbb{C}^3 = \{[x : y : z : 1]\} \subset \mathbb{P}^3$. Moreover, when referring to polynomials, we will sometimes use the abbreviation l.d.t. with the meaning lower-degree terms.

Let us also note here that, in some of these cases, the estimates we obtain for the order of growth of the orbits inside $K^\pm$ may not be sharp. They are, however, good enough for the purpose of introducing the Green's functions. In our forthcoming papers, we will deal with the construction of invariant measures as well as with the study of the dynamics of the maps on $K^+$ and $K^-$ (e.g., with questions like finding the precise rates of growth of the iterates or finding the points or sets at infinity where orbits cluster).

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## 2. The Class $H_1$

The maps of this class are semi-direct products of the form

$$H_1(x, y, z) = (P(x, z) + ay, Q(z) + x, cz + d),$$

where $\max\{\deg(P), \deg(Q)\} = 2$ and $ac \neq 0$. As noticed in [FW], these maps are dynamically interesting if $c \neq 1$ and $P$ has degree 2 in $x$. Then for any fixed $z$, the map is essentially a Hénon map of $\mathbb{C}^2$.

Since $c \neq 1$, we may assume by an affine change of coordinates in $z$ that $d = 0$, so $H_1$ has the form

$$H_1(x, y, z) = (\alpha x^2 + p_1(z)x + p_2(z) + ay, Q(z) + x, cz), \tag{2.1}$$

where $\deg(p_1) \leq 1$, $\deg(p_2) \leq 2$, $\deg(Q) \leq 2$, and $\alpha ac \neq 0$. The inverse map $H_1^{-1}$ has the form

$$H_1^{-1}(x, y, z) = \left(y - Q\left(\frac{z}{c}\right), \tilde{\alpha}y^2 + \tilde{p}_1(z)y + \tilde{p}_2(z) + \frac{x}{a}, \frac{z}{c}\right),$$

where $\tilde{\alpha} \neq 0$, $\deg(\tilde{p}_1) \leq 2$, and $\deg(\tilde{p}_2) \leq 4$.

Hence, in order to understand the dynamics of $H_1$ and $H_1^{-1}$, it is enough to consider maps $H$ of the form

$$\begin{aligned} x_1 &= \alpha x^2 + p_1(z)x + p_2(z) + ay, \\ y_1 &= Q(z) + x, \\ z_1 &= cz, \end{aligned} \tag{2.2}$$

where $\alpha ac \neq 0$, $\deg(Q) \leq 2$, and where $\deg(p_1) = k$ and $\deg(p_2) = l$ are arbitrary. Indeed, the map $H_1^{-1}$ is conjugated to a map $H$ of form (2.2) by the transformation $(x, y, z) \to (y, x, z)$.

It is easy to see that the degree of $H^n$ is $d2^{n-1}$, where $d = \deg(H) = \max\{2, k+1, l\}$. We will need the following lemma.

LEMMA 2.1. *There exists a constant $C > 1$ depending only on the coefficients of $H$ such that, for all $n \geq 1$ and $w \in \mathbb{C}^3$, we have*

$$\|(x_n, y_n)\| \leq \left(C|c|_+^d |z|_+^d \|(x,y)\|_+\right)^{2^n}, \qquad \|w_n\| \leq \left(C|c|_+^d |z|_+^d \|(x,y)\|_+\right)^{2^n}\sqrt{2}.$$

*Proof.* We note that

$$|x_1| \leq (|\alpha| + |p_1(z)| + |p_2(z)| + |a|)\|(x,y)\|_+^2, \qquad |y_1| \leq (|Q(z)|+1)\|(x,y)\|_+^2,$$

hence $\max\{|x_1|, |y_1|\} \leq C'|z|_+^d \|(x,y)\|_+^2$ for some constant $C' > 1$ depending only on the coefficients of $H$. If we let $C = C'\sqrt{2}$ and $C(z) = C|z|_+^d$, it follows that $\|(x_1, y_1)\| \leq C(z)\|(x,y)\|_+^2$ and so

$$\begin{aligned} \|(x_n, y_n)\| &\leq C(c^{n-1}z)(C(c^{n-2}z))^2 \dots (C(z))^{2^{n-1}} \|(x,y)\|_+^{2^n} \\ &= \|(x,y)\|_+^{2^n} \prod_{j=1}^{n} (C(c^{n-j}z))^{2^{j-1}}. \end{aligned}$$

As $C(c^j z) \leq C|c|_+^{jd}|z|_+^d$, we see that

$$\prod_{j=1}^{n} (C(c^{n-j}z))^{2^{j-1}} \leq (C|z|_+^d)^{2^n - 1}(|c|_+^d)^{A_n},$$

where

$$A_n = \sum_{j=1}^{n} (n-j)2^{j-1} \leq 2^{n-2} \sum_{j=1}^{\infty} \frac{j}{2^{j-1}} = 2^n.$$

This yields the first estimate. The second one follows easily from this, as $|z_n| = |c^n z|$ clearly satisfies the same estimate as $\|(x_n, y_n)\|$. □

We now have to consider the three cases $|c| > 1$, $|c| = 1$, and $|c| < 1$ separately because the dynamics of $H$ is different in each case.

*Case 1:* $|c| > 1$

We fix a number $\delta \in (0,1)$ and, for $R > 0$, we define the set

$$V^- = V^-(R) = \{\, w \in \mathbb{C}^3 : |x| > \max\{R, |y|, |z|^{d+1}\}\,\}. \tag{2.3}$$

LEMMA 2.2. *For any number* $\delta \in (0,1)$ *there exists* $R_0 = R_0(\delta) > 0$ *such that, if* $R \geq R_0$*, then* $H(V^-) \subseteq V^-$ *and the following estimates hold for* $w \in V^-$:

$$|x_1 - \alpha x^2| < \delta|\alpha||x|^2, \qquad |y_1 - x| < \delta|x|,$$

$$|\alpha|(1-\delta)|x|^2 < |x_1| < |\alpha|(1+\delta)|x|^2, \qquad (1-\delta)|x| < |y_1| < (1+\delta)|x|,$$

$$[|\alpha|(1-\delta)]^{2^n-1}|x|^{2^n} < |x_n| < [|\alpha|(1+\delta)]^{2^n-1}|x|^{2^n}.$$

*Proof*. We deal at first with the estimates. Let $C$ be a constant depending only on the coefficients such that $\max\{|p_1(z)|, |p_2(z)|, |Q(z)|\} \leq C|z|_+^d$. For $w \in V^-$ we get, by using (2.2) and (2.3),

$$\begin{aligned}|x_1 - \alpha x^2| &\leq C|z|_+^d|x| + C|z|_+^d + |a||y| \\ &< C|x||x|^{d/(d+1)} + C|x|^{d/(d+1)} + |a||x| < \delta|\alpha||x|^2, \\ |y_1 - x| &\leq C|z|_+^d < C|x|^{d/(d+1)} < \delta|x|.\end{aligned}$$

The last inequality of each sequence holds provided that $|x| \geq R_0(\delta)$, with $R_0(\delta)$ sufficiently large. The third and fourth estimates of the lemma are immediate consequences of the first two. The fifth estimate follows by repeated use of the third one, once we have proved the invariance property of $V^-$.

Let now $w \in V^-$. Using the estimates already established, if $R_0(\delta)$ is sufficiently large then each of the following inequalities hold:

$$|x_1| > |\alpha|(1-\delta)|x|^2 > |\alpha|(1-\delta)R^2 > R,$$

$$|x_1| > |\alpha|(1-\delta)|x|^2 > (1+\delta)|x| > |y_1|,$$

$$|z_1|^{d+1} = |c|^{d+1}|z|^{d+1} < |c|^{d+1}|x| < |\alpha|(1-\delta)|x|^2 < |x_1|.$$

These show $H(w) \in V^-$. □

We now define

$$U^+ = \bigcup_{n=0}^{\infty} H^{-n}(V^-), \qquad K^+ = \mathbb{C}^3 \setminus U^+. \tag{2.4}$$

By Lemma 2.2, note that $U^+$ is an increasing union of open sets; in particular, $U^+$ is open and hence $K^+$ is closed. From the estimates of Lemma 2.2 it follows that, on $U^+$, the iterates of $H$ escape to infinity at super-exponential rate: if $w \in U^+$ then

$$|x_n| > \frac{[|\alpha|(1-\delta)R]^{2^n}}{|\alpha|(1-\delta)}, \qquad y_n = x_{n-1} + O(|x_{n-1}|),$$

for all $n$ sufficiently large. We actually see that $y_n/x_n \to 0$, so $H^n(w) \to [1:0:0:0] \in \mathbb{P}^3$ as $n \to \infty$, the convergence being locally uniform on $U^+$.

LEMMA 2.3. *If* $w \in K^+$ *then, for every integer* $n \geq 0$, *we have*

$$\max\{|x_n|, |y_n|\} \leq |c|^{n(d+1)} \max\{R, \tilde{C}, |y|, |z|^{d+1}\},$$

*where* $R$ *is as in the definition* (2.3) *of* $V^-$ *and the constant* $\tilde{C}$ *depends only on* $H$.

*Proof.* If $w \in K^+$ then $(x_n, y_n, z_n) \notin V^-$ for every $n \geq 0$, so

$$|x_n| \leq M_n = \max\{R, |y_n|, |z_n|^{d+1}\}.$$

By the definition of $M_n$ and by (2.2) we have that

$$|z_n|^{d+1} = |c|^{d+1}|z_{n-1}|^{d+1} \leq |c|^{d+1}M_{n-1},$$

$$|y_n| \leq |Q(z_{n-1})| + |x_{n-1}| \leq C|z_{n-1}|_+^d + M_{n-1} \leq CM_{n-1}^{d/(d+1)} + M_{n-1},$$

where $C$ is a constant depending on $H$ as in the proof of Lemma 2.2. It follows that

$$\begin{aligned} M_n &\leq \max\{R, CM_{n-1}^{d/(d+1)} + M_{n-1}, |c|^{d+1}M_{n-1}\} \\ &\leq \max\{R, \tilde{C}, |c|^{d+1}M_{n-1}\}, \end{aligned} \tag{2.5}$$

where $\tilde{C} = [|c|C/(|c|^{d+1} - 1)]^{d+1}$. Indeed, if $CM_{n-1}^{d/(d+1)} + M_{n-1} \geq |c|^{d+1}M_{n-1}$, then $M_{n-1} \leq [C/(|c|^{d+1} - 1)]^{d+1}$ and so $CM_{n-1}^{d/(d+1)} + M_{n-1} \leq \tilde{C}$. The lemma now follows by repeated use of (2.5). □

At this point let us observe that the complex hyperplane $z = 0$ is invariant under $H$, the restriction of $H$ to this hyperplane is a Hénon map of $\mathbb{C}^2$ given by

$$h(x, y) = (\alpha x^2 + p_1(0)x + p_2(0) + ay, Q(0) + x).$$

If we denote by $K_h^+ \subset \mathbb{C}^2$ the set of points with bounded forward orbit, then by [BS] this set is unbounded (hence nonempty), and $K_h^+ \times \{0\} \subset K^+$ consists of points $w \in \mathbb{C}^3$ with bounded forward orbit under $H$. Moreover, if $w \in K^+ \cap \{z = 0\}$, then the $H$-orbit of $w$ is bounded. Indeed, by Lemma 2.3, this orbit could escape to infinity at most at exponential rate, so it must be bounded in view of the results of [BS]. We will see later (Corollary 2.6) that $K^+$ contains points outside the plane $z = 0$ as well. Since $z_n = c^n z$, the iterates of any such point escape to infinity at exponential rate. We summarize our results in the following theorem.

THEOREM 2.4. *For a map $H$ of form* (2.2), *let $U^+$ and $K^+$ be defined by* (2.4). *The orbits of a point $w \in \mathbb{C}^3$ can either escape to infinity at super-exponential rate or grow at most exponentially. The first situation occurs precisely on the open set $U^+$, where the iterates of $H$ converge locally uniformly to* $[1:0:0:0] \in \mathbb{P}^3$. *The iterates of a point $w \in K^+$ are bounded if and only if $w \in K^+ \cap \{z = 0\}$. If $w \in K^+ \setminus \{z = 0\}$ then the iterates escape to infinity at exponential rate.*

We now proceed with the construction of the Green's function, following the methods of [BS]. Recall that $\deg(H^n) = d2^{n-1}$, where $d = \deg(H) \geq 2$. For $n \geq 1$ we let

$$G_n(w) = \frac{1}{d2^{n-1}} \log^+ \|H^n(w)\|, \qquad \tilde{G}_n(w) = \frac{1}{d2^{n-1}} \log^+ |x_n|,$$

and define the Green's function

$$G^+(w) = \lim_{n\to\infty} G_n(w) = \lim_{n\to\infty} \tilde{G}_n(w). \tag{2.6}$$

THEOREM 2.5. *The limits in* (2.6) *exist and are equal, the convergence being locally uniform on* $\mathbb{C}^3$ *in both cases. The limit function* $G^+$ *is nonnegative, continuous and plurisubharmonic on* $\mathbb{C}^3$; $G^+$ *is pluriharmonic on* $U^+$; *and* $K^+ = \{G^+ = 0\}$. *Moreover,* $G^+ \circ H = 2G^+$.

*Proof.* By the estimates of Lemma 2.3 and since $z_n = c^n z$, the two limits exist and are both zero on $K^+$. By Theorem 2.4 we have $y_n/x_n \to 0$ and $z_n/x_n \to 0$ for $w \in U^+$, so $\tilde{G}_n(w) \le G_n(w) \le \tilde{G}(w) + \log 3/(d2^{n-1})$, provided that $n$ is sufficiently large. Thus, if one of the limits exists at $w \in U^+$, then the other does also and they are equal. By Lemma 2.2 we have, for $w \in V^-$, that $|\alpha|(1-\delta)|x_{n-1}|^2 < |x_n| < |\alpha|(1+\delta)|x_{n-1}|^2$ and so

$$|\tilde{G}_n(w) - \tilde{G}_{n-1}(w)| < \frac{\text{const}}{d2^{n-1}}.$$

This shows that the sequence $\{\tilde{G}_n\}$ is uniformly Cauchy on $V^-$ and hence locally uniformly Cauchy on $U^+$. Thus the two limits in (2.6) exist and are equal on $\mathbb{C}^3$. Moreover, the function $G^+$ is pluriharmonic on $U^+$, since $\tilde{G}_n$ are pluriharmonic and the convergence is locally uniform. By Lemma 2.2 we have $G^+ > 0$ on $U^+$, so $K^+ = \{G^+ = 0\}$. Relation (2.6) implies that

$$G^+ \circ H(w) = \lim_{n\to\infty} G_n(H(w)) = \lim_{n\to\infty} 2G_{n+1}(w) = 2G^+(w).$$

The estimate on $\|w_n\|$ from Lemma 2.1 gives, for all $w$ and $n$,

$$G_n(w) \le M + 2\log^+|z| + \log^+\|(x, y)\| \tag{2.7}$$

for some constant $M$ depending on $H$. So the upper semicontinuous regularization $G^+_\star$ of $G^+$ is plurisubharmonic on $\mathbb{C}^3$ and satisfies (2.7) as well (see [K]). We have $G^+ = G^+_\star$ on $U^+ \cup \operatorname{int} K^+$, where $G^+$ was already continuous. Since $G^+ \circ H = 2G^+$, the same holds for $G^+_\star$. Hence, for every $w \in K^+$ and $n \ge 1$, we get by (2.7) together with the estimates of Lemma 2.3 that $G^+_\star(w) = G^+_\star(H^n(w))/2^n \sim n/2^n$. It follows that $G^+_\star = 0$ on $K^+$, so $G^+ = G^+_\star$ is plurisubharmonic on $\mathbb{C}^3$. Note that $G_n$ and $\tilde{G}_n$ converge locally uniformly to $G^+$ on $U^+ \cup \operatorname{int} K^+$. We now use the upper semicontinuity of $G^+$ together with Hartogs' lemma to see that, for every $\varepsilon > 0$ and every $w \in K^+$, there exists an open ball $B_w$ centered at $w$ and $n_0 = n_0(w, \varepsilon)$ such that $0 \le G_n(w') \le \varepsilon$ and $0 \le \tilde{G}_n(w') \le \varepsilon$ for every $w' \in B_w$ and $n \ge n_0$. This shows that the convergence is locally uniform on $\mathbb{C}^3$, and hence $G^+$ is continuous. □

COROLLARY 2.6. $K^+ \setminus \{z = 0\} \ne \emptyset$.

*Proof.* If $K^+ \subseteq \{z = 0\}$ then—by the removable singularity theorem—the function $G^+$, which is pluriharmonic on $\mathbb{C}^3 \setminus K^+$ and continuous on $\mathbb{C}^3$, would be pluriharmonic on $\mathbb{C}^3$. As $G^+ \ge 0$, this implies that $G^+$ is constant, which is impossible. □

Now we can define the closed positive current of bi-degree $(1, 1)$, $\mu^+ = dd^c G^+$, which satisfies

$$H^\star \mu^+ = dd^c(G^+ \circ H) = 2\mu^+.$$

As in [BS, Lemma 3.6], we have that supp $\mu^+ = \partial K^+$.

*Case 2:* $|c| = 1$

For $Q(z) = a_1 z^2 + a_2 z + a_3$ as in (2.1), we set $|Q|(z) = |a_1| z^2 + |a_2| z + |a_3|$. Note that in this case $|z_1| = |z|$. For $R > 0$ we define

$$V^- = V^-(R) = \{\, w \in \mathbb{C}^3 : |x| > \max\{R, |y| - |Q|(|z|), |z|^{d+1}\}\,\}.$$

It is easy to see that all of the conclusions of Lemma 2.2 hold in this case as well, with the above choice of $V^-$. Next, we let $U^+ = \bigcup_{n=0}^{\infty} H^{-n}(V^-)$ and $K^+ = \mathbb{C}^3 \setminus U^+$. As before, we have that on $U^+$ the iterates of $H$ escape to infinity at super-exponential rate, converging locally uniformly to $[1:0:0:0]$. On the other hand, this time $K^+$ is the set of points with bounded forward orbit as follows.

LEMMA 2.7. *If $w \in K^+$ then, for all $n \geq 0$, we have*

$$\max\{|x_n|, |y_n|\} \leq \max\{R, |y|, |z|^{d+1}\} + |Q|(|z|).$$

*Proof.* If $w \notin K^+$ then, since $|z_n| = |z|$, for all $n \geq 0$ we have

$$|x_n| \leq M_n = \max\{R, |y_n| - |Q|(|z|), |z|^{d+1}\}.$$

Note by (2.2) that

$$|y_n| - |Q|(|z|) \leq |Q|(|z_{n-1}|) + |x_{n-1}| - |Q|(|z|) = |x_{n-1}| \leq M_{n-1},$$

so $M_n \leq \max\{R, M_{n-1}, |z|^{d+1}\}$. This implies

$$M_n \leq \max\{R, |y| - |Q|(|z|), |z|^{d+1}\},$$

which yields the estimate of the lemma. □

The Green's function $G^+$ is then defined as in (2.6), and Theorem 2.5 holds in this case as well. Similarly, the current $\mu^+ = dd^c G^+$ has the same properties as before. We also see from the above remarks that the iterates $\{H^n\}$ are a normal family on $U^+ \cup \operatorname{int} K^+$, which is the Fatou set of $H$. Hence $\partial K^+ = J^+$ is the Julia set of $H$.

*Case 3:* $|c| < 1$

The new feature in this case is that $z_n = c^n z \to 0$ as $n \to \infty$, so the orbit of any point will approach the invariant hyperplane $z = 0$. If $Q(z) = a_1 z^2 + a_2 z + a_3$ then for $R > 0$ we let

$$V^- = V^-(R) = \{\, w \in \mathbb{C}^3 : |x| > \max\{R, |y| - |a_3|\},\ |z| < 1\,\}.$$

Then Lemma 2.2 holds for this set $V^-$, and the set $U^+$, defined as in (2.4), has the same properties as before. The complement $K^+$ of $U^+$ is the set of points with bounded forward orbits, as the next lemma shows.

LEMMA 2.8. *Let* $w \in K^+$ *and choose* $n_0(w)$ *such that* $|z_{n_0}| < 1$. *Then, for all* $n \geq n_0$,

$$\max\{|x_n|, |y_n|\} \leq \max\{R, |y_{n_0}|\} + |a_3| + \frac{C|z|}{1-|c|},$$

*where* $C = |a_1| + |a_2|$.

*Proof.* With the lemma's choice of $n_0$, since $w \in K^+$ we have that $|x_n| \leq M_n = \max\{R, |y_n| - |a_3|\}$ for all $n \geq n_0$. As $|Q|(|z|) - |a_3| < C|z|$ holds for $|z| < 1$, we have

$$|y_n| - |a_3| \leq |x_{n-1}| + C|z_{n-1}| \leq M_{n-1} + C|z_{n-1}|.$$

Hence $M_n \leq \max\{R, M_{n-1}\} + C|z_{n-1}|$, which yields the desired estimate. □

The construction of $G^+$ and $\mu^+$ is accomplished as before, and Theorem 2.5 holds in this case as well.

INVARIANT MEASURES. We now consider the problem of constructing invariant measures for the maps $H_1$ of form (2.1).

If $|c| \neq 1$, and say without loss of generality $|c| > 1$, then the inverse map $H_1^{-1}$ has form (2.2) with $z_1 = z/c$. Then $K^- = K^+(H_1^{-1})$ is the set of points with bounded backward orbit. Let $G^-$ be the Green's function for $H_1^{-1}$, constructed as in Case 3, and let $\mu^- = dd^c G^-$. We have $(H_1^{-1})^\star \mu^- = 2\mu^-$, so $H_1^\star \mu^- = \frac{1}{2}\mu^-$. By Theorem 2.4, the set of points with bounded forward orbit is an invariant subset of $K^+$, namely $K^+ \cap \{z = 0\}$. Thus $K^+ \cap K^- \cap \{z = 0\}$ is the set of points with bounded full orbit (i.e., forward and backward). Since $H_1^\star(dd^c \log|z|) = dd^c \log|z_1| = dd^c \log|z|$, an invariant measure for $H_1$ is given by

$$\mu = \mu^+ \wedge \mu^- \wedge dd^c \log|z|. \tag{2.8}$$

It is useful to give the following alternative description of $\mu$. If $h$ denotes the restriction of $H_1$ to $z = 0$, then $h$ is a Hénon map of $\mathbb{C}^2$ and $h^{-1} = H_1^{-1}|_{z=0}$. Let $\nu$ be the invariant probability measure for $h$ constructed in [BS]. Note that the Green's functions $g^\pm$ of $h$ are just the restrictions of $G^\pm$ to $z = 0$, and $\nu = dd^c g^+ \wedge dd^c g^-$. Then the measure $\mu$ in (2.8) satisfies

$$\int_{\mathbb{C}^3} \phi \, d\mu = \int_{\mathbb{C}^2} \phi(x, y, 0)\, d\nu \tag{2.9}$$

for any $\phi \in C_0(\mathbb{C}^3)$. This shows that $\mu$ is a probability measure supported on the compact set $\partial K^+ \cap \partial K^- \cap \{z = 0\}$. Relation (2.9) follows easily by considering two sequences $\{u_j\}$ and $\{v_j\}$ of smooth plurisubharmonic functions such that $u_j \searrow G^+$ and $v_j \searrow G^-$. Then the measures $\mu_j = dd^c u_j \wedge dd^c v_j \wedge dd^c \log|z|$ converge weakly to $\mu$, by a result of Bedford and Taylor (see [K]). On the other hand, if $u_j^0(x, y) = u_j(x, y, 0)$ and $v_j^0(x, y) = v_j(x, y, 0)$ are regarded as plurisubharmonic functions on $\mathbb{C}^2$, then the measures $\nu_j = dd^c u_j^0 \wedge dd^c v_j^0$ converge weakly to $\nu$. As $dd^c \log|z|$ is the current of integration along the hyperplane $z = 0$ and since $u_j, v_j$ are smooth, we have for $\phi \in C_0^\infty(\mathbb{C}^3)$ that $\int_{\mathbb{C}^3} \phi \, d\mu_j = \int_{\mathbb{C}^2} \phi(x, y, 0)\, d\nu_j$. By the preceding remarks this gives (2.9) as $j \to \infty$.

This construction also works in the case $|c| = 1$, but it does not seem to be dynamically natural anymore, as both $K^{\pm}$ consist only of points with bounded orbit. In this case we can also define an invariant measure by $\mu = \mu^+ \wedge \mu^- \wedge i\,dz \wedge d\bar{z}$, since $H_1^\star(i\,dz \wedge d\bar{z}) = |c|^2 i\,dz \wedge d\bar{z} = i\,dz \wedge d\bar{z}$.

## 3. The Class $H_2$

The second class consists of maps $H_2$, with inverse $H_2^{-1}$, of the form

$$\begin{aligned} H_2(x,y,z) &= (P(y,z) + ax,\, Q(y) + bz,\, y), \\ H_2^{-1}(x,y,z) &= \left(\frac{1}{a}x - \frac{1}{a}P\left(z, \frac{1}{b}y - \frac{1}{b}Q(z)\right), z, \frac{1}{b}y - \frac{1}{b}Q(z)\right), \end{aligned} \tag{3.1}$$

where $\max\{\deg(P), \deg(Q)\} = 2$ and $ab \neq 0$. The dynamics of these maps is interesting when $\deg(Q) = 2$ and $\deg(P) > 0$. If $Q(y) = \beta_1 y^2 + \beta_2 y + \beta_3$ and $\beta_1 \neq 0$, it is easy to see (by an affine change of coordinates in $y$ and $z$) that we may assume $\beta_1 = 1$ and $\beta_3 = 0$.

Hence it is enough to consider maps $H$ of the form

$$\begin{aligned} x_1 &= ax + P(y,z), \\ y_1 &= y^2 + \beta y + bz, \\ z_1 &= y, \end{aligned} \tag{3.2}$$

where $N = \deg(P) > 0$ and $ab \neq 0$. Then the inverse $H_2^{-1}$ is conjugated to a map $H$ of form (3.2), essentially by the transformation $(x,y,z) \to (x,z,y)$.

Let $h$ denote the Hénon map

$$h(y,z) = (y^2 + \beta y + bz,\, y); \tag{3.3}$$

then $H$ is a semidirect product over $h$. We start by finding the degree of the iterates $H^n$. By Proposition 4.2 of [BS], there exists $n_0 \geq 0$ such that $P \circ h^{n_0}$ has $cy^k$ as the unique term of highest total degree: $P \circ h^{n_0}(y,z) = cy^k + \text{l.d.t.}$, where $k > 0$ and $c \neq 0$. We choose the smallest such $n_0$ and define the numbers

$$\gamma = k/2^{n_0+1} \quad \text{and} \quad \gamma_+ = \max\{\gamma, 1\}, \tag{3.4}$$

which determine the degree of the iterates $H^n$ for $n$ sufficiently large. Since $k \leq N2^{n_0}$ we have $0 < \gamma \leq N/2$, so $\gamma \leq 1$ in the quadratic case $N = \deg(P) \leq 2$. It is easy to see that $\gamma$ could be larger than 1 as well—for instance, when $P(y,z) = y^N + \text{l.d.t.}$ and $N \geq 3$. The dynamics of $H$ is particularly interesting in such cases, as we can see from the discussion following Lemma 3.4.

LEMMA 3.1. *There exists a number $n_1 \geq n_0$ such that $\deg(H^n) = \gamma_+ 2^n$ for all $n \geq n_1$.*

*Proof.* We have $\deg(y_n) = 2\deg(z_n) = 2^n$ for all $n$. For $j \geq 0$ let $l_j = \deg(x_{n_0+j})$. As $\deg(P \circ h^{n_0+j}) = k2^j$ and $x_{n_0+j+1} = ax_{n_0+j} + P \circ h^{n_0+j}(y,z)$, we have $l_{j+1} \leq \max\{l_j, k2^j\}$ and hence, by induction, $l_j \leq \max\{l_0, k2^{j-1}\}$ for all $j \geq 1$. We fix $j_0$ such that $k2^{j_0-1} \geq l_0$. In view of the above, we have $l_j = k2^{j-1}$ for all $j > j_0$. Thus for $n \geq n_1 = n_0 + j_0 + 1$ we conclude that

$$\deg(H^n) = \max\{\deg(x_n), \deg(y_n)\} = \max\{k2^{n-n_0-1}, 2^n\} = \gamma_+ 2^n. \qquad \square$$

We now proceed to find the set $U^+$ where the iterates escape to infinity at super-exponential rate. With the usual notation, for $R > 0$ we let

$$V^- = V^-(R) = \{\, w \in \mathbb{C}^3 : |y| > \max\{R, |z|, |x_{n_0}|^{4/(4k-1)}\}\,\}. \tag{3.5}$$

LEMMA 3.2. *For any $\delta \in (0,1)$ there exists $R_0(\delta) > 0$ such that, for $R > R_0$, the following statements hold.*

(i) *For $w \in V^-$, we have*:

$$(1-\delta)|y|^2 < |y_1| < (1+\delta)|y|^2,$$

$$[(1-\delta)|y|]^{2^n} < |y_n| < [(1+\delta)|y|]^{2^n},$$

$$(1-\delta)|c||y|^k < |x_{n_0+1}| < (1+\delta)|c||y|^k.$$

(ii) $H(V^-) \subseteq V^-$.

(iii) *There exist constants $C_1, C_2$, depending on $H$ and $\delta$, such that for $w \in V^-$ and $n > n_0$ we have $C_1|y_n|^\gamma < |x_n| < C_2|y_n|^\gamma$. Moreover, if $\gamma = 1$ then $x_n/y_n \to c$ as $n \to \infty$.*

*Proof*. (i) By (3.2) and (3.5) we have

$$|y_1 - y^2| < (|\beta| + |b|)|y| < \delta|y|^2,$$

$$|x_{n_0+1} - cy^k| \leq |a||x_{n_0}| + C|y|^{k-1} < |a||y|^{k-1/4} + C|y|^{k-1} < |c|\delta|y|^k,$$

where $C$ depends only on the coefficients of $P \circ h^{n_0}$ and where the last inequality of each sequence holds provided that $|y| > R_0$, with $R_0(\delta)$ sufficiently large. The remaining inequality in (i) follows by repeated use of the first, once we have proved (ii).

(ii) For $R_0(\delta)$ sufficiently large, we have

$$|y_1| > (1-\delta)|y|^2 > |y| = |z_1|,$$

$$|x_{n_0+1}| < |c|(1+\delta)|y|^k < [(1-\delta)|y|^2]^{k-1/4} < |y_1|^{k-1/4};$$

hence $H(V^-) \subseteq V^-$.

(iii) Using the invariance of $V^-$ and the estimates in (i) we derive

$$(1-\delta)|c||y_l|^k < |x_{l+n_0+1}| < (1+\delta)|c||y_l|^k,$$

$$(1-\delta)^{2^{n_0+1}}|y_l|^{2^{n_0+1}} < |y_{l+n_0+1}| < (1+\delta)^{2^{n_0+1}}|y_l|^{2^{n_0+1}}.$$

The second inequality gives $(1-\delta)^k|y_l|^k < |y_{l+n_0+1}|^\gamma < (1+\delta)^k|y_l|^k$. This, combined with the first of the preceding two inequalities, yields

$$\frac{(1-\delta)|c|}{(1+\delta)^k} < \frac{|x_n|}{|y_n|^\gamma} < \frac{(1+\delta)|c|}{(1-\delta)^k},$$

where $n = l + n_0 + 1 > n_0$.

Assume now that $\gamma = 1$, so $k = 2^{n_0+1}$. As in the proof of (i), we have

$$|x_{l+n_0+1} - cy_l^k| \le |a||x_{l+n_0}| + C|y_l|^{k-1}$$

and so

$$\left|\frac{x_{l+n_0+1}}{y_l^k} - c\right| \le \frac{|a|}{|y_l|^{1/4}} + \frac{C}{|y_l|} \to 0 \quad \text{as } l \to \infty.$$

By writing $y_{n_0+1} = \tilde{P}(y, z) = y^k + \text{l.d.t.}$ we obtain

$$\left|\frac{y_{l+n_0+1}}{y_l^k} - 1\right| \le \frac{C'(\tilde{P})}{|y_l|} \to 0 \quad \text{as } l \to \infty.$$

Thus we have shown that $x_n/y_n \to c$ as $n \to \infty$. □

Define

$$U^+ = \bigcup_{n=0}^{\infty} H^{-n}(V^-), \qquad K^+ = \mathbb{C}^3 \setminus U^+. \tag{3.6}$$

By Lemma 3.2, the orbit of any point $w \in U^+$ escapes at infinity at super-exponential rate (eventually like $(\text{const})^{2^n}$). Moreover, by Lemma 3.2(iii), the iterates $H^n$ converge locally uniformly on $U^+$ to

$$[1:0:0:0] \text{ if } \gamma > 1, \quad [c:1:0:0] \text{ if } \gamma = 1, \quad [0:1:0:0] \text{ if } \gamma < 1. \tag{3.7}$$

For the Hénon map $h$ of (3.3), we let $K_h^+$ be the set of points with bounded forward orbit and $U_h^+ = \mathbb{C}^2 \setminus K_h^+$ the set of points whose orbit escapes to infinity.

LEMMA 3.3. *We have $K^+ = \mathbb{C} \times K_h^+$ and $U^+ = \mathbb{C} \times U_h^+$. Moreover, the $H$-orbit of any point $w \in K^+$ can escape to infinity at most at exponential rate.*

*Proof*. Note that $H^n(w) = (x_n, y_n, z_n) = (x_n, h^n(y, z))$. Thus if $w \in \mathbb{C} \times K_h^+$ then the iterates $h^n(y, z)$ are locally uniformly bounded on $K_h^+$, so $w \notin U^+$ by Lemma 3.2. This shows that $\mathbb{C} \times K_h^+ \subseteq K^+$.

Let now $w \in K^+$. Then, for all $n \ge 0$,

$$|y_n| \le M_n = \max\{R, |z_n|, |x_{n+n_0}|^{4/(4k-1)}\}.$$

If we let $P \circ h^{n_0}(y, z) = cy^k + \sum_{j+l\le k-1} c_{jl} y^j z^l$ then we have

$$\begin{aligned} |x_{n+n_0}| &\le |a||x_{n+n_0-1}| + c|y_{n-1}|^k + \sum_{j+l\le k-1} |c_{jl}||y_{n-1}|^j |z_{n-1}|^l \\ &\le |a| M_{n-1}^{k-1/4} + |c| M_{n-1}^k + C M_{n-1}^{k-1} \le \tilde{C} M_{n-1}^k, \end{aligned}$$

where $C = \sum |c_{jl}|$ and $\tilde{C} = |a| + |c| + C + 1$. Using this together with $|z_n| = |y_{n-1}| \leq M_{n-1}$, we obtain

$$M_n \leq \max\{M_{n-1}, \tilde{C} M_{n-1}^{4k/(4k-1)}\} \leq \tilde{C} M_{n-1}^{4/3}$$

for all $n \geq 1$. This shows that $M_n \leq (\tilde{C} M_0)^{(4/3)^n}$ and so $|y_n| \leq (\text{const})^{(4/3)^n}$, which implies $(y, z) \in K_h^+$ and so $w \in \mathbb{C} \times K_h^+$. We conclude that $K^+ = \mathbb{C} \times K_h^+$; hence, $U^+ = \mathbb{C} \times U_h^+$. Since

$$x_n = a^n x + a^{n-1} P(y, z) + a^{n-2} P(y_1, z_1) + \cdots + P(y_{n-1}, z_{n-1}),$$

we see that, on $K^+$, $|x_n|$ can grow to infinity at most at exponential rate. □

We postpone for the moment our discussion of the precise behavior of the iterates $H^n$ on $K^+$ in order that we may introduce the Green's function. Denote by $g^+(y, z)$ the Green's function for the Hénon map $h$. We define

$$G^+(w) = \lim_{n\to\infty} \frac{1}{\gamma_+ 2^n} \log^+ \|H^n(w)\| = \lim_{n\to\infty} \frac{1}{2^n} \log^+ |y_n|. \tag{3.8}$$

By [BS], the second limit of (3.8) equals $g^+(y, z)$, the convergence being locally uniform on $\mathbb{C}^2$. The two limits are equal on $K^+$, as they are both zero there by Lemma 3.3, and they are also equal on $U^+$ in view of Lemma 3.2(iii). We conclude that

$$G^+(x, y, z) = g^+(y, z)$$

is continuous and plurisubharmonic on $\mathbb{C}^3$ and pluriharmonic on $U^+$, that $K^+ = \{G^+ = 0\}$, and that the convergence is locally uniform on $\mathbb{C}^3$ in both limits. Moreover, we have $G^+ \circ H = 2G^+$ and hence $H^\star \mu^+ = 2\mu^+$, where $\mu^+ = dd^c G^+$.

In order to discuss the behavior of the iterates $H^n$ on $K^+$, we must consider three cases.

*Case 1:* $|a| < 1$

Here the iterates $H^n$ are locally uniformly bounded on $K^+$. Indeed, we have the following formula for $x_n$:

$$x_n = a^n x + \sum_{j=0}^{n-1} a^{n-1-j} P(y_j, z_j). \tag{3.9}$$

Let $D$ be an open relatively compact subset of $\mathbb{C}^3$. Since (by [BS]) the orbits $\{h^n(y, z)\}$ are uniformly bounded on $K_h^+ \cap D$, it follows that there exists a constant $M > 0$, depending on $D$, such that $|P(y_j, z_j)| < M$ for all $w \in K^+ \cap D$ and all $j$. Then, by (3.9), $|x_n| \leq |x| + M/(1 - |a|)$ for all $n \geq 0$.

*Case 2:* $|a| = 1$

In this case we have the following locally uniform estimates on $K^+$: For any open relatively compact $D \subset \mathbb{C}^3$, there exists a constant $M$ such that $|x_n| \leq |x| + nM$

for all $w \in K^+ \cap D$ and all $n \geq 0$. This follows from (3.9), as in Case 1. For example, assume that $a = 1$ and that $(y_0, z_0) \neq (0, 0)$ is an attractive fixed point of $h$, with basin of attraction $B \subset \mathbb{C}^2$. If $P(y_0, z_0) \neq 0$ then it is easy to see that, for $w \in \mathbb{C} \times B$, we have $x_n = nP(y_0, z_0) + O(1)$. If $P(y_0, z_0) = 0$ then the iterates $H^n$ are locally uniformly bounded on $\mathbb{C} \times B$. On the other hand, if $P(0, 0) = 0$ and $P(y_0, z_0) \neq 0$ hold simultaneously, then we see that there are points of $K^+$ with bounded orbit (the origin is fixed by $H$) as well as points $w \in K^+$ with $x_n = n\alpha + O(1)$, $\alpha \neq 0$.

*Case 3:* $|a| > 1$

We introduce the following notation:

$$\begin{aligned} S_n(y,z) &= \sum_{j=0}^{n-1} \frac{1}{a^j} P(y_j, z_j), \qquad T_n(w) = x + \frac{1}{a} S_n(y,z), \\ S(y,z) &= \sum_{j=0}^{\infty} \frac{1}{a^j} P(y_j, z_j), \qquad T(w) = x + \frac{1}{a} S(y,z). \end{aligned} \tag{3.10}$$

Because the iterates of $h$ are locally uniformly bounded on $K_h^+$, we see that $\{S_n\}$ converges locally uniformly on $K_h^+$ to $S$ and that $\{T_n\}$ converges locally uniformly on $K^+$ to $T$; therefore, $S \in C(K_h^+) \cap O(\operatorname{int} K_h^+)$ and $T \in C(K^+) \cap O(\operatorname{int} K^+)$. We remark that $\operatorname{int} K^+$ is, in some cases, empty. By (3.9) we have

$$x_n = a^n \left( x + \frac{1}{a} S_n(y,z) \right) = a^n T_n(x, y, z).$$

We denote by $X$ the following subset of $K^+$:

$$X = \{ w \in K^+ : T(w) = 0 \} = \left\{ w \in K^+ : x = -\frac{1}{a} S(y,z) \right\}.$$

LEMMA 3.4. *For $w \in K^+$ we have $T \circ H(w) = aT(w)$, so $H(X) = X$. If $w \in K^+ \setminus X$ then $\{H^n(w)\} \sim \{a^n\}$ escapes to infinity at exponential rate, the iterates $H^n$ converging locally uniformly on $K^+ \setminus X$ to* $[1 : 0 : 0 : 0]$. *The orbits of the points $w \in X$ are locally uniformly bounded relative to the invariant set $X$.*

*Proof*. We have

$$T \circ H(x, y, z) = ax + P(y, z) + \sum_{j=0}^{\infty} \frac{1}{a^{j+1}} P(y_{j+1}, z_{j+1}) = aT(x, y, z),$$

which gives $H(X) = X$. If $w_0 \in K^+ \setminus X$, we fix a relatively open neighborhood $D$ of $w$ in $K^+ \setminus X$ such that $M > |T(w)| > \varepsilon > 0$ on $D$. For $n$ sufficiently large we have $|T_n(w) - T(w)| < \varepsilon/2$ for all $w \in D$. So, by (3.9) and (3.10), $|x_n/a^n - T(w)| = |T_n(w) - T(w)| < \varepsilon/2$, which gives $\varepsilon|a|^n/2 < |x_n| < (M + \varepsilon/2)|a|^n$ for all $w \in D$. Finally, if $w \in X$ then, since $T(w) = 0$, we obtain

$$|x_n| = |a^n(T_n(w) - T(w))| \leq \sum_{j=n}^{\infty} \frac{1}{|a|^{j-n+1}} |P(y_j, z_j)|,$$

which shows that the iterates $H^n$ are locally uniformly bounded relatively to $X$.

□

Let us recall the description of the dynamics of $H$ on $U^+$ given in (3.7). By Lemma 3.4 and by (3.7), if $\gamma > 1$ then $H^n(w) \to [1:0:0:0]$ as $n \to \infty$ for all $w \in U^+ \cup (K^+ \setminus X)$. The convergence is locally uniform on the open set $U^+$ and locally uniform (in the relative sense) on $K^+ \setminus X$, but it is not locally uniform on the whole union. We also note that, when $\operatorname{int} K^+ \neq \emptyset$, the Julia set $J^+$ of $H$ (defined with normal families) is equal to $\partial K^+ \cup X$. In this case $X$ is a complex submanifold of $\operatorname{int} K^+$. Hence $J^+ \neq \operatorname{supp} \mu^+$.

Invariant Measures. If $H$ is of form (3.2) then (a) we set $K^- = K^+(H^{-1})$—see (3.6); (b) we denote by $G^-$ the Green's function of the inverse map $H^{-1}$; and (c) we let $\mu^- = dd^c G^-$. In view of the preceding results, if $g^-(y,z)$ is the Green's function of $h^{-1}$ (the inverse of the Hénon map $h$ in (3.3)) then $G^-(x,y,z) = g^-(y,z)$. Moreover, $K^- = \mathbb{C} \times K_h^-$ and $H^\star \mu^- = \frac{1}{2}\mu^-$. As we noticed at the beginning, the map $H^{-1}$ has form (3.2) (after a suitable change of coordinates in $y$ and $z$), with $1/a$ as the coefficient of $x$.

We have that the $(1,1)$ bidimensional current $\mu^+ \wedge \mu^-$ is $H$-invariant:

$$H^\star(\mu^+ \wedge \mu^-) = H^\star\mu^+ \wedge H^\star\mu^- = 2\mu^+ \wedge \tfrac{1}{2}\mu^- = \mu^+ \wedge \mu^-.$$

In the case when $|a| = 1$, the measure

$$\mu = \mu^+ \wedge \mu^- \wedge i\, dx \wedge d\bar{x}$$

is $H$-invariant. This is because

$$\mu^+ \wedge \mu^- \wedge i\, dx_1 \wedge d\bar{x}_1 = \mu^+ \wedge \mu^- \wedge i\, d(ax) \wedge d(\overline{ax}) = \mu^+ \wedge \mu^- \wedge i\, dx \wedge d\bar{x}.$$

Let us consider now the case $|a| \neq 1$. Besides the invariant currents $\mu^\pm$, we also have the invariant set $X$. If $|a| > 1$ then $X \subset K^+$ is precisely the set of points with bounded forward orbit, whereas if $|a| < 1$ then $X$ is constructed using the inverse $H^{-1}$ (with $|1/a| > 1$ as the coefficient of $x$ as before), and $X$ is the set of points with bounded backward orbit. Hence $K^+ \cap K^- \cap X$ equals the set of points with bounded full orbit. Let $\nu$ be the invariant measure on $\mathbb{C}^2$ constructed in [BS] for the Hénon map $h$. Since $G^\pm = g^\pm$, we actually have $\nu = \mu^+ \wedge \mu^-$, regarded as a measure on $\mathbb{C}^2$, with support on $L = \partial K_h^+ \cap \partial K_h^-$. In direct analogy to the construction of invariant measures in Section 2 (see (2.9)), we define an invariant measure $\mu$ on $\mathbb{C}^3$ by its action on $\phi \in C_0(\mathbb{C}^3)$:

$$\int_{\mathbb{C}^3} \phi\, d\mu = \int_L \phi\left(-\frac{1}{a} S(y,z), y, z\right) d\nu, \tag{3.11}$$

where $S$ is as in (3.10). Because $\nu$ is a probability measure, it follows that $\mu$ is a probability measure supported on the compact set

$$\left\{ w : x = -\frac{1}{a}S(y,z),\ (y,z) \in \partial K_h^+ \cap \partial K_h^- \right\} = X \cap \partial K^+ \cap \partial K^-.$$

The invariance of $\mu$ also follows from (3.11): If for $\phi$ as above we write $\psi_\phi(y,z) = \phi\big(-\frac{1}{a}S(y,z), y, z\big)$, then since $\psi_{\phi\circ H} = \psi_\phi \circ h$ and $\nu$ is $h$-invariant we obtain

$$(H_\star\mu)(\phi) = \int_{\mathbb{C}^3} \phi \circ H\, d\mu = \int_L \psi_\phi \circ h\, d\nu = \int_L \psi_\phi\, d\nu = \int_{\mathbb{C}^3} \phi\, d\mu.$$

We also note that $\mu = \lim_{n\to\infty} \mu_n$, where $\mu_n = \mu^+ \wedge \mu^- \wedge dd^c \log|T_n|$. Indeed, using a smoothing argument as at the end of Section 2, we see that for $\phi \in C_0^\infty(\mathbb{C}^3)$ we have

$$\int_{\mathbb{C}^3} \phi\, d\mu_n = \int_{\mathbb{C}^2} \phi\left(-\frac{1}{a}S_n(y,z), y, z\right) d\nu,$$

so by (3.11) the measures $\mu_n$ converge weakly to $\mu$. Since $T_n \circ H = aT_{n+1}$, we have $H^\star\mu_n = \mu_{n+1}$, which also gives the invariance of $\mu$.

## 4. The Class $H_3$

This class contains maps $H_3$ of the form

$$H_3(x,y,z) = (P(x,z) + ay,\ Q(x) + z,\ x), \tag{4.1}$$

where $\max\{\deg(P), \deg(Q)\} = 2$ and $a \neq 0$. For simplicity we will write $H = H_3$. The inverse of this map is given by

$$H^{-1}(x,y,z) = \left(z,\ \frac{1}{a}x - \frac{1}{a}P(z, y - Q(z)),\ y - Q(z)\right). \tag{4.2}$$

We start by discussing the dynamics of $H$ and write

$$P(x,z) = \alpha x^2 + \alpha' xz + \alpha'' z^2 + \text{l.d.t.},$$
$$Q(x) = \beta x^2 + \text{l.d.t.} \tag{4.3}$$

We remark that if $\alpha = 0$ and $\alpha' \neq 0$ then $\deg(x_n) = \deg(x_{n-1}) + \deg(x_{n-2})$, so the degrees of the iterates $H^n$ are given by Fibonacci's numbers (such maps are considered in [B]).

Hence we will work under the generic assumption $\alpha \neq 0$ (so the case $\beta = 0$ is also covered). For $\varepsilon > 0$ and $R > 1/\varepsilon$, we define

$$V^- = V^-_{\varepsilon,R} = \{ w \in \mathbb{C}^3 : |x| > \max\{R, |z|/\varepsilon, (|y|/\varepsilon)^{1/2}\}\}. \tag{4.4}$$

LEMMA 4.1. *For any $\delta \in (0,1)$, there exist $\varepsilon = \varepsilon(\delta) \in (0,1)$ and $R_0(\delta) > 1/\varepsilon$ such that, for any $R > R_0$, the set $V^-$ has the following properties*:

(i) *the following estimates hold for $w \in V^-$*:

$$|\alpha|(1-\delta)|x|^2 < |x_1| < |\alpha|(1+\delta)|x|^2, \qquad |y_1| < (|\beta|+1)|x|^2,$$

$$[|\alpha|(1-\delta)]^{2^n-1}|x|^{2^n} < |x_n| < [|\alpha|(1+\delta)]^{2^n-1}|x|^{2^n};$$

(ii) $H(V^-) \subseteq V^-$;
(iii) *the iterates* $H^n$ *converge uniformly on* $V^-$ *to* $[\alpha : \beta : 0 : 0]$.

*Proof*. (i) For $w \in V^-$ we have

$$|x_1 - \alpha x^2| < C\varepsilon |x|^2 \quad \text{and} \quad |y_1| < (|\beta| + C\varepsilon)|x|^2,$$

where $C$ depends only on the coefficients of $H$. So the first two inequalities hold if we choose $\varepsilon(\delta)$ sufficiently small and $R > 1/\varepsilon$. The third estimate follows from the first one and from part (ii) of the lemma.

(ii) If $w \in V^-$ then, by the estimates of (i), we obtain

$$|x_1| > |\alpha|(1-\delta)R^2 > R,$$

$$|z_1|/\varepsilon = |x|/\varepsilon < |\alpha|(1-\delta)|x|^2 < |x_1|,$$

$$(|y_1|/\varepsilon)^{1/2} < [(|\beta|+1)/\varepsilon]^{1/2}|x| < |\alpha|(1-\delta)|x|^2 < |x_1|,$$

provided that $|x| > R_0(\delta)$, with $R_0$ sufficiently large. This shows $H(w) \in V^-$.

(iii) Using $x_n = ay_{n-1} + P(x_{n-1}, x_{n-2})$ together with the estimate $|y_{n-1}| < (|\beta|+1)|x_{n-2}|^2$, we have

$$\left|\frac{x_n}{x_{n-1}^2} - \alpha\right| \le C\frac{|x_{n-2}|}{|x_{n-1}|} < \frac{C}{|\alpha|(1-\delta)}\,\frac{1}{|x_{n-2}|} \to 0 \quad \text{as } n \to \infty,$$

where $C$ is a constant depending on coefficients. Similarly, since

$$y_n = Q(x_{n-1}) + x_{n-2}$$

we obtain

$$\left|\frac{y_n}{x_{n-1}^2} - \beta\right| \le C\frac{|x_{n-2}|}{|x_{n-1}|} \to 0 \quad \text{as } n \to \infty,$$

so $\lim_{n\to\infty} y_n/x_n = \beta/\alpha$ uniformly on $V^-$. □

We define

$$U^+ = \bigcup_{n=0}^{\infty} H^{-n}(V^-), \qquad K^+ = \mathbb{C}^3 \setminus U^+. \tag{4.5}$$

By Lemma 4.1, the orbit of any point in $U^+$ escapes at infinity at super-exponential rate and the iterates $H^n$ converge locally uniformly on $U^+$ to $[\alpha : \beta : 0 : 0]$. On the other hand, the orbit of any point $w \in K^+$ can escape to infinity at most at exponential rate, as follows.

LEMMA 4.2. *If* $w \in K^+$ *then the following estimate holds for all* $n \ge 0$:

$$\max\{|x_n|, |y_n|, |z_n|\} \le \frac{1}{\varepsilon^{2n+1}} \max\{\varepsilon^2 R^2, |z|^2, \varepsilon|y|\}.$$

*Proof*. Since $w \in K^+$, by (4.4) we have $|x_n| \le M_n = \max\{R, |z_n|/\varepsilon, (|y_n|/\varepsilon)^{1/2}\}$ for all $n \ge 0$. We note that $|z_n|/\varepsilon = |x_{n-1}|/\varepsilon \le M_{n-1}/\varepsilon$ and $|y_n| \le |Q(x_{n-1})| + |z_{n-1}| \le CM_{n-1}^2$, where $C > 1$ depends only on $H$. Thus $(|y_n|/\varepsilon)^{1/2} \le (C/\varepsilon)^{1/2}M_{n-1}$ and so we conclude that $M_n \le M_{n-1}/\varepsilon$ and hence $M_n \le M_0/\varepsilon^n$.

It follows that $|x_n| \le M_0/\varepsilon^n$, $|y_n| \le M_0^2/\varepsilon^{2n-1}$, and $|z_n| \le M_0/\varepsilon^{n-1}$, which together imply the estimate of the lemma. $\square$

In order to introduce the Green's function $G^+$, we need the following simple observation.

LEMMA 4.3. $\deg(H^n) = 2^n$, *and* $\|H^n(w)\| \le (C\|w\|_+)^{2^n}$ *holds for all* $n \ge 0$ *and* $w \in \mathbb{C}^3$, *where* $C$ *is a constant depending only on* $H$.

*Proof*. There exists a constant $C > 1$, depending on coefficients, such that $\|H(w)\|_+ \le C\|w\|_+^2$. By induction, this gives the desired estimate. $\square$

We now define the Green's function $G^+$ of $H$ by

$$G^+(w) = \lim_{n\to\infty} \frac{1}{2^n} \log^+\|H^n(w)\| = \lim_{n\to\infty} \frac{1}{2^n} \log^+|x_n|. \tag{4.6}$$

Note that by Lemma 4.3 we have $(1/2^n)\log^+\|H^n(w)\| \le \log^+\|w\| + \log C$ for all $n$ and for all $w \in \mathbb{C}^3$. Lemma 4.2 shows that the limits are both 0 on $K^+$. By Lemma 4.1(i), we have $|y_n| = O(|x_n|)$ and $|z_n| = |x_{n-1}| = o(|x_n|)$ for $w \in V^-$ and $n \ge 0$, so both limits exist and are equal on $U^+$ provided that one of them exists. We now proceed exactly as in the proof of Theorem 2.5: we first show that the second limit exists locally uniformly on $U^+$. Then, using the estimate derived above from Lemma 4.3, we show that $G_\star^+ = G^+$. We conclude that all the assertions of Theorem 2.5 hold for the function $G^+$ in (4.6): $G^+ \in \mathrm{PSH} \cap C(\mathbb{C}^3)$, $K^+ = \{G^+ = 0\}$, $G^+$ is pluriharmonic on $U^+$, and $G^+ \circ H = 2G^+$. Hence, if $\mu^+ = dd^cG^+$ then we have $H^\star\mu^+ = 2\mu^+$.

We now study the dynamics of the inverse map $H^{-1}$ given in (4.2). Consider the change of coordinates given by $w' = (x', y', z') = F(w)$, where

$$x' = x, \quad y' = y - Q(z), \quad z' = z. \tag{4.7}$$

The map $H^{-1}$ is then conjugated to the map $\tilde{H}(w') = F \circ H^{-1} \circ F^{-1}(w')$ given by

$$\begin{aligned} x_1' &= z', \\ y_1' &= \frac{1}{a}x' - \frac{1}{a}P(z', y') - Q(y') \\ &= \frac{1}{a}x' + p(z', y') + \gamma'' z'^2 + \gamma' y' z' + \gamma y'^2, \\ z_1' &= y', \end{aligned} \tag{4.8}$$

where $\deg(p) \le 1$, $\gamma'' = -\alpha/a$, $\gamma' = -\alpha'/a$, and $\gamma = -\beta - \alpha''/a$ (recall the form of $P$, $Q$ from (4.3)). We will study the map $\tilde{H}$ under the generic assumption $\gamma \ne 0$ (recall that the map $H$ was studied under the generic assumption $\alpha \ne 0$). One reason for doing so is that when $\gamma = 0$ and $\gamma' \ne 0$ we have $\deg(y_n') = \deg(y_{n-1}') + \deg(y_{n-2}')$, so the degrees of the iterates are once again given by Fibonacci's numbers.

We proceed to find the dynamically relevant sets $U^+ = U^+(\tilde{H})$ and $K^+ = K^+(\tilde{H})$ and to define the Green's function $G^+$ for the map $\tilde{H}$ of (4.8). We will then relate these to the corresponding sets $U^-$, $K^-$ and to the Green's function $G^-$ of the map $H^{-1}$.

For $\varepsilon > 0$ and $R > 1/\varepsilon$, let

$$V^- = V^-_{\varepsilon,R}(\tilde{H}) = \{\, w' \in \mathbb{C}^3 : |y'| > \max\{R, |x'|, |z'|/\varepsilon\}\,\}.$$

The following lemma is proved in a similar way to Lemma 4.1.

LEMMA 4.4. *For every $\delta \in (0,1)$, there exist $\varepsilon = \varepsilon(\delta) \in (0,1)$ and $R_0(\delta) > 1/\varepsilon$ such that, for all $R > R_0$, we have $\tilde{H}(V^-) \subseteq V^-$ and the following estimates hold for $w' \in V^-$:*

$$|\gamma|(1-\delta)|y'|^2 < |y'_1| < |\gamma|(1+\delta)|y'|^2,$$

$$[|\gamma|(1-\delta)]^{2^n-1}|y'|^{2^n} < |y'_n| < [|\gamma|(1+\delta)]^{2^n-1}|y'|^{2^n}.$$

We next let

$$U^+ = \bigcup_{n=0}^{\infty} \tilde{H}^{-n}(V^-), \qquad K^+ = \mathbb{C}^3 \setminus U^+. \tag{4.9}$$

From Lemma 4.4 it follows that the orbit $\{w'_n\}$ of any point $w' \in U^+$ escapes to infinity at super-exponential rate and that the iterates $\tilde{H}^n$ converge locally uniformly on $U^+$ to $[0:1:0:0]$. On the set $K^+$, the orbits can escape to infinity at most at exponential rate, as we now show.

LEMMA 4.5. *For any $w \in K^+$ and any $n \geq 0$, we have*

$$\max\{|x'_n|, |y'_n|, |z'_n|\} \leq \frac{1}{\varepsilon^n} \max\{R, |x'|, |z'|/\varepsilon\}.$$

*Proof.* If $w' \in K^+$ and $n \geq 0$, then $|y'_n| \leq M_n = \max\{R, |x'_n|, |z'_n|/\varepsilon\}$. But $|x'_n| = |z'_{n-1}| \leq \varepsilon M_{n-1}$ and $|z'_n|/\varepsilon = |y'_{n-1}|/\varepsilon \leq M_{n-1}/\varepsilon$, so $M_n \leq M_{n-1}/\varepsilon$ and the lemma follows. □

The Green's function of $\tilde{H}$ can now be defined by

$$\begin{aligned} G^+(w') &= \lim_{n\to\infty} \frac{1}{2^n} \log^+ \|\tilde{H}^n(w')\| \\ &= \lim_{n\to\infty} \frac{1}{2^n} \log^+|y'_n| = \lim_{n\to\infty} \frac{1}{2^{n-1}} \log^+|z'_n|. \end{aligned} \tag{4.10}$$

It has the same properties as before: the convergence in the above limits is locally uniform on $\mathbb{C}^3$, $G^+ \in \mathrm{PSH} \cap C(\mathbb{C}^3)$, $K^+ = \{G^+ = 0\}$, $G^+$ is pluriharmonic on $U^+$, and $G^+ \circ \tilde{H} = 2G^+$.

We now return to the map $H^{-1}(w) = F^{-1} \circ \tilde{H} \circ F(w)$. We note that if $w' = F(w)$ and if $w_n = H^{-n}(w)$ and $w'_n = \tilde{H}^n(w')$ for $n \geq 0$, then $F(w_n) = w'_n$; hence, by (4.7) we have

$$x'_n = x_n, \quad z'_n = z_n, \quad y'_n = y_n - Q(z_n). \tag{4.11}$$

Using the sets $U^+ = U^+(\tilde{H})$ and $K^+ = K^+(\tilde{H})$ of (4.9), we define the corresponding sets for the map $H^{-1}$ by

$$U^- = F^{-1}(U^+), \qquad K^- = F^{-1}(K^+) = \mathbb{C}^3 \setminus U^-.$$

Using Lemmas 4.4 and 4.5, we easily obtain the following.

LEMMA 4.6. *The orbits $\{H^{-n}(w)\}_{n\geq 0}$ of points $w \in U^-$ escape to infinity at super-exponential rate, whereas the $H^{-1}$-orbit of any point $w \in K^-$ can escape to infinity at most at exponential rate.*

*Proof.* If $w \in U^-$ then $w' = F(w) \in U^+(\tilde{H})$, so $|z_n| = |z'_n|$ increases super-exponentially to infinity. For $w \in K^-$ we have $w' = F(w) \in K^+(\tilde{H})$, and by (4.11) we obtain $x_n = x'_n$, $z_n = z'_n$, and $y_n = y'_n + Q(z'_n)$. The estimate on $\|w'_n\|$ from Lemma 4.5 implies that $\|w_n\|$ grows at most exponentially to infinity. □

We will be more precise about the behavior of the iterates $\{H^{-n}\}_{n\geq 0}$ on $U^-$ when we discuss the Green's function $G^-$ of the map $H^{-1}$. We now introduce the function $\hat{G}^- = G^+ \circ F$. Using (4.10), (4.11), and $F \circ H^{-n} = \tilde{H}^n \circ F$, we have

$$\hat{G}^-(w) = \lim_{n\to\infty} \frac{1}{2^n} \log^+\|F(H^{-n}(w))\| = \lim_{n\to\infty} \frac{1}{2^{n-1}} \log^+|z_n|. \tag{4.12}$$

The properties of $G^+$ mentioned before imply that $\hat{G}^- \in \mathrm{PSH} \cap C(\mathbb{C}^3)$, $K^- = \{\hat{G}^- = 0\}$, $\hat{G}^-$ is pluriharmonic on $U^-$, $\hat{G}^- \circ H^{-1} = 2\hat{G}^-$, and the convergence of the two sequences defining $\hat{G}^-$ is locally uniform on $\mathbb{C}^3$. Hence, if $\mu^- = dd^c\hat{G}^-$ then $H^\star\mu^- = \frac{1}{2}\mu^-$.

Finally, consider the usual Green's function $G^-$ of $H^{-1}$:

$$G^-(w) = \lim_{n\to\infty} \frac{1}{\deg(H^{-n})} \log^+\|H^{-n}(w)\|.$$

THEOREM 4.7. *The limit of the above sequence exists, the convergence being locally uniform on $\mathbb{C}^3$. The function $G^-$ is equal to the function $\hat{G}^-$ of* (4.12), *up to multiplication by a constant. Moreover, the iterates $H^{-n}$ converge locally uniformly on the set $U^-$ to a point of $\mathbb{P}^3$ which depends on $H$.*

*Proof.* As the degree of $H^{-1}$ can be 2, 3, or 4, we must consider the individual cases.

*Case 1: $\beta = 0$.* Our generic assumption $\gamma \neq 0$ implies that $\alpha'' \neq 0$, so $\deg(H^{-n}) = 2^n$ for all $n \geq 1$ (by (4.2), $y_n$ has the term $y^{2^n}$). Using (4.11), we see that $y_n/z_n^2 = (y'_n + Q(z'_n))/(z'_n)^2 = y'_n/(y'_{n-1})^2 + o(1)$ holds on $U^-$. Hence Lemma 4.4 and (4.12) imply that, for $w \in \mathbb{C}^3$,

$$G^-(w) = \lim_{n\to\infty} \frac{1}{2^n} \log^+\|H^{-n}(w)\| = \lim_{n\to\infty} \frac{1}{2^{n-1}} \log^+|z_n| = \hat{G}^-(w),$$

and the iterates $H^{-n}$ converge locally uniformly on $U^-$ to $[0:1:0:0]$.

*Case 2: $\beta \neq 0$ and $\alpha' = \alpha'' = 0$.* Then, by (4.2), $\deg(H^{-n}) = 2^n$, and $z_n$ has the term $z^{2^n}$ for all $n \geq 1$. By Lemma 4.4 and by (4.11) we have $|y_n| =$

$|y_n' + Q(z_n')| = O(|z_n'|^2) = O(|z_n|^2)$ on $U^-$. Using this and (4.2) we get in fact that $|y_{n+1}| = O(|z_n|^2) = O(|z_{n+1}|)$, so $\|H^{-n}(w)\| = O(|z_n|)$ on $U^-$. Thus

$$G^-(w) = \lim_{n\to\infty} \frac{1}{2^n} \log^+\|H^{-n}(w)\| = \lim_{n\to\infty} \frac{1}{2^n} \log^+|z_n| = \frac{1}{2}\hat{G}^-(w),$$

for any $w \in \mathbb{C}^3$. Since $H^{-1}$ has the form

$$x_1 = z, \quad y_1 = cz^2 + \text{l.d.t.}, \quad z_1 = \beta z^2 + \text{l.d.t.}$$

for some constant $c \in \mathbb{C}$, it follows that $y_{n+1}/z_n^2 \to c$ and $z_{n+1}/z_n^2 \to \beta$; hence $y_n/z_n \to c/\beta$ on $U^-$. We conclude that the iterates $H^{-n}$ converge locally uniformly on $U^-$ to $[0 : c : \beta : 0]$.

*Case 3:* $\beta \neq 0$, $\alpha'' = 0$, *and* $\alpha' \neq 0$. In this case we have $\deg(H^{-n}) = 3(2^{n-1})$, and $y_n$ has the term $z^{3(2^{n-1})}$ for all $n \geq 1$. As in Case 2, we have $|y_n| = O(|z_n|^2)$ on $U^-$. Therefore, by (4.2), $|y_{n+1}| < M|z_n|^3 < M'|z_{n+1}|^{3/2}$ holds on $V^+ = F^{-1}(V^-(\tilde{H}))$, with constants $M$, $M'$ independent on $w$. Using (4.2) again we get in fact that $0 < m < |y_{n+1}|/|z_n|^3 < M''$, so $|y_n| \sim |z_n|^{3/2}$ for $w \in V^+$. (Here $V^-(\tilde{H})$ is as in Lemma 4.4.) This, combined with (4.12), shows that

$$G^-(w) = \lim_{n\to\infty} \frac{1}{3(2^{n-1})} \log^+\|H^{-n}(w)\| = \lim_{n\to\infty} \frac{1}{2^n} \log^+|z_n| = \frac{1}{2}\hat{G}^-(w)$$

for all $w \in \mathbb{C}^3$. Moreover, the iterates $H^{-n}$ converge locally uniformly on $U^-$ to $[0 : 1 : 0 : 0]$.

*Case 4:* $\beta \neq 0$ *and* $\alpha'' \neq 0$. Since $\gamma \neq 0$ we have $\deg(\tilde{H}^n) = \deg(y_n') = 2^n$, and $y_n'$ has the term $(y')^{2^n}$. Hence $\deg(\tilde{H}^n \circ F) = \deg(y_n - Q(z_n)) = 2^{n+1}$, and $y_n - Q(z_n)$ has the term $z^{2^{n+1}}$ for all $n \geq 1$. As $\alpha'' \neq 0$, these imply $\deg(H^{-n}) = \deg(y_n) = 2\deg(z_n) = 2^{n+1}$. By Lemma 4.4 and (4.11), we see that $|y_{n+1}| \sim |y_n - Q(z_n)|^2 = |z_{n+1}|^2$ holds on $V^+ = F^{-1}(V^-(\tilde{H}))$ and so $\|H^{-n}(w)\| \sim |z_n|^2$ (here $\sim$ is used in the same sense as in Case 3). It follows that, on $\mathbb{C}^3$,

$$G^-(w) = \lim_{n\to\infty} \frac{1}{2^{n+1}} \log^+\|H^{-n}(w)\| = \lim_{n\to\infty} \frac{1}{2^n} \log^+|z_n| = \frac{1}{2}\hat{G}^-(w).$$

The iterates $H^{-n}$ converge locally uniformly on $U^-$ to $[0 : 1 : 0 : 0]$. □

## 5. The Class $H_4$

The maps $H = H_4$ of this class have the form

$$H(x, y, z) = (P(x, y) + az, Q(y) + x, y), \tag{5.1}$$

where $\max\{\deg(P), \deg(Q)\} = 2$ and $a \neq 0$. The inverse map is given by

$$\begin{aligned} H^{-1}(x, y, z) &= \left(y - Q(z), z, \frac{1}{a}x + \tilde{P}(y, z)\right), \\ \tilde{P}(y, z) &= -\frac{1}{a}P(y - Q(z), z). \end{aligned} \tag{5.2}$$

As before, we write

$$P(x, y) = \alpha x^2 + \alpha' xy + \alpha'' y^2 + \text{l.d.t.},$$
$$Q(y) = \beta y^2 + \text{l.d.t.} \tag{5.3}$$

We recall from [S] that a map $H$ is regular if $I^+ \cap I^- = \emptyset$, where $I^\pm$ denote the indeterminacy sets of $H$ and $H^{-1}$, respectively. It is easy to check that a map $H$ of form (5.1) is regular if and only if $\beta \neq 0$ and $\alpha \neq 0$ (in which case $I^+ = [0 : 0 : 1 : 0]$ and $I^- = \{t = z = 0\}$).

In studying the dynamics of the maps $H$ in (5.1) that are not regular, we must discuss several cases. Before doing so, let us consider the action of $H$ on the hyperplane at infinity of $\mathbb{P}^3$. The extension of $H$ to $\mathbb{P}^3$ is given by $H[x : y : z : t] = [t^2 H(x/t, y/t, z/t) : t^2]$, so $H[x : y : z : 0] = [\alpha x^2 + \alpha' xy + \alpha'' y^2 : \beta y^2 : 0 : 0]$. When $\beta \neq 0$ (so $\alpha = 0$), we have $H(\{t = 0\} \setminus I^+) \subset I^- = \{t = z = 0\}$. Moreover, if $\alpha' \neq 0$ then on $I^-$, with coordinate $u = x/y$, the map is affine: $H[u : 1 : 0 : 0] = \left[\frac{\alpha'}{\beta} u + \frac{\alpha''}{\beta} : 1 : 0 : 0\right]$. On the other hand, when $\alpha' = 0$ or $\beta = 0$, $H$ maps $\{t = 0\} \setminus I^+$ to a single point. Note that this was also the case for the maps $H_3$ of (4.1): with the notation of (4.3) and $\alpha \neq 0$, we have $H_3^2[x : y : z : 0] = [\alpha : \beta : 0 : 0]$.

*Case 1:* $\beta \neq 0$, $\alpha = 0$, $\alpha' \neq 0$

We begin by discussing the map $H$. For $\varepsilon > 0$ and $R > 1/\varepsilon$, let

$$V^- = V^-_{\varepsilon, R} = \{\, w \in \mathbb{C}^3 : |y| > \max\{R, |z|, (|x|/\varepsilon)^{1/2}\}\,\}.$$

LEMMA 5.1. *For any $\delta \in (0, 1)$, there exist $\varepsilon = \varepsilon(\delta) \in (0, 1)$ and $R_0(\delta) > 1/\varepsilon$ such that, for any $R > R_0$, we have $H(V^-) \subseteq V^-$, and the following estimates hold on $V^-$:*

$$|\beta|(1 - \delta)|y|^2 < |y_1| < |\beta|(1 + \delta)|y|^2, \qquad |x_1| < \delta |y|^3,$$
$$[|\beta|(1 - \delta)]^{2^n - 1}|y|^{2^n} < |y_n| < [|\beta|(1 + \delta)]^{2^n - 1}|y|^{2^n}.$$

*Proof.* The first two estimates hold on $V^-$ provided that $\varepsilon = \varepsilon(\delta)$ is sufficiently small and $R_0 > 1/\varepsilon$. As before, they imply $H(V^-) \subset V^-$ if $R_0$ is chosen sufficiently large. The third estimate follows by induction from the first. □

We let

$$U^+ = \bigcup_{n=0}^{\infty} H^{-n}(V^-), \qquad K^+ = \mathbb{C}^3 \setminus U^+. \tag{5.4}$$

By Lemma 5.1, the orbit $H^n(w)$ of $w \in U^+$ escapes to infinity at super-exponential rate $(\text{const})^{2^n}$, because $|y_n|$ does so. The orbits of points in $K^+$ can escape to infinity at most at a slower super-exponential rate.

LEMMA 5.2. *If $M(w) = \max\{R, |z|, (|x|/\varepsilon)^{1/2}\}$, then*

$$\max\{|y_n|, |z_n|\} < [CM(w)]^{(3/2)^n} \quad \text{and} \quad |x_n| < [CM(w)]^{2(3/2)^n}$$

*hold for all $w \in K^+$ and $n \geq 0$, where $C > 1$ is a suitable constant.*

*Proof*. If $w \in K^+$ then $|y_n| \le M_n = \max\{R, |z_n|, (|x_n|/\varepsilon)^{1/2}\}$ holds for all $n$. We have $|z_n| = |y_{n-1}| \le M_{n-1}$ and, by the definition of $M_{n-1}$, $|x_{n-1}| \le \varepsilon M_{n-1}^2$ and $|z_{n-1}| \le M_{n-1}$. Using $x_n = P(x_{n-1}, y_{n-1}) + a z_{n-1}$ (see (5.1)), these imply $|x_n| \le C' M_{n-1}^3$, where $C'$ depends only on coefficients. We conclude that $M_n \le C'' M_{n-1}^{3/2}$ where $C'' = (C'/\varepsilon)^{1/2}$ and so, by induction,

$$M_n \le (C'')^{1+3/2+\cdots+(3/2)^{n-1}} M_0^{(3/2)^n} \le [(C'')^2 M_0]^{(3/2)^n}.$$

The conclusion of the lemma follows. □

We now turn our attention to the dynamics of $H$ on $U^+$. We recall in this case that $I^+ = \{t = y = 0\}$, $I^- = \{t = z = 0\}$, and $H(\{t = 0\} \setminus I^+) \subset I^-$; also, on $I^- \setminus [1:0:0:0]$ with coordinate $u = x/y$, the map $H$ is given by $h(u) = \eta u + \eta'$, where $\eta = \alpha'/\beta$ and $\eta' = \alpha''/\beta$. Note that the map $h$ on $I^- = \mathbb{P}^1$ always has a fixed point at infinity, $[1:0:0:0] \in I^+$. If $\eta \ne 1$, or if $\eta = 1$ and $\eta' = 0$, then the map $h = H$ also has the fixed point $[u_0 : 1 : 0 : 0] \notin I^+$, where $u_0 = \alpha''/(\beta - \alpha')$ (resp. $u_0 = 0$). With this setting, we have the following theorems.

THEOREM 5.3. *If $\eta \ne 1$, or if $\eta = 1$ and $\eta' = 0$, then the sequence of functions $F_n(w) = (1/\eta^n)(x_n/y_n - u_0)$ converges locally uniformly on $U^+$ to a nonconstant holomorphic function $F$. We have $F \circ H = \eta F$ on $U^+$, $X = \{ w \in U^+ : F(w) = 0 \} \ne \emptyset$, and $H(X) = X$. The dynamics of $H$ on $U^+$ is as follows.*

(i) *If $|\eta| < 1$ then the iterates $H^n$ converge locally uniformly on $U^+$ to* $[u_0 : 1 : 0 : 0]$.
(ii) *If $|\eta| = 1$ then the iterates $H^n$ are a normal family on $U^+$, with limit functions of the form $w = [x : y : z : 1] \to [u_0 + e^{i\theta}F(w) : 1 : 0 : 0]$.*
(iii) *If $|\eta| > 1$, the iterates $H^n$ converge locally uniformly on $U^+ \setminus X$ to* $[1:0:0:0]$ *and locally uniformly along $X$ to* $[u_0 : 1 : 0 : 0]$. *In particular, $\{H^n\}_n$ is not a normal family on $U^+$.*

COROLLARY 5.4. *The current $\tilde{\mu}$ of integration along the analytic hypersurface $X \subset U^+$ satisfies $H^\star \tilde{\mu} = \tilde{\mu}$.*

*Proof*. This follows because $\tilde{\mu} = dd^c \log|F|$ and $F \circ H = \eta F$. □

THEOREM 5.5. *If $\eta = 1$ and $\eta' \ne 0$, then the sequence of functions $E_n(w) = x_n/ny_n$ converges locally uniformly on $U^+$ to $\eta'$. In particular, the iterates $H^n$ converge locally uniformly on $U^+$ to* $[1:0:0:0]$.

We postpone the proofs of Theorems 5.3 and 5.5 for the moment in order to discuss the Green's function $G^+$. Let

$$G^+(w) = \lim_{n\to\infty} \frac{1}{2^n} \log^+ \|H^n(w)\| = \lim_{n\to\infty} \frac{1}{2^n} \log^+ |y_n|.$$

PROPOSITION 5.6. *The above limits exist and are equal, the convergence being locally uniform on $\mathbb{C}^3$. We have $G^+ \in \mathrm{PSH} \cap C(\mathbb{C}^3)$, $G^+$ is pluriharmonic on $U^+$, $K^+ = \{G^+ = 0\}$, and $G^+ \circ H = 2G^+$. If $\mu^+ = dd^c G^+$ then $H^\star \mu^+ = 2\mu^+$.*

*Proof*. By Lemma 5.2, the limits are both zero on $K^+$; by Lemma 5.1, the second limit exists locally uniformly on $U^+$. By Theorems 5.3 and 5.5,

$$|x_n| \le [(|F(w)|+1)|\eta|^n + |u_0|]|y_n| \quad \text{or} \quad |x_n| \le (|\eta'|+1)n|y_n|$$

holds for $w \in U^+$, provided that $n$ is sufficiently large. This implies that the first limit exists on $U^+$ and is equal to the second. As in Lemma 4.3, we have $\|H^n(w)\| \le (C\|w\|_+)^{2^n}$ where $C$ is a constant, so the upper semicontinuous regularization $G_\star^+$ of $G^+$ satisfies $G_\star^+(w) \le \log^+\|w\| + \log C$. Using this together with $G_\star^+ \circ H = 2G_\star^+$ and Lemma 5.2, we obtain $G_\star^+ = 0$ on $K^+$. The proof now continues as for that of Theorem 2.5. □

*Proof of Theorem 5.3.* In order to obtain good estimates on $F_n$, we must consider instead of $V^-$ the set

$$W^- = \{\, w \in \mathbb{C}^3 : |y| > \max\{R, |z|, |x|^{2/3}\}\,\}.$$

LEMMA 5.7. *There exists $R_0 > 0$ such, that for any $R > R_0$, we have $H(W^-) \subseteq W^-$. The estimates*

$$\frac{|\beta|}{2}|y|^2 < |y_1| < \frac{3|\beta|}{2}|y|^2 \quad \text{and} \quad \left(\frac{|\beta|}{2}\right)^{2^n-1}|y|^{2^n} < |y_n| < \left(\frac{3|\beta|}{2}\right)^{2^n-1}|y|^{2^n}$$

*hold for $w \in W^-$ and $n \ge 1$. Moreover, $U^+ = \bigcup_{n=0}^{\infty} H^{-n}(W^-)$, where $U^+$ is defined in* (5.4).

*Proof*. The estimates and the invariance of $W^-$ are proved as in Lemma 5.1. The estimate on $|y_n|$ implies $\bigcup_{n=0}^{\infty} H^{-n}(W^-) \subseteq U^+$. We now take $w \notin \bigcup_{n=0}^{\infty} H^{-n}(W^-)$. Then $|y_n| \le M_n = \max\{R, |z_n|, |x_n|^{2/3}\}$ for all $n \ge 0$. Using $|z_n| = |y_{n-1}| \le M_{n-1}$, $|x_{n-1}| \le M_{n-1}^{3/2}$, and $|z_{n-1}| \le M_{n-1}$, we obtain $|x_n| \le C'M_{n-1}^{5/2}$ and hence $M_n \le C''M_{n-1}^{5/3}$. It follows that $M_n \le (CM_0)^{(5/3)^n}$, which implies $w \notin U^+$. □

We now continue with the proof of the theorem. For a fixed $m$, note that $F_n$ is holomorphic on $H^{-m}(W^-)$ if $n \ge m$ and that it satisfies $F_n \circ H = \eta F_{n+1}$. It suffices to show that the sequence $\{F_n\}$ converges locally uniformly on $W^-$. This implies that the sequence converges locally uniformly on $U^+$ to $F \in O(U^+)$ and $F \circ H = \eta F$, hence $H(X) = X$.

On $W^-$, let us write $u = x/y$ and $u_n = x_n/y_n$. Since $u_0$ is a fixed point for $h(u) = \eta u + \eta'$, we have $h(u) - u_0 = \eta(u - u_0)$. Using (5.1) and (5.3) (in which we write $p(x, y)$ for the terms of $P$ of degree lower than 2), we obtain

$$\frac{x_1}{y_1} = \frac{x_1}{\beta y^2}\,\frac{\beta y^2}{y_1} = \left[h\left(\frac{x}{y}\right) + \frac{p(x,y)}{\beta y^2}\right]\frac{\beta y^2}{y_1},$$

so

$$\begin{aligned} u_1 - u_0 &= \eta\frac{\beta y^2}{y_1}(u - u_0) + u_0\left(\frac{\beta y^2}{y_1} - 1\right) + \frac{p(x,y)}{y_1} \\ &= \eta A(w)(u - u_0) + B(w). \end{aligned} \tag{5.5}$$

We have $A, B \in O(W^-)$ and, by the definition of $W^-$ and Lemma 5.7,

$$|A(w) - 1| < C|y|^{-1/2}, \qquad |B(w)| < C|y|^{-1/2} \tag{5.6}$$

for some constant $C$ and for all $w \in W^-$. In particular, $A(w) \neq 0$ on $W^-$ if $R$ is chosen sufficiently large. We introduce the following notation:

$$\begin{gathered} A_j(w) = A \circ H^j(w) = A(w_j), \qquad B_j(w) = B \circ H^j(w), \\ A_0 = A, \qquad B_0 = B, \\ \Gamma_n(w) = A_0(w)A_1(w)\cdots A_n(w). \end{gathered} \tag{5.7}$$

Using (5.5) inductively we obtain, after a straightforward calculation, the following formula for $F_n$:

$$\begin{gathered} F_n(w) = \frac{1}{\eta^n}(u_n - u_0) = \Gamma_{n-1}(w)\tilde{F}_n(w), \quad \text{where} \\ \tilde{F}_n(w) = u - u_0 + f_n(w), \quad f_n(w) = \sum_{j=0}^{n-1} \frac{B_j(w)}{\eta^{j+1}\Gamma_j(w)}. \end{gathered} \tag{5.8}$$

LEMMA 5.8. *The sequences* $\{\Gamma_n\}$, $\{f_n\}$, $\{\tilde{F}_n\}$, *and hence* $\{F_n\}$, *converge uniformly on* $W^-$ *to the holomorphic functions* $\Gamma$, $f$, $\tilde{F}$, *and* $F$, *respectively. We have that* $\Gamma$ *is bounded and nowhere vanishing on* $W^-$, $\tilde{F}(w) = u - u_0 + f(w)$, *and* $F = \Gamma\tilde{F}$ *is not identically zero. Moreover,* $f$ *satisfies the estimate* $|f(w)| < C|y|^{-1/2}$ *for all* $w \in W^-$ *and for some constant* $C$.

*Proof*. By Lemma 5.7 we have (choosing $R$ sufficiently large) that

$$|y_n| > \left(\frac{|\beta|}{2}|y|\right)^{2^n} \frac{2}{|\beta|} > c^{2^n} \quad \text{for all } \ n \geq 0 \ \text{ and } \ w \in W^-,$$

with some constant $c \gg 1$. It follows from (5.6) that $|A_j(w) - 1| < Cc^{-2^{j-1}}$ for all $w \in W^-$ and $j \geq 0$. Hence $\{\Gamma_n\}$ converges uniformly to $\Gamma \in O(W^-)$, which satisfies

$$0 < \prod_{j=0}^{\infty}(1 - Cc^{-2^{j-1}}) < |\Gamma(w)| < \prod_{j=0}^{\infty}(1 + Cc^{-2^{j-1}}) < +\infty$$

on $W^-$. Using the estimate on $|y_j|$ from Lemma 5.7 and (5.6), we obtain

$$|B_j(w)| < C\left(\frac{|\beta|}{2}|y|\right)^{-2^{j-1}} < C|y|^{-1/2}\left(\frac{|\beta|}{2}|y|\right)^{-2^{j-2}} < C|y|^{-1/2}c^{-2^{j-2}}$$

for $j \geq 1$ and $w \in W^-$. This, together with the foregoing estimate on $|\Gamma|$, shows that $\{f_n\}$ converges uniformly to $f \in O(W^-)$ and $|f(w)| < C|y|^{-1/2}$ on $W^-$. To see that $F \not\equiv 0$, we choose $w \in W^-$ such that $x/y \neq u_0$. By the definition of $W^-$ we have $\lambda w \in W^-$ for all $\lambda \geq 1$. Since $\tilde{F}(\lambda w) = x/y - u_0 + f(\lambda w)$, the preceding estimate on $|f|$ implies that $\tilde{F}(\lambda w) \neq 0$ for $\lambda$ sufficiently large. $\square$

LEMMA 5.9. *There exists an open neighborhood $D$ of $[u_0 : 1 : 0 : 0]$ in $\mathbb{P}^3$, with $D \cap \mathbb{C}^3 \subset W^-$, such that $\tilde{F}$ extends holomorphically to $D$ by $\tilde{F}[u : 1 : v : 0] = u - u_0$.*

*Proof.* For $w = [x : y : z : 1] \in W^-$ we have $y \neq 0$, so we can change the coordinates to $[u : 1 : v : t]$, where $u = x/y$, $v = z/y$, and $t = 1/y$. In these coordinates we have

$$W^- = \{ [u : 1 : v : t] : t \neq 0, \max\{R|t|, |u|^{2/3}|t|^{1/3}, |v|\} < 1 \},$$

so we can find an open neighborhood $D \subset W^- \cup \{t = 0\}$ of $[u_0 : 1 : 0 : 0]$ in $\mathbb{P}^3$. By Lemma 5.8 we have, in the new coordinates, $|f(u, v, t)| < C|t|^{1/2}$ on $W^-$, so $f$ extends holomorphically to $D$ by $f(u, v, 0) = 0$. Hence $\tilde{F}(u, v, t) = u - u_0 + f(u, v, t)$ extends holomorphically to $D$ by $\tilde{F}(u, v, 0) = u - u_0$. □

COROLLARY 5.10. *$X \neq \emptyset$; hence $F$ is nonconstant.*

*Proof.* With the notation of Lemma 5.9, let $\tilde{X} = \{ w \in D : \tilde{F}(w) = 0 \}$. By Lemma 5.9, $\tilde{X} \cap \{t = 0\} = \{t = u - u_0 = 0\}$ has dimension 1, so $\tilde{X} \cap W^- \neq \emptyset$. As $F = \Gamma\tilde{F}$ on $W^-$, this shows $X \neq \emptyset$. □

REMARK. The point $[u_0 : 1 : 0 : 0]$ is fixed by $H$, and the derivative $H'[u_0 : 1 : 0 : 0]$—computed in the coordinates from the proof of Lemma 5.9—is a $3 \times 3$ upper triangular matrix with diagonal entries $\eta$, 0, and 0.

We now prove the assertions of Theorem 5.3 regarding the dynamics of $H$ on $U^+$. Parts (i) and (ii) follow if we write $H^n[x : y : z : 1] = [x_n/y_n : 1 : z_n/y_n : 1/y_n]$ and notice that $z_n/y_n, 1/y_n$ converge locally uniformly to zero on $U^+$, by Lemma 5.1. For (iii), we first fix $w \in U^+ \setminus X$, an open neighborhood $B \subset U^+ \setminus X$ of $w$, and a constant $c > 0$ such that $|F(w)| > c$ on $B$. It follows that $x_n/y_n \to \infty$, hence $H^n(w) \to [1 : 0 : 0 : 0]$ uniformly on $B$. Let now $w \in X$. By taking iterates of $w$, we may assume that $w \in W^-$. Since $F = \Gamma\tilde{F}$ and $\Gamma(w) \neq 0$, we have $\tilde{F}(w) = 0$, so $u - u_0 = -f(w)$. Using (5.8), we obtain

$$\begin{aligned} \frac{x_n}{y_n} - u_0 &= \eta^n \Gamma_{n-1}(w)\left[-f(w) + \sum_{j=0}^{n-1} \frac{B_j(w)}{\eta^{j+1}\Gamma_j(w)}\right] \\ &= -\Gamma_{n-1}(w) \sum_{j=n}^{\infty} \frac{B_j(w)}{\eta^{j-n+1}\Gamma_j(w)}. \end{aligned}$$

Proceeding as in the proof of the estimate on $|f|$ in Lemma 5.8, we conclude from this that $|x_n/y_n - u_0| = O(|y_n|^{-1/2})$ and hence $H^n(w) \to [u_0 : 1 : 0 : 0]$ locally uniformly along $X$. The proof of Theorem 5.3 is now complete. □

*Proof of Theorem 5.5.* It suffices to show that $\{E_n\}$ converges uniformly on $W^-$ to $\eta'$, where $W^-$ is as in Lemma 5.7. Writing again $u = x/y$ and $u_n = x_n/y_n$

and using $h(u) = u + \eta'$, we obtain in a similar way with (5.5) that $u_1 = A(w)u + \eta' A(w) + B(w)$, where $A, B \in O(W^-)$ satisfy the estimates (5.6). With the notation of (5.7), this gives

$$E_n(w) = \frac{u_n}{n} = \frac{1}{n}\Gamma_{n-1}(w)\left[u + \sum_{j=0}^{n-1} \frac{B_j(w)}{\Gamma_j(w)}\right] + \eta'\Gamma_{n-1}(w)\frac{1}{n}\left[1 + \sum_{j=0}^{n-2} \frac{1}{\Gamma_j(w)}\right].$$

As in the proof of Lemma 5.8, we have that $\Gamma_n \to \Gamma$ uniformly on $W^-$, $0 < c < |\Gamma(w)| < C$ on $W^-$, and $\sum_{j=0}^{n-1} B_j(w)/\Gamma_j(w) \to f(w)$ uniformly on $W^-$. These imply that $E_n \to \eta'$ uniformly on $W^-$. □

We now consider the dynamics of the inverse map $H^{-1}$ given in (5.2). We write $\tilde{P}(y, z) = \gamma z^3 + \tilde{p}(y, z)$, where $\gamma \neq 0$ and $\deg(\tilde{p}) = 2$. Note that $\deg(H^{-n}) = \deg(z_n) = 3^n$ for all $n \geq 1$. For $\varepsilon > 0$ and $R > 1/\varepsilon$, we let

$$V^+ = V^+_{\varepsilon,R} = \{\, w \in \mathbb{C}^3 : |z| > \max\{R, |y|, (|x|/\varepsilon)^{1/3}\}\,\}.$$

The following lemma is proved in a similar way as previous analogous result.

LEMMA 5.11. *For any* $\delta \in (0, 1)$, *there exist* $\varepsilon = \varepsilon(\delta) \in (0, 1)$ *and* $R_0(\delta) > 1/\varepsilon$ *such that, for any* $R > R_0$, *we have* $H^{-1}(V^+) \subseteq V^+$, *and the estimates*

$$|\beta|(1-\delta)|z|^2 < |x_1| < |\beta|(1+\delta)|z|^2,$$

$$|\gamma|(1-\delta)|z|^3 < |z_1| < |\gamma|(1+\delta)|z|^3, \qquad (C_1|z|)^{3^n} < |z_n| < (C_2|z|)^{3^n}$$

*hold on* $V^+$, *with constants* $C_1, C_2$ *depending on* $H^{-1}$ *and* $\delta$.

We define

$$U^- = \bigcup_{n=0}^{\infty} H^n(V^+), \qquad K^- = \mathbb{C}^3 \setminus U^-.$$

By Lemma 5.11, the iterates $H^{-n}(w)$ of points $w \in U^-$ escape to infinity at super-exponential rate $(\text{const})^{3^n}$, converging locally uniformly on $U^-$ to $[0 : 0 : 1 : 0]$. This time, $K^-$ is the set of points with bounded backward orbit, as follows.

LEMMA 5.12. *The iterates* $H^{-n}$ *are locally uniformly bounded on* $K^-$.

*Proof*. If $w \in K^-$ then $|z_n| \leq M_n = \max\{R, |y_n|, (|x_n|/\varepsilon)^{1/3}\}$ for all $n \geq 0$. As $|y_n| = |z_{n-1}| \leq M_{n-1}$ and $|y_{n-1}| \leq M_{n-1}$, we obtain by (5.2) that $|x_n| \leq C' M_{n-1}^2$; thus $M_n \leq \max\{R, M_{n-1}, CM_{n-1}^{2/3}\}$, where $C$ depends on coefficients and $\varepsilon$. This implies $M_n \leq \max\{R, C^3, M_{n-1}\}$ and hence, by induction, $M_n \leq \max\{R, C^3, M_0\}$. The conclusion follows. □

The Green's function $G^-$ of $H^{-1}$ is now defined by

$$G^-(w) = \lim_{n\to\infty} \frac{1}{3^n} \log^+ \|H^{-n}(w)\| = \lim_{n\to\infty} \frac{1}{3^n} \log^+ |z_n|.$$

It can be shown as before (see e.g. Proposition 5.6) that the convergence in the above limits is locally uniform on $\mathbb{C}^3$, $G^- \in \mathrm{PSH} \cap C(\mathbb{C}^3)$, $G^-$ is pluriharmonic on $U^-$, $K^- = \{G^- = 0\}$, and $G^- \circ H^{-1} = 3G^-$. Hence, if $\mu^- = dd^cG^-$ we have $H^\star\mu^- = \frac{1}{3}\mu^-$ and $\operatorname{supp}\mu^- = \partial K^-$.

*Case 2:* $\beta \neq 0$, $\alpha = \alpha' = 0$

With the notation of (5.2) and (5.3), we have $P(x,y) = \alpha'' y^2 + p(x,y)$, $Q(y) = \beta y^2 + q(y)$, and $\tilde{P}(y,z) = \gamma z^2 + \tilde{p}(y,z)$, where the degrees of $p, q, \tilde{p}$ are $\leq 1$. The maps are given by

$$\begin{aligned} H(x,y,z) &= (\alpha'' y^2 + p(x,y) + az, \beta y^2 + q(y) + x, y), \\ H^{-1}(x,y,z) &= \left(y - \beta z^2 - q(z), z, \frac{1}{a}x + \gamma z^2 + \tilde{p}(y,z)\right). \end{aligned} \tag{5.9}$$

If $\gamma = 0$ then the map $H^2$ is regular. Indeed, if $\tilde{p}_1 = \partial\tilde{p}/\partial y$, it is easy to check that $I^+(H^2) = \{t = y = 0\}$ and $I^-(H^2) = \{t = z = 0,\ x/a + \tilde{p}_1 y = 0\}$, so $I^+(H^2) \cap I^-(H^2) = \emptyset$.

Hence we assume $\gamma \neq 0$. Then the dynamics of both $H$ and $H^{-1}$ is similar to that of the maps $H_3$ of (4.1), and the methods used there work in this case as well. Briefly, the situation is as follows: For $\varepsilon > 0$ and $R > 1/\varepsilon$, for the map $H$ we define the sets

$$V^- = \{\, w \in \mathbb{C}^3 : |y| > \max\{R, |z|, (|x|/\varepsilon)^{1/2}\}\,\},$$

$$U^+ = \bigcup_{n=0}^{\infty} H^{-n}(V^-), \qquad K^+ = \mathbb{C}^3 \setminus U^+;$$

for the map $H^{-1}$ we define

$$V^+ = \{\, w \in \mathbb{C}^3 : |z| > \max\{R, |y|, (|x|/\varepsilon)^{1/2}\}\,\},$$

$$U^- = \bigcup_{n=0}^{\infty} H^{n}(V^+), \qquad K^- = \mathbb{C}^3 \setminus U^-.$$

As usual, for a given $\delta \in (0,1)$ we can find $\varepsilon \in (0,1)$ and $R_0 > 1/\varepsilon$ such that, for any $R > R_0$, we have $H(V^-) \subset V^-$ and $H^{-1}(V^+) \subset V^+$, and the following estimates hold:

$$\text{on } V^-\text{:}\ |\beta|(1-\delta)|y|^2 < |y_1| < |\beta|(1+\delta)|y|^2,\ |x_1| < (|\alpha''|+1)|y|^2,$$

$$\text{on } V^+\text{:}\ |\gamma|(1-\delta)|z|^2 < |z_1| < |\gamma|(1+\delta)|z|^2,\ |x_1| < (|\beta|+1)|z|^2.$$

Hence the forward iterates of points in $U^+$ escape to infinity at super-exponential rate, converging locally uniformly to $[\alpha'' : \beta : 0 : 0]$. The backward iterates of points in $U^-$ escape to infinity at super-exponential rate and converge locally uniformly to $[-\beta : 0 : \gamma : 0]$. On the other hand, one can show using the same techniques as before that the forward (resp. backward) orbits of points in $K^+$ (in

$K^-$, resp.) can escape to infinity at most at exponential rate. The Green's functions and the invariant currents are given by

$$G^+(w) = \lim_{n\to\infty} \frac{1}{2^n} \log^+\|H^n(w)\| = \lim_{n\to\infty} \frac{1}{2^n} \log^+|y_n|, \qquad \mu^+ = dd^c G^+,$$

$$G^-(w) = \lim_{n\to\infty} \frac{1}{2^n} \log^+\|H^{-n}(w)\| = \lim_{n\to\infty} \frac{1}{2^n} \log^+|z_n|, \qquad \mu^- = dd^c G^-.$$

They are continuous plurisubharmonic on $\mathbb{C}^3$, and pluriharmonic on $U^+$ (resp. $U^-$). Moreover, $K^\pm = \{G^\pm = 0\}$ and $G^\pm \circ H = 2^{\pm 1} G^\pm$.

*Case 3:* $\beta = 0$

For this case, it follows that $P(x, y) = \alpha x^2 + \alpha' xy + \alpha'' y^2 + p(x, y)$ and $\tilde{P}(y, z) = \gamma z^2 + \gamma' yz + \gamma'' y^2 + \tilde{p}(y, z)$, with the only restriction that $\deg(P) = 2$.

In the generic situation when $\alpha \neq 0$ and $\gamma \neq 0$, the dynamics is very similar to the one of the maps in (5.9). There is a slight difference in the choice of the sets $V^\pm$; in this case, we let

$$V^- = \{\, w \in \mathbb{C}^3 : |x| > \max\{R, |z|, |y|/\varepsilon\}\,\},$$

$$V^+ = \{\, w \in \mathbb{C}^3 : |z| > \max\{R, |x|, |y|/\varepsilon\}\,\}.$$

With $U^\pm$ and $K^\pm$ defined in the usual way, we have that the forward iterates $H^n$ converge super-exponentially on $U^+$ to $[1 : 0 : 0 : 0]$ and the backward iterates $H^{-n}$ converge super-exponentially on $U^-$ to $[0 : 0 : 1 : 0]$. Again, on $K^+$ and $K^-$ the forward (resp. backward) orbits can escape to infinity at most at exponential rate. The Green's functions $G^\pm$ are defined and have the same properties as in Case 2, with the only difference that this time $G^+(w) = \lim_{n\to\infty}(\log^+|x_n|)/2^n$.

If $\alpha = 0$ and $\alpha' \neq 0$ then it follows by induction that $\deg(y_n) < \deg(x_n)$ and $\deg(x_{n+1}) = \deg(x_n) + \deg(x_{n-1})$ for all $n \geq 1$. So the degrees of the forward iterates are given by Fibonacci's numbers (see [B]). The same holds for the inverse map, when $\gamma = 0$ and $\gamma' \neq 0$.

We finally look at the case when $\alpha = \alpha' = 0$ (so $\alpha'' \neq 0$, as $\deg(P) = 2$). Here we again have that $H^2$ is regular, since $I^+(H^2) = [0 : 0 : 1 : 0]$ and $I^-(H^2) = \{t = z = 0\}$.

## 6. The Class $H_5$

This class contains maps $H = H_5$ of the form

$$H(x, y, z) = (P(x, y) + az,\ Q(x) + by,\ x), \tag{6.1}$$

where $\max\{\deg(P), \deg(Q)\} = 2$ and $a \neq 0 \neq b$. The inverse map is given by

$$\begin{aligned} H^{-1}(x, y, z) &= \left(z,\ \frac{y - Q(z)}{b},\ \frac{x}{a} + \tilde{P}(y, z)\right), \\ \tilde{P}(y, z) &= -\frac{1}{a} P\left(z,\ \frac{y - Q(z)}{b}\right). \end{aligned} \tag{6.2}$$

We write

$$P(x,y)=\alpha x^2+\alpha' xy+\alpha'' y^2+\text{l.d.t.},\qquad Q(x)=\beta x^2+\text{l.d.t.} \tag{6.3}$$

The map $H$ is regular if and only if $\beta\neq 0$ and $\alpha''\neq 0$. In the study of the dynamics of maps $H$ that are not regular, we will consider several cases—of which the first is the most interesting.

*Case 1:* $\beta\neq 0$, $\alpha''=0$, $\alpha'\neq 0$

Let us begin by discussing the dynamics of the inverse map $H^{-1}$. This is similar to the dynamics of the map $H^{-1}$ in (5.2), of the corresponding case for the class $H_4$, so we will state the results without providing the proofs. We write $\tilde{P}(y,z)=\gamma z^3+\tilde{p}(y,z)$, where $\gamma\neq 0$, $\deg(\tilde{p})=2$, and $\tilde{p}$ does not contain $y^2$. Then $\deg(H^{-n})=3^n$ for all $n\geq 1$. For $\varepsilon>0$ and $R>1/\varepsilon$, we let

$$V^+=V^+_{\varepsilon,R}=\{\,w\in\mathbb{C}^3 : |z|>\max\{R,|x|,(|y|/\varepsilon)^{1/2}\}\,\}.$$

LEMMA 6.1. *For any $\delta\in(0,1)$, there exist $\varepsilon=\varepsilon(\delta)\in(0,1)$ and $R_0(\delta)>1/\varepsilon$ such that, for any $R>R_0$, we have $H^{-1}(V^+)\subseteq V^+$, and the estimates*

$$\frac{|\beta|}{|b|}(1-\delta)|z|^2<|y_1|<\frac{|\beta|}{|b|}(1+\delta)|z|^2,$$

$$|\gamma|(1-\delta)|z|^3<|z_1|<|\gamma|(1+\delta)|z|^3,\qquad (C_1|z|)^{3^n}<|z_n|<(C_2|z|)^{3^n}$$

*hold on $V^+$, with constants $C_1$, $C_2$ depending on $H^{-1}$ and $\delta$.*

We define

$$U^-=\bigcup_{n=0}^{\infty}H^n(V^+),\qquad K^-=\mathbb{C}^3\setminus U^-.$$

By Lemma 6.1, the $H^{-1}$-orbits of points in $U^-$ escape to infinity at super-exponential rate, converging locally uniformly on $U^-$ to $[0:0:1:0]$. On $K^-$ we have the following behavior.

LEMMA 6.2. *There exist both a positive continuous function $M$ on $\mathbb{C}^3$ and a constant $C>1$ such that* $\max\{|x_n|,|y_n|,|z_n|\}\leq C^nM(w)$ *holds for any $w\in K^-$ and $n\geq 0$.*

The Green's function $G^-$ of $H^{-1}$ is given by

$$G^-(w)=\lim_{n\to\infty}\frac{1}{3^n}\log^+\|H^{-n}(w)\|=\lim_{n\to\infty}\frac{1}{3^n}\log^+|z_n|.$$

The convergence in the above limits is locally uniform on $\mathbb{C}^3$, $G^-\in\text{PSH}\cap C(\mathbb{C}^3)$, $G^-$ is pluriharmonic on $U^-$, $K^-=\{G^-=0\}$, and $G^-\circ H^{-1}=3G^-$. If $\mu^-=dd^cG^-$ we have $H^\star\mu^-=\frac{1}{3}\mu^-$ and $\operatorname{supp}\mu^-=\partial K^-$.

We now consider the dynamics of $H$. By a change of coordinates of the form $(x, y, z) \to (a'x, b'y, a'z)$, we may assume that $H$ is given by

$$\begin{aligned} x_1 &= \alpha x^2 + xy + p(x, y) + az, \\ y_1 &= x^2 + q(x) + by, \\ z_1 &= x, \end{aligned} \tag{6.4}$$

where $\deg(p), \deg(q) \le 1$. The indeterminacy sets of $H$ are $I^+ = \{t = x = 0\}$ and $I^- = \{t = z = 0\}$. We have $H(\{t = 0\} \setminus I^+) \subseteq I^-$; on $I^-$, with coordinate $u = y/x$, the map is given by

$$H[1 : u : 0 : 0] = [1 : h(u) : 0 : 0], \quad h(u) = \frac{1}{u + \alpha}.$$

It is easy to check that, when $\alpha = \pm 2i$, the Möbius map $h$ has a double fixed point given by $u_0 = -\alpha/2 = \mp i$. If $\alpha \neq \pm 2i$ then $h$ has distinct fixed points $u_0, u_0'$, and we have $h'(u_0) = -u_0^2$ and $h'(u_0') = -(u_0')^2$. If $\Re\alpha = 0$ and $\Im\alpha \in (-2, 2)$ then $|u_0| = |u_0'| = 1$, and both fixed points are neutral; otherwise, $|u_0| < 1 < |u_0'|$, $u_0$ is attracting, and $u_0'$ is repelling.

We now assume $\alpha \neq \pm 2i$. Note that the change of coordinates $u' = (u - u_0)/(u - u_0')$ puts $h$ into the form $\tilde{h}(u') = (-u_0^2)u'$. By a projective change of coordinates

$$w' = S(w) : [x' : y' : z' : t'] = [a'(y - u_0'x) : a'(y - u_0x) : b'z : t]$$

with inverse

$$w = S^{-1}(w') : x = \frac{x' - y'}{a'(u_0 - u_0')}, \ y = \frac{u_0x' - u_0'y'}{a'(u_0 - u_0')}, \ z = \frac{z'}{b'}$$

(where $a', b'$ are suitably chosen), the map $H$ is conjugated to the map $\tilde{H}$ given by

$$\begin{aligned} x_1' &= x'(x' - y') + \tilde{p}(x', y', z'), \\ y_1' &= \eta y'(x' - y') + \tilde{q}(x', y', z'), \\ z_1' &= x' - y', \end{aligned} \tag{6.5}$$

where $\deg(\tilde{p}), \deg(\tilde{q}) \le 1$, $\tilde{p}, \tilde{q}$ both contain $z'$, and $\eta = -u_0^2 = h'(u_0)$. We have $0 < |\eta| \le 1$ and $\eta \neq 1$ (otherwise $u_0^2 = -1$ and $u_0 = u_0' = \pm i$).

Let us denote by $I_\infty^+ = \bigcup_{n=0}^\infty \tilde{H}^{-n}(I^+)$ the extended indeterminacy set of $\tilde{H}$, that is, the minimal set away from which all the forward iterates $\tilde{H}^n$ are well-defined. The new phenomenon occurring for this class is that $I_\infty^+$ is larger than $I^+$. In fact, in the new coordinates we have $I^+ = \{t' = 0, \ x' = y'\}$ while $I_\infty^+ = \bigcup_{j=0}^\infty \{t' = 0, \ x' = \eta^j y'\}$.

We study the dynamics of maps $H$ in (6.4) that correspond to the rationally neutral case when $\eta$ is a primitive root of unity of order $k \ge 2$: $\eta^k = 1$. For instance, $\alpha = 0$ corresponds to $\eta = -1$. In this case $I_\infty^+ = \bigcup_{j=0}^{k-1} \{t' = 0, x' = \eta^j y'\}$. We will construct the dynamically relevant sets $V^-$, $U^+$, $K^+$ for the map $H$ by first defining suitable sets for the map $\tilde{H}$ in (6.5), in the $w'$-coordinates, and then by using the transformation $S^{-1}$. For this purpose it is useful to write $S^{-1}$ in terms of $\eta$:

$$S^{-1}(w') = w : x = a''(x' - y'),\ y = b''(x' - \eta^{k-1}y'),\ z = z'/b', \tag{6.6}$$

where $|a''| = |b''|$. Here we used $u_0' = -1/u_0$, $\eta = -u_0^2$, and $\eta^k = 1$.

For $\varepsilon > 0$ and $R > 1/\varepsilon$ we let

$$\left.\begin{aligned} \tilde{V}^- &= \Big\{ w' \in \mathbb{C}^3 : \min_{j=0,\dots,k-1} |x' - \eta^j y'| \\ &\qquad\qquad > \max\{R, |z'|, (|x'|/\varepsilon)^{1/2}, (|y'|/\varepsilon)^{1/2}\} \Big\}, \\ \tilde{U}^+ &= \bigcup_{n=0}^{\infty} \tilde{H}^{-n}(\tilde{V}^-), \qquad \tilde{K}^+ = \mathbb{C}^3 \setminus \tilde{U}^+. \end{aligned}\right\} \tag{6.7}$$

PROPOSITION 6.3. (i) *For any* $\delta \in (0,1)$, *there exists* $\varepsilon = \varepsilon(\delta)$ *and* $R_0 = R_0(\delta) > 1/\varepsilon$ *such that, for all* $R > R_0$, *we have* $\tilde{H}(\tilde{V}^-) \subseteq \tilde{V}^-$, *and the estimates*

$$(1-\delta)|x' - \eta^{j+1}y'||x' - y'| < |x_1' - \eta^j y_1'| < (1+\delta)|x' - \eta^{j+1}y'||x' - y'| \tag{6.8}$$

*hold for all* $w' \in \tilde{V}^-$ *and* $j \in \{0, \dots, k-1\}$.

(ii) *There exists a positive continuous function* $M$ *on* $\mathbb{C}^3$ *such that, for all* $w' \in \tilde{K}^+$ *and all* $n \geq 0$, *we have* $\max\{|x_n'|, |y_n'|, |z_n'|\} \leq [M(w')]^{\nu^n}$, *where* $\nu = 2\left(\frac{3}{4}\right)^{1/k} \in (\frac{3}{2}, 2)$.

*Proof*. (i) By (6.5), for a fixed $j \in \{0, \dots, k-1\}$ we have

$$x_1' - \eta^j y_1' = (x' - \eta^{j+1}y')(x' - y') + \text{l.d.t.}$$

Because, on $\tilde{V}^-$, $|x'| < \varepsilon|x' - \eta^{j+1}y'|^2$ and $|x'| < \varepsilon|x' - y'|^2$, we conclude that $|x'| < \varepsilon|x' - \eta^{j+1}y'||x' - y'|$, and the same holds for $|y'|$. Moreover, $|z'| < |x' - y'| < \varepsilon|x' - y'||x' - \eta^{j+1}y'|$, since $R > 1/\varepsilon$. Thus

$$|(x_1' - \eta^j y_1') - (x' - \eta^{j+1}y')(x' - y')| < C\varepsilon|x' - y'||x' - \eta^{j+1}y'|,$$

where $C$ depends on coefficients, which yields (6.8) if we let $\varepsilon \leq \delta/C$. Using $\max\{|x'|, |y'|\} < \varepsilon|x' - \eta^{j+1}y'|^2$ and (6.5), we get

$$\max\{|x_1'|, |y_1'|\} < C\varepsilon|x' - \eta^{j+1}y'|^2|x' - y'|,$$

so if $R$ is sufficiently large then by (6.8) we have

$$\begin{aligned} \max\{(|x_1'|/\varepsilon)^{1/2}, (|y_1'|/\varepsilon)^{1/2}\} &< C|x' - \eta^{j+1}y'||x' - y'|^{1/2} \\ &< (1-\delta)|x' - \eta^{j+1}y'||x' - y'| < |x_1' - \eta^j y_1'|. \end{aligned}$$

Since $|z_1'| = |x' - y'| < (1-\delta)|x' - \eta^{j+1}y'||x' - y'|$, it follows by combining all these that $\tilde{H}(\tilde{V}^-) \subseteq \tilde{V}^-$.

(ii) If $w' \in \tilde{K}^+$ then for all $n \geq 0$ we have

$$\min_{j=0,\dots,k-1} |x_n' - \eta^j y_n'| \leq M_n = \max\{R, |z_n'|, (|x_n'|/\varepsilon)^{1/2}, (|y_n'|/\varepsilon)^{1/2}\}. \tag{6.9}$$

As in the proof of (6.8), and since $\eta^k = 1$, we have the following implications:

$$\begin{aligned} |x' - y'| > M_0 \;\&\; |x' - \eta^j y'| > M_0 &\Rightarrow |x'_1 - \eta^{j-1} y'_1| > M_1, \\ |x' - y'| > M_0 &\Rightarrow |x'_1 - \eta^{k-1} y'_1| > M_1. \end{aligned} \tag{6.10}$$

We fix $w' \in \tilde{K}^+$ and claim that, for any $n \geq k$, there exists $p = p(n) \in \{1, \ldots, k\}$ such that

$$|x'_{n-p} - y'_{n-p}| \leq M_{n-p}. \tag{6.11}$$

Indeed, assume by way of contradiction that, for any $p \in \{1, \ldots, k\}$, (6.11) does not hold. Since $|x'_{n-p} - y'_{n-p}| > M_{n-p}$, it follows by (6.10) that

$$|x'_{n-p+1} - \eta^{k-1} y'_{n-p+1}| > M_{n-p+1}.$$

This, together with $|x'_{n-p+1} - y'_{n-p+1}| > M_{n-p+1}$ and (6.10), implies that

$$|x'_{n-p+2} - \eta^{k-2} y'_{n-p+2}| > M_{n-p+2}.$$

Continuing like this we conclude $|x'_{n-1} - \eta^{k-p+1} y'_{n-1}| > M_{n-1}$ for all $p \in \{1, \ldots, k\}$. Since this contradicts (6.9), our claim follows.

Let now $n \geq k$ and $p = p(n) \in \{1, \ldots, k\}$ such that (6.11) holds. Then

$$M_n \leq C M_{n-p}^{3(2^{p-2})}, \tag{6.12}$$

where $C > 1$ is a constant depending on $H$. Indeed, by the definition of $M_j$, we have $|z_j| \leq M_j$ and $|x_j|, |y_j| \leq \varepsilon M_j^2$. Using these together with (6.5) and (6.11), we have

$$\max\{|x'_{n-p+1}|, |y'_{n-p+1}|\} \leq C\varepsilon M_{n-p}^3, \qquad |z'_{n-p+1}| \leq M_{n-p}.$$

Since $\|\tilde{H}^n(w')\| \leq (C\|w'\|_+)^{2^n}$, it follows for all $j \in \{1, \ldots, p\}$ that

$$\|\tilde{H}^{n-j+1}(w')\| = \|\tilde{H}^{p-j}(w'_{n-p+1})\| \leq (C\|w'_{n-p+1}\|_+)^{2^{p-j}};$$

hence $\max\{|x'_n|, |y'_n|\} \leq (C\varepsilon)^{2^{p-1}} M_{n-p}^{3(2^{p-1})}$. Moreover, if $p = 1$ then $|z'_n| = |x'_{n-1} - y'_{n-1}| \leq M_{n-1}$, and if $p \geq 2$ then $|z'_n| \leq |x'_{n-1}| + |y'_{n-1}| \leq (C\varepsilon)^{2^{p-2}} M_{n-p}^{3(2^{p-2})}$. These give (6.12).

We let $\tilde{M}(w') = \max\{M_0(w'), \ldots, M_{k-1}(w')\}$. The assertion of part (ii) follows if we show that, for all $n \geq 0$,

$$M_n \leq [C^4 \tilde{M}^{4/3}]^{\nu^n}, \tag{6.13}$$

where $C$ is the constant in (6.12). This clearly holds for $n < k$, since $\nu > 1$. If $n \geq k$ then we apply (6.12) repeatedly to get

$$M_n \leq C^{1+r_1+(r_1 r_2)+\cdots+(r_1 r_2 \ldots r_{l-1})} (M_{n-p_1-\cdots-p_l})^{r_1 r_2 \cdots r_l}, \tag{6.14}$$

where $r_j = 3(2^{p_j-2})$ and $p_j \in \{1, \ldots, k\}$ are such that $n - p_1 - p_2 - \cdots - p_{l-1} \geq k$ and $0 \leq n - p_1 - p_2 - \cdots - p_l < k$. Hence $n - k < p_1 + \cdots + p_l \leq lk$, so $l > (n/k) - 1$. Moreover, $p_1 + \cdots + p_{l-1} + p_l \leq n - k + k = n$. We conclude that

$$r_1 r_2 \dots r_l = \left(\frac{3}{4}\right)^l 2^{p_1+\cdots+p_l} \le \left(\frac{3}{4}\right)^{(n/k)-1} 2^n = \frac{4}{3}\nu^n.$$

Since $r_j \ge 3/2$, it is easy to check by induction that $1+r_1+r_1r_2\dots+r_1r_2\dots r_l \le 3r_1r_2\dots r_l$, so (6.14) implies $M_n \le (C^3\tilde{M})^{r_1r_2\cdots r_l}$. This gives (6.13), using our estimate on $r_1r_2\dots r_l$. □

We now return to the dynamics of the map $H$ in (6.4) and define

$$\begin{aligned} V^- &= S^{-1}(\tilde{V}^-), \\ U^+ &= \bigcup_{n=0}^{\infty} H^{-n}(V^-) = S^{-1}(\tilde{U}^+), \\ K^+ &= \mathbb{C}^3 \setminus U^+ = S^{-1}(\tilde{K}^+), \end{aligned} \tag{6.15}$$

where $\tilde{V}^-$ is given in (6.7) and $\delta$, $\varepsilon$, $R$ are chosen as in Proposition 6.3. The above identities hold since $H = S^{-1}\circ\tilde{H}\circ S$.

PROPOSITION 6.4. (i) *There exists a positive continuous function $M$ on $\mathbb{C}^3$ such that, for all $w \in K^+$ and all $n \ge 0$, we have* $\max\{|x_n|,|y_n|,|z_n|\} \le [M(w)]^{\nu^n}$, *where* $\nu = 2(\frac{3}{4})^{1/k} < 2$.

(ii) *We have $H(V^-) \subseteq V^-$ and $C$* $\min\{|x_n|,|y_n|\} > [(1-\delta)R]^{2^n}$ *for all $w \in V^-$ and $n \ge 0$, where $C^{-1} = |a''| = |b''|$ and $a''$, $b''$ are given in* (6.6). *In particular, the $H$-orbits of points in $U^+$ escape to infinity at super-exponential rate* $(\text{const})^{2^n}$.

(iii) *The following estimates hold for all $w \in V^-$ and all $n \ge k$:* $|z_1| < C_1|y_1|$,

$$C(1-\delta)|x|^2 < |y_1| < C(1+\delta)|x|^2, \qquad \max\{|x_1|,|y_1|\} < C_1(\max\{|x|,|y|\})^2,$$

$$\begin{aligned} C^k(1-\delta)^k|x_{n-1}|\dots|x_{n-k+1}||x_{n-k}|^2 &< |x_n| \\ &< C^k(1+\delta)^k|x_{n-1}|\dots|x_{n-k+1}||x_{n-k}|^2, \end{aligned}$$

$$C^k(1-\delta)^k|x_{n-1}|\dots|x_{n-k}||y_{n-k}| < |y_n| < C^k(1+\delta)^k|x_{n-1}|\dots|x_{n-k}||y_{n-k}|,$$

$$\frac{|x_n|}{|y_n|} < C_1^{n/k}\max_{j=0,\dots,k-1}\frac{|x_j|}{|y_j|}, \qquad \frac{|y_n|}{|x_n|} < C_1^{n/k}\max_{j=0,\dots,k-1}\frac{|y_j|}{|x_j|},$$

*where $C$ is as in part* (ii) *and $C_1 > 1$ is a constant depending on coefficients.*

*Proof*. (i) This follows directly from Proposition 6.3, the formula (6.6) of $S^{-1}$, and the definition (6.15) of $K^+$.

(ii) The $H$-invariance of $V^-$ follows directly from the $\tilde{H}$-invariance of $\tilde{V}^-$. Using (6.8) repeatedly we see that $|x'_n - y'_n|$ and $|x'_n - \eta^{k-1}y'_n|$ are larger than $(1-\delta)^{2^n-1}$ times a product of $2^n$ factors of the form $|x' - \eta^j y'|$, which are each larger than $R$. Since $|x_n| = C^{-1}|x'_n - y'_n|$ and $|y_n| = C^{-1}|x'_n - \eta^{k-1}y'_n|$, this gives the estimate in part (ii).

(iii) As in (ii), relation (6.8) applied for $j = k-1$ yields the estimate for $|y_1|$. The estimate for $|z_1|$ is then trivial, since $|z_1| = |x| < |x|^2$. The third estimate follows from (6.4), since $\min\{C|x|, C|y|\} > \max\{R, |z|\}$ holds on $V^-$. Using (6.8), we have

$$\begin{aligned}|x'_n - y'_n| &\sim |x'_{n-1} - y'_{n-1}||x'_{n-1} - \eta y'_{n-1}| \\ &\sim |x'_{n-1} - y'_{n-1}||x'_{n-2} - y'_{n-2}||x'_{n-2} - \eta^2 y'_{n-2}| \sim \cdots \\ &\sim |x'_{n-1} - y'_{n-1}| \ldots |x'_{n-k} - y'_{n-k}||x'_{n-k} - \eta^k y'_{n-k}|.\end{aligned}$$

Since $\eta^k = 1$, this yields the fourth inequality of (iii). Similarly,

$$\begin{aligned}|x'_n - \eta^{k-1} y'_n| &\sim |x'_{n-1} - y'_{n-1}|^2 \\ &\sim |x'_{n-1} - y'_{n-1}| \ldots |x'_{n-k} - y'_{n-k}||x'_{n-k} - \eta^{k-1} y'_{n-k}|,\end{aligned}$$

which yields the estimate on $|y_n|$. Using the estimates on $|x_n|$ and $|y_n|$, we obtain

$$\left(\frac{1-\delta}{1+\delta}\right)^k \frac{|x_{n-k}|}{|y_{n-k}|} < \frac{|x_n|}{|y_n|} < \left(\frac{1+\delta}{1-\delta}\right)^k \frac{|x_{n-k}|}{|y_{n-k}|}.$$

This implies the remaining inequalities of the proposition. □

We define the Green's function $G^+$ of $H$ by

$$G^+(w) = \lim_{n\to\infty} \frac{1}{2^n} \log^+ \|H^n(w)\| = \lim_{n\to\infty} \frac{1}{2^n} \log^+|x_n| = \lim_{n\to\infty} \frac{1}{2^n} \log^+|y_n|.$$

THEOREM 6.5. *The above limits exist and are equal, the convergence being locally uniform on* $\mathbb{C}^3$. *We have* $G^+ \in \mathrm{PSH} \cap C(\mathbb{C}^3)$, $G^+$ *is pluriharmonic on* $U^+$, $K^+ = \{G^+ = 0\}$, *and* $G^+ \circ H = 2G^+$. *If* $\mu^+ = dd^c G^+$ *then* $H^\star \mu^+ = 2\mu^+$.

*Proof*. By Proposition 6.4(i), the limits are zero on $K^+$. By the last two inequalities of Proposition 6.4(iii), it follows that if one of the three limits exists at $w \in U^+$ then the other two also exist and they are all equal. Let us write $G_n(w) = (\log^+|x_n|)/2^n$ and fix $w_0 \in V^-$ and a relatively compact open ball $B \subset V^-$ centered at $w_0$. By Proposition 6.4(iii) for $w \in B$ we have $|y_n| < C_1^n |x_n|$ for some constant $C_1 > 1$, so

$$G_{n+1}(w) \le \frac{1}{2^{n+1}} \log^+(C_1 \max\{|x_n|, |y_n|\})^2 \le \frac{n+1}{2^n} \log C_1 + G_n(w).$$

If we define $m_{n+1} = 2^{-n}(n+3)^2 \log C_1$, then $m_{n+1} + (n+1)2^{-n} \log C_1 \le m_n$ for all $n \ge 1$, so $\{G_n + m_n\}$ is a decreasing sequence of positive pluriharmonic functions on $B$. It follows by Harnack's theorem that the functions $G_n$ converge locally uniformly to a positive pluriharmonic function on $B$. Hence $G^+$ is well-defined on $\mathbb{C}^3$ and is pluriharmonic and positive on $U^+$. If $G^+_\star$ is the upper semicontinuous regularization of $G^+$ then, using Proposition 6.4(i) and $G^+_\star \circ H = 2G^+_\star$, it follows that $G^+_\star = 0$ on $K^+$. The proof now continues in the standard way. □

We now discuss the dynamics of $H$ on $U^+$. Recall that, on $I^-$ and with coordinate $u = y/x$, $H$ is given by $h(u) = 1/(u + \alpha)$.

THEOREM 6.6. *The sequence of functions* $F_n(w) = y_{kn}/x_{kn}$ *converges locally uniformly on* $U^+$ *to a nonconstant nowhere-vanishing holomorphic function* $F$, *which satisfies* $F \circ H = h \circ F$ *and, in particular,* $F \circ H^k = F$. *Moreover, for any* $r \in \{0, \dots, k-1\}$, *the sequence of iterates* $\{H^{kn+r}\}_n$ *converges locally uniformly on* $U^+ \subset \mathbb{P}^3$ *to the function* $w = [x : y : z : 1] \to [1 : h^r \circ F(w) : 0 : 0]$.

*Proof.* Instead of the set $\tilde{V}^-$ in (6.7) we must consider the set

$$\tilde{W}^- = \left\{ w' \in \mathbb{C}^3 : \min_{j=0,\dots,k-1} |x' - \eta^j y'| > \max\{R, |z'|, |x'|^{2/3}, |y'|^{2/3}\} \right\}.$$

LEMMA 6.7. *If* $R$ *is sufficiently large then* $\tilde{H}(\tilde{W}^-) \subseteq \tilde{W}^-$ *and, for all* $w' \in \tilde{W}^-$ *and* $j \in \{0, \dots, k-1\}$, *we have*

$$x_1' - \eta^j y_1' = (x' - y')(x' - \eta^{j+1} y')[1 + O(|x' - y'|^{-1/4} |x' - \eta^{j+1} y'|^{-1/4})]. \quad (6.16)$$

*Moreover,* $\tilde{U}^+ = \bigcup_{n=0}^\infty \tilde{H}^{-n}(\tilde{W}^-)$, *where* $\tilde{U}^+$ *is as in* (6.7).

*Proof.* The first part of the lemma follows as in the proof of Proposition 6.3. Note that repeated use of (6.8) implies $[(1-\delta)R]^{2^n} < |x_n' - y_n'| \le 2\max\{|x_n'|, |y_n'|\}$ and so, by Proposition 6.3, $\tilde{U}^+$ is the set where the iterates $\tilde{H}^n$ escape to infinity at the highest super-exponential rate. Hence, by (6.16), $\bigcup_{n=0}^\infty \tilde{H}^{-n}(\tilde{W}^-) \subseteq \tilde{U}^+$. For the reversed inclusion, we show that there exists a positive continuous function $M$ on $\mathbb{C}^3$ such that, if $w' \notin \bigcup_{n=0}^\infty \tilde{H}^{-n}(\tilde{W}^-)$ and $n \ge 0$, then $\max\{|x_n'|, |y_n'|, |z_n'|\} \le [M(w')]^{\nu_1^n}$, where $\nu = 2\left(\frac{5}{6}\right)^{1/k} < 2$. This is done by the same arguments as in the proof of Proposition 6.3(ii): For all $n \ge 0$, we have

$$\min_{j=0,\dots,k-1} |x_n' - \eta^j y_n'| \le M_n = \max\{R, |z_n'|, |x_n'|^{2/3}, |y_n'|^{2/3}\}.$$

Implications (6.10) and inequality (6.11) hold in this setting. Inequality (6.12) becomes $M_n \le C M_{n-p}^{(5/6)2^p}$, where $n \ge k$ and $p = p(n) \in \{1, \dots, k\}$. Inequality (6.14) holds with $p_1, \dots, p_l$ as before, and $r_j = \frac{5}{6} 2^{p_j}$. Hence

$$r_1 r_2 \dots r_l = \left(\frac{5}{6}\right)^l 2^{p_1 + \dots + p_l} \le \frac{6}{5} \nu_1^n,$$

and the conclusion follows. □

LEMMA 6.8. *Let* $W^- = S^{-1}(\tilde{W}^-)$.

(i) *We have* $H(W^-) \subseteq W^-$ *and* $U^+ = \bigcup_{n=0}^\infty H^{-n}(W^-)$, *where* $U^+$ *is as in* (6.15). *Moreover,* $\min\{|x_n|, |y_n|\} > (R')^{2^n}$ *holds on* $W^-$ *for some constant* $R' \gg 1$, *and* $y_n/x_n^2$ *and* $1/x_n$ *converge to zero locally uniformly on* $W^-$.

(ii) *If* $w \in W^-$ *then* $y_k/x_k = (y/x)A(w)$, *where* $A \in O(W^-)$ *satisfies* $|A(w) - 1| < C(|x|^{-1/2} + |y|^{-1/2})$ *on* $W^-$ *for some constant* $C$.

*Proof.* (i) These follow directly from Lemma 6.7, Proposition 6.4, and the estimate $\|H^n(w)\| \le (C\|w\|_+)^{2^n}$.

(ii) By (6.16) and since $\eta^k = 1$, we have the following:

$$\frac{x'_k - y'_k}{(x'_{k-1} - y'_{k-1})(x'_{k-1} - \eta y'_{k-1})} = 1 + O(|x'_{k-1} - y'_{k-1}|^{-1/4}|x'_{k-1} - \eta y'_{k-1}|^{-1/4}),$$

$$\frac{x'_{k-1} - \eta y'_{k-1}}{(x'_{k-2} - y'_{k-2})(x'_{k-2} - \eta^2 y'_{k-2})} = 1 + O(|x'_{k-2} - y'_{k-2}|^{-1/4}|x'_{k-2} - \eta^2 y'_{k-2}|^{-1/4}),$$

$$\vdots$$

$$\frac{x'_1 - \eta^{k-1} y'_1}{(x' - y')^2} = 1 + O(|x' - y'|^{-1/2}).$$

By (6.6), $x_j = a''(x'_j - y'_j)$, so multiplying the foregoing identities yields

$$\frac{(a'')^k x_k}{x_{k-1} \dots x_1 x^2} = 1 + O(|x|^{-1/2}).$$

Here we used the fact that, in view of (6.16), $|x'_j - \eta^l y'_j|$ is larger than a degree-$2^j$ product of factors of the form $|x' - \eta^m y'|$, in which $|x' - y'|$ has degree $2^{j-1}$. Thus $|x'_j - \eta^l y'_j|^{-1/4} \ll |x' - y'|^{-1/4}$. In a similar way, we obtain

$$\frac{(a'')^k y_k}{x_{k-1} \dots x_1 x y} = 1 + O(|x|^{-1/2} + |y|^{-1/2}).$$

These yield part (ii) of the lemma. □

We now proceed with the proof of Theorem 6.6. By Lemma 6.8, it suffices to show that $\{F_n\}$ converges locally uniformly on $W^-$. Let $A_n(w) = A(w_{kn})$, $A_0 = A$, and $\Gamma_n = A_0 A_1 \dots A_{n-1}$. Then the estimates of Lemma 6.8 imply that $\{\Gamma_n\}$ converges uniformly to $\Gamma \in O(W^-)$, which satisfies $C_1 < |\Gamma(w)| < C_2$ for all $w \in W^-$ ($C_1, C_2$ are positive constants). By Lemma 6.8(ii), $F_n(w) = F_{n-1}(w)A(w_{k(n-1)}) = F_{n-1}(w)A_{n-1}(w)$, hence by induction $F_n(w) = (y/x)\Gamma_n(w)$ holds on $W^-$. So $\{F_n\}$ converges locally uniformly on $U^+$ to $F \in O(U^+)$, and $F(w) = (y/x)\Gamma(w)$ holds on $W^-$. We have $F_n \circ H^k = F_{n+1}$ and so, on $U^+$, $F \circ H^k = F$. Since $C_1 < |\Gamma| < C_2$ it follows that $F$ is nowhere vanishing on $W^-$, hence on $U^+$.

Note that, for $\tau \in (0, 1/3)$, $W^-$ contains points $w = (x, y, 0)$ with $|y| = |x|^{1-\tau}$, provided that $|x|$ is sufficiently large. Indeed, using (6.6) to write $x'$, $y'$, and $x' - \eta^j y'$ in terms of $x$, $y$, we have that

$$\max\{|x'|, |y'|\} < (\text{const})|x|, \qquad \min_{j=0,\dots,k-1} |x' - \eta^j y'| > (\text{const})|y|,$$

so $w' = S(w) \in \tilde{W}^-$. For such points $w \in W^-$ we have $y/x \to 0$ as $|x| \to \infty$; and since $C_1 < |\Gamma| < C_2$, it follows that $F(w) = (y/x)\Gamma(w)$ cannot be constant.

By Lemma 6.8 and (6.4), for $w \in W^-$ we have

$$F_n \circ H(w) = \frac{y_{kn+1}}{x_{kn+1}} = \frac{1 + o(1)}{F_n(w) + \alpha + o(1)}.$$

As $n \to \infty$, this implies that $F \circ H = h \circ F$ holds on $U^+$ and hence $F \circ H^r = h^r \circ F$ for all $r \in \{1, \dots, k\}$. We conclude that

$$H^{kn+r}(w) = [1 : y_{kn+r}/x_{kn+r} : z_{kn+r}/x_{kn+r} : 1/x_{kn+r}] \to [1 : h^r \circ F(w) : 0 : 0]$$

locally uniformly on $U^+$ as $n \to \infty$, and the proof of the theorem is complete. $\square$

*Case 2:* $\beta \neq 0$, $\alpha' = \alpha'' = 0$

With the notation of (6.1) and (6.2), we have $P(x, y) = \alpha x^2 + \text{l.d.t.}$ and $\tilde{P}(y, z) = \gamma z^2 + \text{l.d.t.}$ The generic case $\alpha \neq 0 \neq \gamma$ is covered in Case 2 for the class $H_4$ by switching the roles of $x$ and $y$ in the formulas of $V^\pm, G^\pm, \dots$ given there. There are also interesting examples when $P \equiv 0$. Then it is easy to check that $H^2$ is given by

$$x_2 = ax, \quad y_2 = b^2 y + Q(az) + bQ(x), \quad z_2 = az;$$

hence $\deg(H^{2n}) \leq 2$ for all integers $n$.

*Case 3:* $\beta = 0$

For this case, $P(x, y) = \alpha x^2 + \alpha' xy + \alpha'' y^2 + \text{l.d.t.}$, $\tilde{P}(y, z) = \gamma z^2 + \gamma' yz + \gamma'' y^2 + \text{l.d.t.}$, and $\deg(P) = \deg(\tilde{P}) = 2$. The generic situation when $\alpha \neq 0 \neq \gamma$ is covered by Case 3 for the class $H_4$. If $\alpha = 0$ and $\alpha' \neq 0$ then, as for the class $H_4$, the degrees of the forward iterates $H^n$ are given by Fibonacci's numbers. If $\alpha = \alpha' = 0$ (hence $\alpha'' \neq 0$) there are some interesting examples, as when $P(x, y) = \alpha'' y^2$ and $Q \equiv 0$. Then it is easy to see that

$$H^n(w) = (c_n y^2 + p_n(x, z), b^n y, d_n y^2 + q_n(x, z))$$

for all integers $n$, where $\deg(p_n) \leq 1$ and $\deg(q_n) \leq 1$. Hence all forward and backward iterates of $H$ have degree at most 2.

## 7. Conclusions

We summarize here the main dynamical features of the five classes of automorphisms studied throughout the paper. The goal is to highlight the new phenomena that occur in dimension 3 and also to point out dynamical differences and analogies between these five classes.

We recalled in the introduction a few facts about the dynamics of Hénon maps in $\mathbb{C}^2$ and of regular polynomial autmorphisms in $\mathbb{C}^N$. In order to emphasize the dynamical differences between these maps and the ones studied here, we first mention a few more properties of Hénon maps, regular polynomial automorphisms, and the shift-like automorphisms of [BP].

For Hénon maps $h$ in $\mathbb{C}^2$, the indeterminacy sets consist each of one point: $I^+ = [0 : 1 : 0]$ and $I^- = [1 : 0 : 0]$. The extension of $h$ to $\mathbb{P}^2$ maps the line $\{t = 0\}$ at infinity to $[1 : 0 : 0]$, and this point is a super-attracting fixed point of $h$ with basin of attraction $U^+ \cup (\{t = 0\} \setminus I^+)$.

For regular polynomial automorphisms $h$ of $\mathbb{C}^N$ studied in [S], the situation is analogous to the Hénon maps from the following point of view: $h(\{t=0\}\setminus I^+) \subset I^-$, $I^-$ is an attractor with basin $U^+$ in $\mathbb{C}^N$, and the set $K = K^+ \cap K^-$ is compact and is the set of points with bounded full orbit. Moreover, the extended indeterminacy set $I^+_\infty = I^+$. Sibony also showed that if $d = \deg(h)$, $d' = \deg(h^{-1})$, and $\dim(I^-) = l - 1$, then $\deg(h^n) = d^n$, $\dim(I^+) = N - l - 1$, $d^l = (d')^{N-l}$, and $\mu = (\mu^+)^l \wedge (\mu^-)^{N-l}$ is an invariant probability measure supported on $K$.

Similar dynamical behavior with Hénon maps and regular automorphisms is exhibited by the shift-like polynomial automorphisms of $\mathbb{C}^N$ studied in [BP]. A shift-like automorphism of type $\nu \in \{1, \dots, N-1\}$ has the form

$$f(x_1, \dots, x_N) = (x_2, \dots, x_N, P(x_{N-\nu+1}) - ax_1),$$

where $\deg(P) = d \geq 2$ and $a \neq 0$. Then $K^+$ and $K^-$ are again the sets of points with bounded forward (resp. backward) orbit, $K = K^+ \cap K^-$ is compact in $\mathbb{C}^N$, and $\mu = (\mu^+)^\nu \wedge (\mu^-)^{N-\nu}$ is an invariant probability measure supported on $K$. We note that these similarities come from the fact that $h = f^{\nu(N-\nu)}$ is regular: $\deg(h) = d^{N-\nu}$, $\deg(h^{-1}) = d^\nu$, $I^+(h) = \{x_{N+1-\nu} = \cdots = x_N = t = 0\}$, $\dim(I^+) = N - \nu - 1$, $I^-(h) = \{x_1 = \cdots = x_{N-\nu} = t = 0\}$, and $\dim(I^-) = \nu - 1$.

The automorphisms we consider are not regular, but the degrees of their forward and backward iterates are always given by (const)$2^n$ or $3^n$ for $n$ sufficiently large. With the vanishing of some coefficients it is possible to obtain maps $H$ in these classes with "irregular" growth of degree (i.e., like Fibonacci's numbers—see [B], or with $\deg(H^{2n}) \leq 2$) or such that $H^2$ is regular (see e.g. the class $H_4$).

For all the maps studied here, the set $U^+$ of points whose orbit escapes to infinity at the highest super-exponential rate is always open. Hence $K^+ = \mathbb{C}^3 \setminus U^+$ is closed, as in the case of regular automorphisms, but it no longer consists only of points with bounded forward orbit; we have only that $K^+ = \{G^+ = 0\}$ is the set of points whose orbit escapes to infinity at rates slower than the one corresponding to $U^+$. Similar statements hold for $U^-$ and $K^-$. The Green's functions $G^+$ and $G^-$ are, in all these cases, pluriharmonic on the sets $U^+$ (resp. $U^-$). With the usual notation $\mu^\pm = dd^cG^\pm$, we have that $\partial K^+ \cap \partial K^- = \operatorname{supp}(\mu^+ \wedge \mu^-)$ cannot be compact, hence $K^+ \cap K^-$ is unbounded. Moreover, one cannot construct invariant measures using only the currents $\mu^+$ and $\mu^-$, as is done for regular automorphisms.

The dynamics on $U^+$ of the automorphisms we consider is determined by the behavior of their extension to $\mathbb{P}^3$ along the hyperplane $\{t = 0\}$ at infinity. For the maps $H$ in the first three classes—given by (2.2), (3.2), and (4.1)—we have that the second iterate $H^2$ maps all the points at infinity (in $\mathbb{P}^3$, where it is well-defined) to a single point. The iterates $H^n$ form a normal family on $U^+$, since they converge locally uniformly to that point. Similar statements hold in these cases for the inverse maps $H^{-1}$ on the corresponding sets $U^-$.

For the maps we considered from the classes $H_4$ and $H_5$ (see (5.1) and (6.1)), the behavior at infinity is in general different than as just described (i.e., in Case 1 for each of these classes). Such maps map $\{t = 0\} \setminus I^+$ to $I^- = \{t = z = 0\}$, where they are given by

$$\begin{aligned} H_4[u:1:0:0] &= [h_4(u):1:0:0], \quad h_4(u) = \eta u + \eta', \\ H_5[1:u:0:0] &= [1:h_5(u):0:0], \quad h_5(u) = 1/(u+\alpha). \end{aligned} \tag{7.1}$$

In Theorems 5.3, 5.5, and 6.6, we showed that the dynamics of $H_4$ and $H_5$ on $U^+$ is determined by the "dynamics at infinity" of $h_4$ and $h_5$, respectively. This was done by constructing suitable holomorphic functions $F_4$ and $F_5$ on the sets $U^+$, which satisfy (resp.) $F_4 \circ H_4 = \eta F_4$ and $F_5 \circ H_5 = h_5 \circ F_5$. (In the case of $H_4$, if $\tilde{F}_4 = F_4 + u_0$, where $F_4$ is the function of Theorem 5.3 and $u_0$ is the finite fixed point of $h_4$, we have $\tilde{F}_4 \circ H_4 = h_4 \circ \tilde{F}_4$.)

The maps in the classes $H_1$ and $H_2$ are essentially semidirect products over Hénon maps in $\mathbb{C}^2$. Let us denote by $K$ the set of points with bounded full orbit. It is natural to expect that $K$ carries some dynamical information. For the maps $H$ in the classes $H_1$ and $H_2$, $K$ is compact and it is not difficult to find it explicitly, roughly speaking as the intersection of $K^+ \cap K^-$ with an analytic hypersurface that is invariant under $H$ ($K = K^+ \cap K^- \cap \{z = 0\}$ for $H_1$, and $K = K^+ \cap K^- \cap X$ for $H_2$—see Section 3). Hence we can construct invariant measures using $\mu^+$, $\mu^-$, and this invariant hypersurface. Similar constructions of invariant measures may work for some of the maps in the remaining classes. We also note the following difference between some maps $H_2$ and Hénon maps. If we denote by $\mathcal{F}^+$ the Fatou set and by $J^+$ the Julia set (defined using normality) then, for Hénon maps, $J^+ = \partial K^+ = \operatorname{supp} \mu^+$. For maps $H_2$ such that $\operatorname{int} K^+ \neq \emptyset$, we have that $J^+ = \partial K^+ \cup X$ is larger than $\partial K^+ = \operatorname{supp} \mu^+$.

Recall that, for Hénon maps, $U^+ \subseteq \mathcal{F}^+$. We have the following new phenomenon for the maps $H_4$ with $|\eta| > 1$ (see (7.1)): The iterates $H_4^n$ converge locally uniformly on $U^+ \setminus X$ to $[1:0:0:0]$, where $X$ is an analytic hypersurface in $U^+$, invariant under $H_4$ (with the above notation, $X = \{F_4 = 0\} = \{\tilde{F}_4 = u_0\}$). Since along $X$ the iterates $H_4^n$ converge locally uniformly to the finite fixed point $[u_0 : 1:0:0]$ of $h_4$, we conclude that $\{H_4^n\}_n$ is not a normal family on $U^+$. Note also that $[1:0:0:0] \in I^- \cap I^+$.

The dynamics of maps in class $H_5$, Case 1, seems to be the most complicated among the five classes. This is because the maps are nontrivial along the hyperplane at infinity in $\mathbb{P}^3$ and, at the same time, the extended indeterminacy set $I_\infty^+$ is larger than $I^+$. We only consider the "rationally neutral" case, corresponding to $\eta^k = 1$ (see (6.5)). In this case $I_\infty^+ \cap I^-$ consists of finitely many points, a fact that allowed our construction of the set $V^-$ to work out. The remaining cases—when $I_\infty^+ \cap I^-$ is a countable set—will be the subject of forthcoming papers.

We also have the following new situation arising for maps $H$ in Case 1 of each of the classes $H_4$ and $H_5$. For such maps, $H^\star\mu^+ = 2\mu^+$ whereas $H^\star\mu^- = \frac{1}{3}\mu^-$. Moreover, the Green's functions $G^+$ and $G^-$ are pluriharmonic on $U^+$ (resp. $U^-$). Because of these facts, the construction of invariant measures using the currents $\mu^+$ and $\mu^-$ seems to be complicated.

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