The Bergman Kernel on Monomial Polyhedra

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0. Introduction

In order to understand the Bergman kernel for a complex domain Ω in $\mathbb{C}^n$ at z close to the boundary $\partial\Omega$, we usually insert the biholomorphic image of a polydisc $\mathcal{D}$ centered at z in Ω to generate the upper bound for the Bergman kernel on Ω:

$$K_\Omega(z,z) \le K_{\mathcal{D}}(z,z) = \frac{1}{\operatorname{Vol}(\mathcal{D})}.$$

On the other hand, Catlin [3] showed by using a $\bar\partial$ estimate that, on a finite type pseudoconvex domain Ω in $\mathbb{C}^2$, there exists a polydisc $\mathcal{D}$ such that

$$K_\Omega(z,z) \ge c \cdot \frac{1}{\operatorname{Vol}(\mathcal{D})};$$

the same formula was later shown by McNeal [8] on convex domains in $\mathbb{C}^n$. A question arises: Are polydiscs enough to describe the Bergman kernel for smooth bounded domains?

For a general domain in $\mathbb{C}^n$, it is not always possible to find a polydisc D that models the domain. Consider $\Omega \subset \mathbb{C}^3$ defined by $|z_1|^{10} + |z_2|^{10} + |z_1 z_2|^2 + |z_3|^2 < 1$, and let $z = (0,0,1-\varepsilon)$. It is easy to show that all polydiscs centered at z in Ω have maximal volume of approximately ε^4; thus, the upper bound of the Bergman kernel at z obtained by inserting polydiscs is roughly ε^{-4}. But consider a Reinhardt domain $\mathcal{R}$ centered at z bounded by $|z_1| < 1$, $|z_2| < 1$, $|z_3 - (1-\varepsilon)| < \varepsilon/2$, and $|z_1 z_2| < \varepsilon/2$. The volume of $\mathcal{R}$ is roughly $\varepsilon^4(-\log\varepsilon + 1)$, which is much larger than ε^4 when $\varepsilon \ll 1$; therefore, the upper bound at z given by $\mathcal{R}$ is $1/\varepsilon^4(-\log\varepsilon + 1)$, much smaller than the ones given by any polydiscs.

The preceding example shows that polydiscs do not provide a good enough way of estimating upper bounds for the Bergman kernel. Instead of trying to fit a polydisc $\mathcal{D}$ about the point z into Ω, it seems better to try to fit the largest "monomial polyhedron" P about z into Ω, where a monomial polyhedron P associated with a finite subcollection $\mathcal{B}$ of index space $\mathcal{N}^n$, $\mathcal{N} = \mathbb{N} \cup \{0\}$, is defined as follows.

Definition 1.1. A domain P in $\mathbb{C}^n$ is a *monomial polyhedron* if there exists a subset $\mathcal{B} = \{\alpha_1, \dots, \alpha_m\}$ of $\mathcal{N}^n$ and, for each $\alpha \in \mathcal{B}$, there exists a unique $C_\alpha \in \mathbb{R}$ such that $P = P(\mathcal{B}) = \{z \in \mathbb{C}^n : |z^\alpha| < e^{C_\alpha}, \alpha \in \mathcal{B}\}$.

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Note that $\mathcal{R}$ in the foregoing example is a monomial polyhedron. Also, it is obvious that the log domain of P defined as

$$\log(P) = \{ w \in \mathbb{R}^n : \alpha \cdot w < c_\alpha = \log d_\alpha,\ \alpha \in \mathcal{B} \} \tag{1}$$

is an unbounded polyhedron containing $(-\infty, \ldots, -\infty)$.

It is possible that, among the inequalities in (1) that define $\log(P)$, some may be redundant. However we will show that we can assume $\mathcal{B}$ satisfies: (i) $\mathcal{B}$ is a minimal collection defining P; and (ii) for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathcal{B}$, $\alpha_1, \ldots, \alpha_n$ are relatively prime. Such a set $\mathcal{B}$ is unique with respect to P. We call such a set $\mathcal{B}$ a *regular index set* for P.

In order to precisely define our estimate of the Bergman kernel, we must first define the representation of the faces of the convex monomial polyhedron $\log(P)$. For each $\mathcal{A} \subseteq \mathcal{B}$, define a face $\mathcal{F} = \mathcal{F}(\mathcal{A})$ of $\partial \log(P)$ determined by $\mathcal{A}$ by

$$\mathcal{F}(\mathcal{A}) = \{ w \in \overline{\log(P)} : \alpha \cdot w = c_\alpha\ \forall \alpha \in \mathcal{A} \text{ and } \alpha \cdot w < c_\alpha\ \forall \alpha \in \mathcal{B} - \mathcal{A} \}.$$

Of course, there is no guarantee that such a face $\mathcal{F}$ is not empty. However, if it is not empty then we can conversely determine a subcollection $\mathcal{A} = \mathcal{A}(\mathcal{F})$ of $\mathcal{B}$ by $\mathcal{A}(\mathcal{F}) = \{ \alpha \in \mathcal{B} : \alpha \cdot w = c_\alpha\ \forall w \in \mathcal{F} \}$, and we will show (in Proposition 1.8) that it is a one-to-one correspondence map between non-empty faces and the subcollections that determine non-empty faces.

We will see later that, in order to estimate the Bergman kernel, we need only study the bounded faces of $\log(P)$. Thus we will define $\mathbb{F} = \{ \mathcal{F} : \mathcal{F} \text{ is bounded} \}$ and $\mathcal{U}(\mathcal{F}) = \{ \beta \in \mathcal{N}^n : \beta = \sum_{\alpha \in \mathcal{A}(\mathcal{F})} \lambda_\alpha \alpha,\ 0 < \lambda_\alpha \le 1 \}$. Notice that if we let $|\mathcal{K}|$ be the cardinality of $\mathcal{K}$, then both $|\mathbb{F}|$ and $|\mathcal{U}(\mathcal{F})|$ are finite.

Define P as (M, ε)-*nondegenerate* for some M and $\varepsilon > 0$ if: (i) $\sum_{j=1}^n \alpha_j \le M$ for all $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathcal{B}$; (ii) $\mathcal{A}(\mathcal{F})$ is a linearly independent set for all $\mathcal{F} \in \mathcal{B}$; and (iii) every $\mathcal{F} \in \mathcal{B}$ contains an ε-ball in the corresponding dimension. We will give a more precise description in Definition 1.10. Now let us state our theorem.

THEOREM 3.1. *Let P be an (M, ε)-nondegenerate bounded monomial polyhedron, and let $\zeta_\beta(z) = z^\beta$. Then there are constants $C > c > 0$ depending on M, ε, and n such that the Bergman kernel for P can be estimated as*

$$c \cdot K_P(z, z) < \sum_{\mathcal{F} \in \mathbb{F}} \left(\prod_{\alpha \in \mathcal{A}(\mathcal{F})} \frac{1}{(1 - |(z^\alpha / d_\alpha)|^2)^2} \cdot \sum_{\beta + \bar{\mathbf{1}} \in \mathcal{U}(\mathcal{F})} \frac{|z^\beta|^2}{\|\zeta_\beta\|^2} \right) < C \cdot K_P(z, z).$$

Furthermore, for $\beta + \bar{\mathbf{1}} \in \mathcal{U}(\mathcal{F})$ and with constants $C > c > 0$ depending on ε, M, and n, we have

$$c \cdot \|z^\beta\|^2 < |\tilde{z}^{\beta + \bar{\mathbf{1}}}|^2 \cdot A^{n-k}(\mathcal{F}) < C \cdot \|z^\beta\|^2,$$

where $\tilde{z} = e^{\tilde{w}}$ for any $\tilde{w} \in \mathcal{F}$ and $A^{n-k}(\mathcal{F})$ is the volume of $\mathcal{F}$ in its corresponding dimension, which will be described more precisely in Sections 2 and 3.

REMARK 3.1. $|\tilde{z}^{\beta + \bar{\mathbf{1}}}|$ is independent of the choice of $\tilde{w}$ as long as $\beta + \bar{\mathbf{1}} \in \mathcal{U}(\mathcal{F})$ and $\tilde{w} \in \mathcal{F}$.

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1. Geometry of Bounded Monomial Polyhedrons

In order to describe the Bergman kernel $K(z, w)$ for a Reinhardt domain by utilizing the formula, with $\zeta_\beta(z) = z^\beta$,

$$K(z, w) = \sum_{\beta \in \mathcal{N}^n} \frac{z^\beta \bar{w}^\beta}{\|\zeta_\beta\|^2}, \tag{1.1}$$

where $\mathcal{N} = \mathbb{N} \cup \{0\}$, it is desirable to estimate $\|z^\beta\|$ for all $\beta \in \mathcal{N}^n$. Thus it becomes important to better understand the geometry of the underlying domains.

In our paper, we assume $\mathcal{N}^n = \{ (\alpha_1, \dots, \alpha_n) : \alpha_j \in \mathbb{N} \cup \{0\} \}$. Let Ω be a bounded Reinhardt domain and let

$$\log(\Omega) = \{ (w_1, \dots, w_n) \in \mathbb{R}^n : (e^{w_1}, \dots, e^{w_n}) \in \Omega \}.$$

We want to give a definition for monomial polyhedrons.

Definition 1.1. A domain P in $\mathbb{C}^n$ is a *monomial polyhedron* if there exists a subset $\mathcal{B} = \{\alpha_1, \dots, \alpha_m\}$ of $\mathcal{N}^n$ and, for each $\alpha \in \mathcal{B}$, there exists a unique $C_\alpha \in \mathbb{R}$ such that

$$P = P(\mathcal{B}) = \{ z \in \mathbb{C}^n : |z^\alpha| < e^{C_\alpha}, \ \alpha \in \mathcal{B} \}. \tag{1.2}$$

Because a monomial polyhedron P thus defined is a Reinhardt domain, we see that

$$\log(P) = \{ w \in \mathbb{R}^n : \alpha \cdot w < C_\alpha, \ \alpha \in \mathcal{B} \}.$$

Definition 1.2. We say $\alpha = (a_1, \dots, a_n) \in \mathcal{N}^n$ is *prime* if $a_1, \dots, a_n$ are relatively prime.

Without loss of generality, we can assume that all α in $\mathcal{B}$ are prime. Let us denote

$$\mathcal{W}_\alpha = \{ w \in \mathbb{R}^n : \alpha \cdot w < C_\alpha \}.$$

Definition 1.3. We say α is *essential* in $\mathcal{B}$ if there exists a $w \in \mathbb{R}^n$ such that $\alpha \cdot w \geq C_\alpha$ but $\beta \cdot w < C_\beta$ for all β in $\mathcal{B}$ and $\beta \neq \alpha$. It is equivalent to say that

$$\left(\bigcap_{\beta \in \mathcal{B} - \{\alpha\}} \mathcal{W}_\beta \right) \cap \mathcal{W}_\alpha^C \neq \emptyset. \tag{1.3}$$

Also, we say α is non-essential in $\mathcal{B}$ if it is not essential—that is, if for all $w \in \mathbb{R}^n$ such that $\alpha \cdot w \geq C_\alpha$ there exists a $\beta \in \mathcal{B}$ such that $\beta \cdot w \geq C_\beta$. It is equivalent to say that

$$\bigcap_{\beta \in \mathcal{B} - \{\alpha\}} \mathcal{W}_\beta \subseteq \mathcal{W}_\alpha. \tag{1.4}$$

Let us denote $\operatorname{ess}(\mathcal{B}) = \{\,\alpha \in \mathcal{B} : \alpha \text{ is essential}\,\}$. First we would like to show that essential indices will remain essential even after we take off non-essential indices away from $\mathcal{B}$.

LEMMA 1.4. *Let α be non-essential in $\mathcal{B}$, and let $\mathcal{B}' = \mathcal{B} - \{\alpha\}$. Then* $\operatorname{ess}(\mathcal{B}) = \operatorname{ess}(\mathcal{B}')$.

Proof. It is obvious from (1.3) that $\operatorname{ess}(\mathcal{B}) \subseteq \operatorname{ess}(\mathcal{B}')$. Suppose γ is non-essential in $\mathcal{B}$ but essential in $\mathcal{B}'$, and let $\tilde{P} = \bigcap_{\beta \in \mathcal{B}-\{\alpha,\gamma\}} \mathcal{W}_\beta$.

Since both α and γ are non-essential in $\mathcal{B}$, by (1.4) we have $\tilde{P} \cap \mathcal{W}_\alpha \subseteq \mathcal{W}_\gamma$ and $\tilde{P} \cap \mathcal{W}_\gamma \subseteq \mathcal{W}_\alpha$. By intersecting with $\tilde{P}$, this means $\tilde{P} \cap \mathcal{W}_\gamma = \tilde{P} \cap \mathcal{W}_\alpha$. But the assumption that γ is essential in $\mathcal{B}'$ and (1.3) imply that $\tilde{P} \cap \mathcal{W}_\gamma^C \neq \emptyset$. Notice that $\mathcal{W}_\gamma$ and $\mathcal{W}_\alpha$ are both half-spaces. For any two half-spaces $\mathcal{W}_1$ and $\mathcal{W}_2$, if there exists an open neighborhood $\mathcal{U}$ such that both $\mathcal{W}_1 \cap \mathcal{U}$ and $\mathcal{W}_1^C \cap \mathcal{U}$ are non-empty and if $\mathcal{W}_1 \cap \mathcal{U} = \mathcal{W}_2 \cap \mathcal{U}$, then $\mathcal{W}_1 = \mathcal{W}_2$. Thus, by taking $\mathcal{U} = \tilde{P}$, we have $\mathcal{W}_\gamma = \mathcal{W}_\alpha$. Finally, since all indices in $\mathcal{B}$ are prime, $\gamma = \alpha$, a contradiction. □

Now denote $\operatorname{ess}\log(P) = \{\, w \in \mathbb{R}^n : \alpha \cdot w < C_\alpha,\ \alpha \in \operatorname{ess}(\mathcal{B})\,\}$. It is obvious that

$$\log(P) = \bigcap_{\alpha \in \mathcal{B}} \mathcal{W}_\alpha \quad \text{and} \quad \operatorname{ess}\log(P) = \bigcap_{\alpha \in \operatorname{ess}(\mathcal{B})} \mathcal{W}_\alpha. \tag{1.5}$$

We would like to show the following.

LEMMA 1.5. $\operatorname{ess}\log(P) = \log(P)$.

Proof. From (1.4) and (1.5), we see that $\log(P) = \bigcap_{\beta \in \mathcal{B}-\{\alpha\}} \mathcal{W}_\beta$ for any non-essential index α in $\mathcal{B}$. And Lemma 1.4 shows that the essential indices in $\mathcal{B}$ and $\mathcal{B}'$ are identical, where $\mathcal{B}' = \mathcal{B} - \{\alpha\}$. By repeated application of this procedure for (finitely many times), the result follows. □

From Definition 1.3, there exists an open neighborhood $\mathcal{U}$ for each α in $\operatorname{ess}(\mathcal{B})$ such that $\mathcal{U} \cap \partial\log(P) \neq \emptyset$ and $\mathcal{U} \cap \log(P) = \mathcal{U} \cap \mathcal{W}_\alpha$. Thus, we can think of essential indices as normal vectors to local neighborhoods of $\partial\log(P)$.

Obviously, $\mathcal{B}$ must contain such normal vectors. However, if we take only those normal vectors, Lemma 1.5 implies that we obtain a unique collection of indices, $\mathcal{B}$, which describes the monomial polyhedron P. We would like to give a name to such a collection.

DEFINITION 1.6. $\mathcal{B} \subset \mathcal{N}^n$ is a *regular index set* with respect to P if, for all $\alpha \in \mathcal{B}$, α is prime and essential in $\mathcal{B}$.

It is easy to see that a monomial polyhedron P uniquely corresponds to a regular index $\mathcal{B}$, and we will assume that $\mathcal{B}$ is regular throughout this paper.

REMARK 1.1. Since we require P to be bounded, from (1.2) it is easy to see that $e_j = (0, \dots, 0, 1, 0, \dots, 0) \in \mathcal{B}$ for all $j = 1, \dots, n$. For if not, then

$(0, \ldots, 0, z, 0, \ldots, 0)$ satisfies all the inequalities in (1.2) for all $z \in \mathbb{C}$. Thus P would be unbounded.

Next, we would like to describe the faces of $\overline{\log(P)}$.

DEFINITION 1.7. $\mathcal{F}$ is a face of $\overline{\log(P)}$ if there exists a subcollection $\mathcal{A}$ of $\mathcal{B}$ such that

$$\mathcal{F} = \mathcal{F}(\mathcal{A}) = \{ w \in \overline{\log(P)} : \alpha \cdot w = C_\alpha \text{ for all } \alpha \in \mathcal{A} \text{ and } \alpha \cdot w < C_\alpha \text{ for all } \alpha \in \mathcal{B} - \mathcal{A} \}. \tag{1.6}$$

It follows from (1.6) that if $\mathcal{A}_1$ and $\mathcal{A}_2$ are subcollections of $\mathcal{B}$ such that $\mathcal{A}_1 \neq \mathcal{A}_2$ and one of $\mathcal{F}(\mathcal{A}_1)$ and $\mathcal{F}(\mathcal{A}_2)$ is not empty, then $\mathcal{F}(\mathcal{A}_1) \neq \mathcal{F}(\mathcal{A}_2)$. Thus, for each non-empty face $\mathcal{F}$ there corresponds an index set

$$\mathcal{A} = \mathcal{A}(\mathcal{F}) = \{ \alpha \in \mathcal{B} : \alpha \cdot w = C_\alpha \text{ for all } w \in \mathcal{F} \}.$$

But it follows directly from the definition that

$$\mathcal{A}(\mathcal{F}(\mathcal{A}_0)) = \mathcal{A}_0 \quad \text{and} \quad \mathcal{F}(\mathcal{A}(\mathcal{F}_0)) = \mathcal{F}_0 \tag{1.7}$$

once $\mathcal{F}(\mathcal{A}_0)$ and $\mathcal{F}_0$ are not empty. Thus, by collecting all non-empty faces and corresponding index sets as

$$\bar{\mathbb{F}} = \{ \mathcal{F} \subseteq \overline{\log(P)} : \mathcal{F} \text{ is a non-empty face} \}$$

and

$$\bar{\mathbb{A}} = \{ \mathcal{A} \subseteq \mathcal{B} : \mathcal{F}(\mathcal{A}) \text{ is non-empty} \},$$

we have the following proposition.

PROPOSITION 1.8.

(i) *There exists a one-to-one onto map between* $\bar{\mathbb{F}}$ *and* $\bar{\mathbb{A}}$;
(ii) $\overline{\log(P)}$ *is a disjoint union of all* $\mathcal{F}$ *in* $\mathbb{F}$.

We also need the following.

LEMMA 1.9. *For* $\mathcal{F}_1, \mathcal{F}_2 \in \bar{\mathbb{F}}$, $\bar{\mathcal{F}}_1 = \bar{\mathcal{F}}_2$ *if and only if* $\mathcal{F}_1 = \mathcal{F}_2$.

Proof. The "if" part is obvious. For the "only if" part, first notice that for each $\mathcal{F}$ there exists an $\mathcal{A}$ such that (1.6) holds. But then $\bar{\mathcal{F}}$ becomes

$$\begin{aligned}
\bar{\mathcal{F}} &= \{ w \in \overline{\log(P)} : \alpha \cdot w = C_\alpha \text{ for all } \alpha \in \mathcal{A} \text{ and } \\
&\qquad \alpha \cdot w \leq C_\alpha \text{ for all } \alpha \in \mathcal{B} - \mathcal{A} \} \\
&= \bigcup_{\mathcal{A} \subseteq \mathcal{A}'} \{ w \in \overline{\log(P)} : \alpha \cdot w = C_\alpha \text{ for all } \alpha \in \mathcal{A}' \text{ and } \\
&\qquad \alpha \cdot w < C_\alpha \text{ for all } \alpha \in \mathcal{B} - \mathcal{A}' \} \\
&= \bigcup_{\mathcal{A} \subseteq \mathcal{A}'} \mathcal{F}(\mathcal{A}').
\end{aligned} \tag{1.8}$$

By Proposition 1.8(ii), this expression is a disjoint union. It follows that, if $\mathcal{F}_1 = \mathcal{F}(\mathcal{A}_1)$ and $\mathcal{F}_2 = \mathcal{F}(\mathcal{A}_2)$, then $\bar{\mathcal{F}}_1 = \bar{\mathcal{F}}_2$ implies $\mathcal{F}_1 \subseteq \bar{\mathcal{F}}_2$ and $\mathcal{F}_2 \subseteq \bar{\mathcal{F}}_1$, which implies $\mathcal{A}_1 = \mathcal{A}_2$, which in turn implies $\mathcal{F}_1 = \mathcal{F}_2$. □

To make our computation manageable, we need to impose some extra conditions on our monomial polyhedrons.

DEFINITION 1.10. A monomial polyhedron P is (M, ε)-*nondegenerate* for some $M, \varepsilon > 0$ if all of the following conditions hold.

(i) For all $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathcal{B}$, $\alpha_1 + \cdots + \alpha_n \leq M$.
(ii) For all $\mathcal{A}$ in $\bar{\mathbb{A}}$, the set of elements in $\mathcal{A}$ is linearly independent in $\mathbb{R}^n$.
(iii) For all $\mathcal{F}$ in $\bar{\mathbb{F}}$, let $\mathbf{B}_\varepsilon(w)$ be an Euclidean ball centered at w with radius ε, and let $\mathcal{W}_{\mathcal{F}} = \{ w : \alpha \cdot w = C_\alpha \text{ for all } \alpha \in \mathcal{A}(\mathcal{F}) \}$; then there exists a w in $\mathcal{F}$ such that $\mathbf{B}_\varepsilon(w) \cap \mathcal{W}_{\mathcal{F}} \subseteq \mathcal{F}$.

Notice that, for any monomial polyhedron P, there exist $M, \varepsilon > 0$ that satisfy conditions (i) and (iii). We emphasize the roles played by M and $\varepsilon > 0$ because our result will depend on both M and ε.

In order to sum (1.1), we need to suitably decompose the index space $\mathcal{N}^n$. One natural way of doing so is by decomposing the index space into a finite disjoint union of convex cones generated by elements in $\bar{\mathbb{A}}$, as follows.

Let $\mathcal{A} = \{\alpha_1, \ldots, \alpha_k\} \subset \mathcal{N}^n$. We say α is a *nonnegative* combination of $\mathcal{A}$ if there exist $\lambda_1, \ldots, \lambda_k$ and $\lambda_j \geq 0$ for all $j = 1, \ldots, k$ such that $\alpha = \sum_{j=1}^k \lambda_j \alpha_j$; we say α is a *positive* combination of $\mathcal{A}$ if α is a nonnegative combination of $\mathcal{A}$ as just described while also requiring $\lambda_j > 0$ for all $j = 1, \ldots, k$. We say $\Gamma(\mathcal{A})$ is the *open* convex cone generated by $\mathcal{A}$ if $\Gamma(\mathcal{A}) = \{ \alpha : \alpha \text{ is a positive combination of } \mathcal{A} \}$.

Note that the term "open" used here is not in a traditional sense, for the set $\Gamma(\mathcal{A})$ is discrete. Rather, we use "open" to emphasize that this cone does not contain the boundary.

The following proposition will imply that the index space $\mathcal{N}^n$ is a (finite) disjoint union of all open convex cones $\Gamma(\mathcal{A})$ for all $\mathcal{A}$ in $\bar{\mathbb{A}}$.

PROPOSITION 1.11. *Let P be (M, ε)-nondegenerate. Then the following are equivalent*:

(i) *β in $\mathcal{N}^n$ is a positive combination of $\mathcal{A}$—that is, $\beta \in \Gamma(\mathcal{A})$*;
(ii) *the linear functional f_β defined by $f_\beta(w) = \beta \cdot w$, when restricted to $\overline{\log(P)}$, reaches its maximum at all points and only at points of $\overline{\mathcal{F}(\mathcal{A})}$.*

Proof. For (i) ⇒ (ii), let $\mathcal{A} = \{\alpha_1, \ldots, \alpha_k\} \in \bar{\mathbb{A}}$ and assume there exist $\lambda_1, \ldots, \lambda_k > 0$ such that $\beta = \sum_{j=1}^k \lambda_j \alpha_j$. Let w be any point in $\overline{\log(P)} - \overline{\mathcal{F}(\mathcal{A})}$. From (1.8), we see that there must exist a $j \in \{1, \ldots, k\}$ such that $\alpha_j \cdot w < C_{\alpha_j}$; therefore,

$$f_\beta(w) = \beta \cdot w = \sum_{j=1}^k \lambda_j \alpha_j \cdot w < \sum_{j=1}^k \lambda_j C_{\alpha_j}.$$

But for $w \in \overline{\mathcal{F}(\mathcal{A})}$,

$$f_\beta(w) = \sum_{j=1}^{k} \lambda_j C_{\alpha_j},$$

so the result follows.

For (ii) $\Rightarrow$ (i), first note that $\mathcal{A} = \{\alpha_1, \ldots, \alpha_k\} = \{\alpha \in \mathcal{A} : \alpha \cdot w = C_\alpha$ for all $w \in \bar{\mathcal{F}}\}$, and that $\mathcal{F}$ is non-empty. By Definition 1.10(ii), $\mathcal{A}$ is linearly independent. Thus, by solving the system

$$\begin{aligned} \alpha_1 \cdot w &= C_{\alpha_1} \\ &\vdots \\ \alpha_k \cdot w &= C_{\alpha_k}, \end{aligned}$$

we have an $(n-k)$-dimensional linear affine space $\mathcal{W}$ containing $\mathcal{F}$. Let w_0 be a point in $\mathcal{F}$ satisfying Definition 1.10(iii), and let $\beta_{k+1}, \ldots, \beta_n$ be a basis of $\mathcal{W} - w_0$ that is an $n-k$ dimensional vector space. Then we can express $\mathcal{W}$ as

$$\mathcal{W} = \left\{ w_0 + \sum_{j=k+1}^{n} s_j \beta_j : s_j \in \mathbb{R} \right\},$$

where

(a) $w_0 + \sum_{j=k+1}^{n} s_j \beta_j \in \mathcal{F} = \mathcal{F}(\mathcal{A})$ for $|s_j|$ small;
(b) $\{\alpha_1, \ldots, \alpha_k, \beta_{k+1}, \ldots, \beta_n\}$ is a linear basis.

Thus, by (b), $\beta = \sum_{j=1}^{k} \lambda_j \alpha_j + \sum_{j=k+1}^{n} \lambda_j \beta_j$. But f_β reaches its maximum at all points of $\bar{\mathcal{F}}$, and (a) implies that $\lambda_j = 0$ for $j = k+1, \ldots, n$. For if not (say, $\lambda_j > 0$ for some $j > k$) then we can find $w_1 = w_0 + s_j \beta_j$ where $s_j > 0$ is so small that $w_1 \in \mathcal{F}$. But then $f(w_1) > f(w_0)$, a contradiction. This means that β is a linear combination of $\mathcal{A}$.

Suppose β is not a nonnegative combination of $\mathcal{A}$. We can assume $\beta = \sum_{j=1}^{k} \lambda_j \alpha_j$, $\lambda_1, \ldots, \lambda_l \geq 0$ and $\lambda_{l+1}, \ldots, \lambda_k < 0$. But we can always find w such that

$$\begin{aligned} \alpha_1 \cdot w &= C_{\alpha_1} \\ &\vdots \\ \alpha_l \cdot w &= C_{\alpha_l} \\ \alpha_{l+1} \cdot w &< C_{\alpha_{l+1}} \\ &\vdots \\ \alpha_k \cdot w &< C_{\alpha_k}. \end{aligned}$$

It is easy to see that $w \in \overline{\log(P)}$ and $f_\beta(w) \gneqq f_\beta(w_0)$ for $w_0 \in \mathcal{F}$, a contradiction.

Finally, suppose β is not a positive combination of $\mathcal{A}$; then it is a positive combination of some $\mathcal{A}' \subsetneq \mathcal{A}$. By the proof of (i) $\Rightarrow$ (ii), f_β reaches its maximum on $\overline{\mathcal{F}(\mathcal{A}')} \supsetneq \overline{\mathcal{F}(\mathcal{A})}$, a contradiction. $\square$

Proposition 1.11 will allow us to represent all β in $\mathcal{N}^n$ as a positive combination of a unique $\mathcal{A}$. We thus have the following result.

COROLLARY 1.12. *$\mathcal{N}^n$ can be decomposed into a finite disjoint union of open convex cones $\Gamma(\mathcal{A})$ for $\mathcal{A}$ in $\bar{\mathbb{A}}$. That is,*

$$\mathcal{N}^n = \bigcup_{\mathcal{A}\in\bar{\mathbb{A}}} \Gamma(\mathcal{A}).$$

Proof. For every β in $\mathcal{N}^n$, f_β must reach its maximum at all points of $\bar{\mathcal{F}}$ for some $\mathcal{F}$ in $\bar{\mathbb{F}}$. By Proposition 1.11, β is in $\Gamma(\mathcal{A})$ for some $\mathcal{A}$ in $\bar{\mathbb{A}}$. But f_β, as a linear functional, can reach its maximum on only one $\bar{\mathcal{F}}$, so $\mathcal{A}$ is unique. □

DEFINITION 1.13. We say $\mathcal{A} \in \bar{\mathbb{A}}$ is bounded (unbounded) if $\mathcal{F}(\mathcal{A})$ is bounded (unbounded).

LEMMA 1.14. *Let $\mathcal{A} = \{\alpha_1, \ldots, \alpha_k\}$ and $\alpha_j = (\alpha_j^1, \ldots, \alpha_j^n)$. Then $\mathcal{A}$ is unbounded if and only if there exists an $l \in \{1, \ldots, n\}$ such that $\alpha_j^l = 0$ for all $j = 1, \ldots, k$.*

Proof. ($\Rightarrow$) Since P is bounded, for a face $\mathcal{F}$ in $\overline{\log(P)}$ to be unbounded there must exist an $l \in \{1, \ldots, n\}$ such that if $w_0 = (w_0^1, \ldots, w_0^n) \in \mathcal{F}$ then

$$(w_0^1, \ldots, w^j, \ldots, w_0^n) \in \mathcal{F} \quad \text{for all } w^j < w_0^j.$$

For Remark 1.1 shows that if P is bounded and $w = (w^1, \ldots, w^n) \in \overline{\log(P)}$, then every component w^j is bounded above. Thus, from Definition 1.7, we have $\alpha_j^l = 0$ for all $j = 1, \ldots, k$.

($\Leftarrow$) From Definition 1.7, if there exists an $l \in \{1, \ldots, n\}$ such that $\alpha_j^l = 0$ for all $j = 1, \ldots, k$, then $\mathcal{F}$ must be unbounded. □

Let us define

$$\mathbb{F} = \{\, \mathcal{F} \subseteq \overline{\log(P)} : \mathcal{F} \text{ is a non-empty bounded face}\,\} \tag{1.9}$$

and

$$\mathbb{A} = \{\, \mathcal{A} \subseteq \mathcal{B} : \mathcal{F}(\mathcal{A}) \text{ is a non-empty bounded face}\,\}. \tag{1.10}$$

Then we have the following.

PROPOSITION 1.15.

$$\mathbb{N}^n = \bigcup_{\mathcal{A}\in\mathbb{A}} \Gamma(\mathcal{A}).$$

Moreover, no components of elements in $\mathcal{A} \in \mathbb{A}$ will be simultaneously zero.

Proof. If $\mathcal{A} \in \bar{\mathbb{A}} - \mathbb{A}$ then by Lemma 1.14, for all $\beta = (b^1, \ldots, b^n) \in \Gamma(\mathcal{A})$ there exists an $l \in \{1, \ldots, n\}$ such that $b^l = 0$,—that is $\beta \in \mathcal{N}^n - \mathbb{N}^n$. Therefore, $\Gamma(\mathcal{A}) \subseteq \mathcal{N}^n - \mathbb{N}^n$ and

$$\bigcup_{\mathcal{A}\in\bar{\mathbb{A}}-\mathbb{A}} \Gamma(\mathcal{A}) \subseteq \mathcal{N}^n - \mathbb{N}^n.$$

But for $\beta = (b^1, \ldots, b^n) \in \mathcal{N}^n - \mathbb{N}^n$ there exists an $l \in \{1, \ldots, n\}$ such that $b^l = 0$, and by Corollary 1.12 there exists an $\mathcal{A}$ such that $\beta \in \Gamma(\mathcal{A})$. Yet this means that β is a positive combination of $\mathcal{A}$, so if $\mathcal{A} = \{\alpha_1, \ldots, \alpha_k\}$ and $\alpha_j = (\alpha_j^1, \ldots, \alpha_j^n)$ then $\alpha_j^l = 0$ for all $j = 1, \ldots, k$. By Lemma 1.14, $\mathcal{A} \in \bar{\mathbb{A}} - \mathbb{A}$. Therefore,

$$\bigcup_{\mathcal{A}\in\bar{\mathbb{A}}-\mathbb{A}} \Gamma(\mathcal{A}) \supseteq \mathcal{N}^n - \mathbb{N}^n.$$

This shows

$$\bigcup_{\mathcal{A}\in\bar{\mathbb{A}}-\mathbb{A}} \Gamma(\mathcal{A}) = \mathcal{N}^n - \mathbb{N}^n.$$

Now using Corollary 1.12 again, the result follows. □

2. Estimates for L^2-norms of z^β

We wish to calculate the Bergman kernel for P. Since a monomial polyhedron is a Reinhardt domain, by letting $\bar{\mathbf{1}} = (1, \ldots, 1)$ and $\zeta_\beta(z) = z^\beta$ we have

$$K_P(z_0, w_0) = \sum_{\beta\in\mathcal{N}^n} \frac{z_0^\beta \bar{w}_0^\beta}{\|\zeta_\beta\|^2} = \sum_{\beta+\bar{\mathbf{1}}\in\mathbb{N}^n} \frac{z_0^\beta \bar{w}_0^\beta}{\|\zeta_\beta\|^2} = \sum_{\mathcal{A}\in\mathbb{A}} \sum_{\beta+\bar{\mathbf{1}}\in\Gamma(\mathcal{A})} \frac{z_0^\beta \bar{w}_0^\beta}{\|\zeta_\beta\|^2}. \tag{2.1}$$

Note that the first summation in the last expression is a finite sum.

Let $\beta + \bar{\mathbf{1}}$ be in $\Gamma(\mathcal{A})$ for some $\mathcal{A}$ in $\mathbb{A}$, and let $w_{\beta+\bar{\mathbf{1}}}$ be a point on $\mathcal{F} = \mathcal{F}(\mathcal{A})$. Define

$$S_{\beta+\bar{\mathbf{1}}}(t) = \overline{\log(P)} \cap \{\, w : (\beta+\bar{\mathbf{1}}) \cdot w = (\beta+\bar{\mathbf{1}}) \cdot w_{\beta+\bar{\mathbf{1}}} - t \,\}.$$

Since the function $e^{2(\beta+\bar{\mathbf{1}})\cdot w}$ is a constant on $S_{\beta+\bar{\mathbf{1}}}(t)$ for fixed t, if we define $A_{\beta+\bar{\mathbf{1}}}(t)$ to be the function measuring the $(n-1)$-dimensional area of $S_{\beta+\bar{\mathbf{1}}}(t)$ then

$$\begin{aligned}
\|\zeta_\beta\|^2 &= \int_P |z^\beta|^2 \, dV(z) \\
&= (2\pi)^n \int_{\log(P)} e^{2(\beta+\bar{\mathbf{1}})\cdot w} \, dV(w) \\
&= \frac{(2\pi)^n}{\|\beta+\bar{\mathbf{1}}\|} \int_0^\infty e^{2(\beta+\bar{\mathbf{1}})\cdot w_{\beta+\bar{\mathbf{1}}} - 2t} A_{\beta+\bar{\mathbf{1}}}(t) \, dt.
\end{aligned}$$

The last equality is gained by performing a unitary change of coordinates so that $dt = d(\beta + \bar{\mathbf{1}})$, where $\beta + \bar{\mathbf{1}} = (\beta_1 + 1, \ldots, \beta_n + 1)$.

For the convenience of discussion, let us use the following notation:

$$\Omega_{\beta+\bar{\mathbf{1}}}(t) = \overline{\log(P)} \cap \{\, w : (\beta+\bar{\mathbf{1}}) \cdot w \geq (\beta+\bar{\mathbf{1}}) \cdot w_{\beta+\bar{\mathbf{1}}} - t \,\};$$

$$\Delta_{\beta+\bar{\mathbf{1}}}(\delta) = \{\, w : w = w_{\beta+\bar{\mathbf{1}}} + s \cdot (w' - w_{\beta+\bar{\mathbf{1}}}) \text{ for all } w' \in S_{\beta+\bar{\mathbf{1}}}(\delta) \text{ and } s \geq 0 \,\};$$

$$\Delta_{\beta+\bar{\mathbf{1}}}(\delta, t) = \Delta_{\beta+\bar{\mathbf{1}}}(\delta) \cap \{\, w : (\beta+\bar{\mathbf{1}}) \cdot w \geq (\beta+\bar{\mathbf{1}}) \cdot w_{\beta+\bar{\mathbf{1}}} - t \,\}.$$

Note that $\Delta_{\beta+\bar{\mathbf{1}}}(\delta) = \Delta_{\beta+\bar{\mathbf{1}}}(\delta, \infty)$.

Now notice

$$\int_{\log(P)} = \int_{\Omega_{\beta+\bar{\mathbf{1}}}(\delta)} + \int_{\log(P)-\Omega_{\beta+\bar{\mathbf{1}}}(\delta)},$$

$$\int_{\Omega_{\beta+\bar{\mathbf{1}}}(\delta)} \geq \int_{\Delta_{\beta+\bar{\mathbf{1}}}(\delta,\delta)}, \quad \text{and} \quad \int_{\Delta_{\beta+\bar{\mathbf{1}}}(\delta)-\Delta_{\beta+\bar{\mathbf{1}}}(\delta,\delta)} \geq \int_{\log(P)-\Omega_{\beta+\bar{\mathbf{1}}}(\delta)}.$$

The inequalities are obtained from

$$\Omega_{\beta+\bar{\mathbf{1}}}(\delta) \supseteq \Delta_{\beta+\bar{\mathbf{1}}}(\delta,\delta) \quad \text{and} \quad \Delta_{\beta+\bar{\mathbf{1}}}(\delta) - \Delta_{\beta+\bar{\mathbf{1}}}(\delta,\delta) \supseteq \log(P) - \Omega_{\beta+\bar{\mathbf{1}}}(\delta),$$

which in turn are results of the convexity of $\log(P)$. The first inclusion is easy to show. For the second inclusion, suppose $w \in \log(P) - \Omega_{\beta+\bar{\mathbf{1}}}(\delta)$; the line segment between w and $w_{\beta+\bar{\mathbf{1}}}$ must be in $\log(P)$, which intersects $S_{\beta+\bar{\mathbf{1}}}(\delta)$ at one point. Thus w is a point in $\Delta_{\beta+\bar{\mathbf{1}}}(\delta)$, but it cannot be in $\Delta_{\beta+\bar{\mathbf{1}}}(\delta,\delta)$. The result follows.

If we can show there exists a constant $c = c(\delta) > 0$ such that

$$\int_{\Delta_{\beta+\bar{\mathbf{1}}}(\delta,\delta)} \geq c \int_{\Delta_{\beta+\bar{\mathbf{1}}}(\delta)-\Delta_{\beta+\bar{\mathbf{1}}}(\delta,\delta)}, \tag{2.2}$$

then by

$$\left(1+\frac{1}{c}\right)\int_{\Omega_{\beta+\bar{\mathbf{1}}}(\delta)} \geq \int_{\Omega_{\beta+\bar{\mathbf{1}}}(\delta)} + \int_{\log(P)-\Omega_{\beta+\bar{\mathbf{1}}}(\delta)} = \int_{\log(P)} \geq \int_{\Omega_{\beta+\bar{\mathbf{1}}}(\delta)}$$

we have

$$\int_{\log(P)} \approx \int_{\Omega_{\beta+\bar{\mathbf{1}}}(\delta)}. \tag{2.3}$$

To prove (2.2), we will instead show that there exists a constant $c > 0$ such that

$$\int_{\Delta_{\beta+\bar{\mathbf{1}}}(\delta,\delta)} \geq c \int_{\Delta_{\beta+\bar{\mathbf{1}}}(\delta)}.$$

In general, however,

$$\begin{aligned}\int_{\Delta_{\beta+\bar{\mathbf{1}}}(\delta,s)} e^{2(\beta+\bar{\mathbf{1}})\cdot w}\, dV(w) &= \int_0^s e^{2(\beta+\bar{\mathbf{1}})\cdot w_{\beta+\bar{\mathbf{1}}}-2t} A_{\beta+\bar{\mathbf{1}}}(\delta)\left(\frac{t}{\delta}\right)^{n-1} dt \\ &= \frac{e^{2(\beta+\bar{\mathbf{1}})\cdot w_{\beta+\bar{\mathbf{1}}}} A_{\beta+\bar{\mathbf{1}}}(\delta)}{\delta^{n-1}} \int_0^s e^{-2t} t^{n-1}\, dt.\end{aligned}$$

Set

$$f(s) = \int_0^s e^{-2t} t^{n-1}\, dt$$

and observe that

$$f(\delta) > \frac{(n-1)!}{2^n} \cdot \frac{(2\delta)^n}{e^{2\delta}\cdot n!} \quad \text{and} \quad \lim_{s\to\infty} f(s) = \frac{(n-1)!}{2^n}.$$

With $c(\delta) = (2\delta)^n/(e^{2\delta}\cdot n!)$, we have

$$\int_{\Delta_{\beta+\bar{\mathbf{1}}}(\delta,\delta)} e^{2(\beta+\bar{\mathbf{1}})\cdot w}\, dV(w) > c(\delta) \int_{\Delta_{\beta+\bar{\mathbf{1}}}(\delta)} e^{2(\beta+\bar{\mathbf{1}})\cdot w}\, dV(w).$$

Therefore, by (2.3), we can state the following proposition.

PROPOSITION 2.1. *With $c = c(\delta) = (2\delta)^n/(e^{2\delta}\cdot n!)$, we have*

$$\int_{\Omega_{\beta+\bar{\mathbf{1}}}(\delta)} e^{2(\beta+\bar{\mathbf{1}})\cdot w}\, dV(w) \le \|z^\beta\|^2 \le \left(1+\frac{1}{c}\right) \int_{\Omega_{\beta+\bar{\mathbf{1}}}(\delta)} e^{2(\beta+\bar{\mathbf{1}})\cdot w}\, dV(w). \tag{2.4}$$

Moreover,

$$\int_{\Omega_{\beta+\bar{\mathbf{1}}}(\delta)} e^{2(\beta+\bar{\mathbf{1}})\cdot w}\, dV(w) = \frac{e^{2(\beta+\bar{\mathbf{1}})\cdot w_{\beta+\bar{\mathbf{1}}}}}{\|\beta+\bar{\mathbf{1}}\|} \int_0^\delta e^{-2t} A_{\beta+\bar{\mathbf{1}}}(t)\, dt. \tag{2.5}$$

Notice that Proposition 2.1 is true for all Reinhardt domains, not necessarily monomial polyhedrons.

In order to sum (2.1) we must take a closer look at $A_{\beta+\bar{\mathbf{1}}}(t)$ for $t\in[0,\delta]$, where $\delta = \delta(M,\varepsilon)$ will be determined later, and apply certain elementary linear algebraic computation to carry out a formula for $A_{\beta+\bar{\mathbf{1}}}(t)$ and thus $\|z^\beta\|^2$. Here we will make use of the properties of monomial polyhedra and Definition 1.10 of (M,ε)-nondegeneracy.

Let $\beta+\bar{\mathbf{1}}\in\Gamma(\mathcal{A})$ for some $\mathcal{A}=\{\alpha_1,\dots,\alpha_k\}$ in $\mathbb{A}$, let $\mathcal{F}=\mathcal{F}(\mathcal{A})$, and assume that $A^{n-k}(\mathcal{F})$ is the area of $\mathcal{F}$ measured as an $(n-k)$-dimensional object. Notice that $A^{n-k}(\mathcal{F})$ is never zero (by the definition of (M,ε)-nondegeneracy) whereas $A_{\beta+\bar{\mathbf{1}}}(0)$, measuring the same face $\mathcal{F}$ as a $(n-1)$-dimensional object, is usually zero unless $|\mathcal{A}(\mathcal{F})|=1$.

Our purpose for the rest of this section is to show that

$$A_{\beta+\bar{\mathbf{1}}}(t) \approx \frac{\|\beta+\bar{\mathbf{1}}\| t^{k-1}\cdot A^{n-k}(\mathcal{F})}{\lambda_1\cdots\lambda_k}.$$

Combining (2.4) and (2.5), this implies

$$\|\zeta_\beta\|^2 = \int_P |z^\beta|^2\, dV(z) \approx \frac{e^{2(\beta+\bar{\mathbf{1}})\cdot w_{\beta+\bar{\mathbf{1}}}}\cdot A^{n-k}(\mathcal{F})}{\lambda_1\cdots\lambda_k},$$

where the ratio depends only on M, ε, and n.

First let us simplify the domain. Let

$$\mathbb{A}(\mathcal{A}) = \{\,\mathcal{A}'\in\mathbb{A} : \mathcal{A}\subset\mathcal{A}'\,\}$$

and

$$\mathcal{B}(\mathcal{A}) = \bigcup_{\mathcal{A}'\in\mathbb{A}(\mathcal{A})} \mathcal{A}' = \{\alpha_1,\dots,\alpha_k,\alpha_{k+1},\dots,\alpha_{k+m}\}.$$

Notice that the boundary of $\mathcal{F}(\mathcal{A})$ is a union of all faces $\mathcal{F}(\mathcal{A}')$, where $\mathcal{A}\subsetneq\mathcal{A}'\in\mathbb{A}(\mathcal{A})$.

By a unitary change of coordinates, we can assume $\alpha_j = (\alpha_{j,1},\dots,\alpha_{j,j},0,\dots,0)$ for $j=1,\dots,k$. Thus $\mathcal{F}$ will be defined by

$$\begin{aligned}\alpha_1 \cdot w &= 0\\ &\vdots\\ \alpha_k \cdot w &= 0\\ \alpha_{k+1} \cdot w &< C_1\\ &\vdots\\ \alpha_{k+m} \cdot w &< C_m,\end{aligned}$$

and $\log(P)$ around $\mathcal{F}$ will be defined by

$$\begin{aligned}\alpha_1 \cdot w &< 0\\ &\vdots\\ \alpha_k \cdot w &< 0\\ \alpha_{k+1} \cdot w &< C_1\\ &\vdots\\ \alpha_{k+m} \cdot w &< C_m.\end{aligned}$$

By writing all elements x in $\mathbb{R}^n$ into x' in $\mathbb{R}^k$ and x'' in $\mathbb{R}^{n-k}$, where x' consists of the first k components while x'' consists of the rest, we can see that

$$\begin{aligned}\alpha_1 \cdot w &= \alpha_1' \cdot w'\\ &\vdots\\ \alpha_k \cdot w &= \alpha_k' \cdot w'\end{aligned}$$

and that $\Omega_{\beta+\bar{\mathbf{1}}}(t)$, for $t \in [0, \delta]$, is defined by

$$\begin{aligned}(\beta + \bar{\mathbf{1}})' \cdot w' &> -t\\ \alpha_1' \cdot w' &< 0\\ &\vdots\\ \alpha_k' \cdot w' &< 0\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}\alpha_{k+1}'' \cdot w'' &< C_1 - \alpha_{k+1}' \cdot w'\\ &\vdots\\ \alpha_{k+m}'' \cdot w'' &< C_m - \alpha_{k+m}' \cdot w'.\end{aligned} \tag{2.7}$$

Note that $(\beta + \bar{\mathbf{1}}) \cdot w = (\beta + \bar{\mathbf{1}})' \cdot w'$.

For each w' satisfying (2.6), define $\mathcal{W}(w') = \{\, w'' : (w', w'') \in \log(P) \,\}$. We know that when $w' = 0$, the set of all possible w'' satisfying system (2.7) (i.e., $\mathcal{W}(0)$) is exactly $\mathcal{F}$, and we want to understand by how much the volume of $\mathcal{W}(w')$ can vary when w' changes.

The possible values that w' can take are controlled only by the system of inequalities (2.6) for $t \in [0, \delta]$, which defines a k-simplex. Since all inequalities involved in (2.7) are linear, we know the maximal change of the volume of $\mathcal{W}(w')$ happens on the extreme points of the k-simplex defined by (2.6) with $t \in [0, \delta]$.

The extreme points are either 0 or the solution to the system of equations

$$\begin{aligned}
\alpha_1' \cdot (w^j)' &= 0 \\
&\vdots \\
\alpha_{j-1}' \cdot (w^j)' &= 0 \\
(\beta + \bar{\mathbf{1}})' \cdot (w^j)' &= -t \\
\alpha_{j+1}' \cdot (w^j)' &= 0 \\
&\vdots \\
\alpha_k' \cdot (w^j)' &= 0
\end{aligned}$$

for $j = 1, \ldots, k$. By plugging in $\beta + \bar{\mathbf{1}} = \sum_{j=1}^{k} \lambda_j \alpha_j$ and writing $[\,\alpha\,]_k = [\,\alpha_{i,j}\,]$, $i, j = 1, \ldots, k$, we have a unique solution

$$(w^j)' = [\,\alpha\,]_k^{-1} \left(\frac{-t}{\lambda_j} \right) e_j, \quad \text{where } e_j = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}.$$

But plugging this information back into

$$\begin{aligned}
\alpha_{k+1}'' \cdot w'' &< C_1 - \alpha_{k+1}' \cdot w' \\
&\vdots \\
\alpha_{k+m}'' \cdot w'' &< C_m - \alpha_{k+m}' \cdot w'
\end{aligned}$$

(system (2.7)) when $\alpha_{k+i}' = \sum_{j=1}^{k} \lambda_{i,j} \alpha_j$, we have

$$\alpha_{k+i}' \cdot (w^j)' = \frac{-t\lambda_{i,j}}{\lambda_j}.$$

Thus, for all $j = 1, \ldots, k$,

$$\begin{aligned}
\alpha_{k+1}'' \cdot w'' &< C_1 + t\lambda_{1,j}/\lambda_j \\
&\vdots \\
\alpha_{k+m}'' \cdot w'' &< C_m + t\lambda_{m,j}/\lambda_j.
\end{aligned}$$

Notice that α has positive integer components in the original coordinates and, by Definition 1.10(i), the sum of components of α is less than M. Hence λ_j is positive and bounded away from 0 by a constant depending only on M, and $\lambda_{i,j}$ is positive and bounded by M. Thus there exist $C = C(M) > 0$ such that

$$\left| \frac{\lambda_{i,j}}{\lambda_j} \right| < C(M), \quad i = 1, \ldots, m, \;\; j = 1, \ldots, k.$$

Next, using Definition 1.10(iii) on $\mathcal{F}$ (where $\mathcal{F}$ can be expressed by the system (2.7) when $w' = 0$) and taking $w'' = \varepsilon \cdot \alpha_{k+i}''/\|\alpha_{k+i}''\|$, since $\|w''\| = \varepsilon$ and $w'' \in \bar{\mathcal{F}}$ we have $\varepsilon \cdot \|\alpha_{k+i}''\| \leq C_i$. But $\|\alpha_{k+i}''\| \geq c$ for some $c = c(M) > 0$. Therefore, $C_i \geq \varepsilon \cdot c(M)$.

Now set $\delta = \varepsilon c(M)/2C(M)$, define $\mathcal{F}_d = \{ w'' \in \mathbb{R}^{n-k} : \alpha''_{k+i} \cdot w'' < d \cdot C_i,\ i = 1, \ldots, m \}$, and define $L_{\beta+\bar{\mathbf{1}}}(t) = \{ w' \in \mathbb{R}^n : \alpha'_j \cdot w' < 0$ for $j = 1, \ldots, k$ and $(\beta + \bar{\mathbf{1}})' \cdot w' = -t \}$. First, notice $\mathcal{F} = \mathcal{F}_1$. We also have

$$L_{\beta+\bar{\mathbf{1}}}(t) \times \mathcal{F}_{1/2} \subseteq S_{\beta+\bar{\mathbf{1}}}(t) \subseteq L_{\beta+\bar{\mathbf{1}}}(t) \times \mathcal{F}_2 \quad \text{for } t \in [0, \delta]. \tag{2.8}$$

Now let $P_{\beta+\bar{\mathbf{1}}}(\delta) = \bigcup_{t\in[0,\delta]} L_{\beta+\bar{\mathbf{1}}}(t)$. Then $P_{\beta+\bar{\mathbf{1}}}(\delta)$ is a k-simplex with extreme points at 0 and $(w^j)'$ for $j = 1, \ldots, k$, where $(w^j)' = [\,\alpha\,]_k^{-1}(-t/\lambda_j)e_j$. But for any k-simplex with extreme points at 0 and $a_i = (a_{i,j})$, $i, j = 1, \ldots, k$, the volume is exactly $\frac{1}{k!}\det[\,a_{i,j}\,]$. Thus, the volume of $P_{\beta+\bar{\mathbf{1}}}(t)$ is

$$A^k(P_{\beta+\bar{\mathbf{1}}}(t)) = \frac{t^k}{k! \cdot \det[\,\alpha\,]_k \cdot \lambda_1 \cdots \lambda_k}.$$

However,

$$A^k(P_{\beta+\bar{\mathbf{1}}}(t)) = \frac{1}{\|\beta + \bar{\mathbf{1}}\|} \int_0^t A^{k-1}(L_{\beta+\bar{\mathbf{1}}}(\eta))\, d\eta.$$

By taking derivatives with respect to t on both ends of the preceding equations, we have the volume of $L_{\beta+\bar{\mathbf{1}}}(t)$ as

$$A^{k-1}(L_{\beta+\bar{\mathbf{1}}}(t)) = \frac{\|\beta + \bar{\mathbf{1}}\| \cdot t^{k-1}}{(k-1)! \cdot \det[\,\alpha\,]_k \cdot \lambda_1 \cdots \lambda_k}.$$

Also note that

$$A^{n-k}(\mathcal{F}_d) = d^{n-k} \cdot A^{n-k}(\mathcal{F}_1), \quad \text{where } \mathcal{F}_1 = \mathcal{F}.$$

Thus, by (2.8), for $t \in [0, \delta]$ we have

$$\begin{aligned} \frac{2^{-n} \cdot \|\beta + \bar{\mathbf{1}}\| \cdot t^{k-1} \cdot A^{n-k}(\mathcal{F})}{(k-1)! \cdot \det[\,\alpha\,]_k \cdot \lambda_1 \cdots \lambda_k} &\le A_{\beta+\bar{\mathbf{1}}}(t) \\ &\le \frac{2^{n} \cdot \|\beta + \bar{\mathbf{1}}\| \cdot t^{k-1} \cdot A^{n-k}(\mathcal{F})}{(k-1)! \cdot \det[\,\alpha\,]_k \cdot \lambda_1 \cdots \lambda_k} \end{aligned}$$

or simply

$$A_{\beta+\bar{\mathbf{1}}}(t) \approx \frac{\|\beta + \bar{\mathbf{1}}\| \cdot t^{k-1} \cdot A^{n-k}(\mathcal{F})}{(k-1)! \cdot \det[\,\alpha\,]_k \cdot \lambda_1 \cdots \lambda_k}, \tag{2.9}$$

where the ratio depends only on the total dimension n.

Combining (2.4), (2.5), and (2.9) with $\zeta_\beta(z) = z^\beta$, we have

$$\begin{aligned} \|\zeta_\beta\|^2 &\approx \frac{e^{2(\beta+\bar{\mathbf{1}})\cdot w_{\beta+\bar{\mathbf{1}}}} \cdot A^{n-k}(\mathcal{F})}{(k-1)! \cdot \det[\,\alpha\,]_k \cdot \lambda_1 \cdots \lambda_k} \int_0^\delta e^{-2t} t^{k-1}\, dt \\ &\approx \frac{e^{2(\beta+\bar{\mathbf{1}})\cdot w_{\beta+\bar{\mathbf{1}}}} \cdot A^{n-k}(\mathcal{F})}{\det[\,\alpha\,]_k \cdot \lambda_1 \cdots \lambda_k}, \end{aligned}$$

where the first approximation depends only on n while the second approximation depends on n and δ, which in turn is defined as a function of M and ε.

Let us summarize in the form of a proposition.

PROPOSITION 2.2. *Let* $\log(P)$ *be* (M, ε)*-nondegenerate. For any* β *in* $\mathcal{N}^n$*, there exists a unique* $\mathcal{A}$ *in* $\mathbb{A}$ *such that* $\beta + \bar{\mathbf{1}}$ *is in* $\Gamma(\mathcal{A})$*. For any* $w_{\beta+\bar{\mathbf{1}}}$ *in* $\mathcal{F} = \mathcal{F}(\mathcal{A})$ *we have, with the ratio depending on* M, ε, *and* n,

$$\|z^\beta\|^2 \approx \frac{e^{2(\beta+\bar{\mathbf{1}})\cdot w_{\beta+\bar{\mathbf{1}}}} \cdot A^{n-k}(\mathcal{F})}{\det[\,\alpha\,]_k \cdot \lambda_1 \cdots \lambda_k},$$

where $A^{n-k}(\mathcal{F})$ *is the* $(n-k)$*-dimensional volume of* $\mathcal{F}$*. When* $k = n$, $A^0(\mathcal{F}) = 1$.

REMARK 2.1. Because $(\beta + \bar{\mathbf{1}}) \cdot w$ is constant for w in $\mathcal{F}$, it does not matter which $w_{\beta+\bar{\mathbf{1}}}$ we choose in $\mathcal{F}$ for Proposition 2.2. Also, the log term we usually see in the Bergman kernel will come out naturally from the calculation of the term $A^{n-k}(\mathcal{F})$ when $k < n$.

3. Estimate for the Bergman Kernel on a Diagonal

Using (2.1) on the diagonal with $\zeta_\beta(z) = z^\beta$, we have

$$K_P(z_0, z_0) = \sum_{\mathcal{A}\in\mathbb{A}} \sum_{\beta+\bar{\mathbf{1}}\in\Gamma(\mathcal{A})} \frac{|z_0^\beta|^2}{\|\zeta_\beta\|^2}. \tag{3.1}$$

The first summation is a finite sum, for there are only finitely many $\mathcal{A}$ in $\mathbb{A}$. We want to express the second one as a finite sum, too, by considering the fundamental set $\mathcal{U}(\mathcal{A})$ of the open cone $\Gamma(\mathcal{A})$, where

$$\mathcal{U}(\mathcal{A}) = \Gamma(\mathcal{A}) \cap \left\{ \alpha : \alpha = \sum_{j=1}^{k} \lambda_j \alpha_j,\ 0 < \lambda_j \le 1 \right\}.$$

Notice that $\mathcal{U}(\mathcal{A})$ contains only finitely many indices, and that the open cone $\Gamma(\mathcal{A})$ can be decomposed into the fundamental set $\mathcal{U}(\mathcal{A})$ and its integral multiple translations $\mathcal{U}_{(m_1,\ldots,m_n)}(\mathcal{A})$, where

$$\mathcal{U}_{(m_1,\ldots,m_k)}(\mathcal{A}) = \sum_{j=1}^{k} m_j \alpha_j + \mathcal{U}(\mathcal{A})$$

for $m_j = 0, 1, 2, \ldots$ $(j = 1, \ldots, k)$. That is, for $\beta + \bar{\mathbf{1}} \in \Gamma(\mathcal{A})$, there exist nonnegative integers $m_1, \ldots, m_k$ and nonnegative numbers $\lambda_1, \ldots, \lambda_k$ such that $0 < \lambda_j \le 1$ and

$$\beta + \bar{\mathbf{1}} = \sum_{j=1}^{k} (m_j + \lambda_j)\alpha_j.$$

But notice that, since all components in $\beta + \bar{\mathbf{1}}$ take positive integer values and the sum of components for all elements in $\mathcal{A}$ is bounded by M, the value for λ_j $(j = 1, \ldots, k)$ is bounded away from 0 where the lower bound depends only on M. Thus we have, with the ratio depending only on M,

$$m_j + \lambda_j \approx m_j + 1.$$

By letting $w_0 = \log|z_0|$ and $z_{\mathcal{F}} = e^{w_{\mathcal{F}}}$, where $w_{\mathcal{F}}$ is any point on $\mathcal{F} = \mathcal{F}(\mathcal{A})$, we have

$$\begin{aligned}
&\sum_{\beta+\bar{\mathbf{1}}\in\Gamma(\mathcal{A})} \frac{|z_0^\beta|^2}{\|z^\beta\|^2} \\
&\quad \approx \sum_{m_1=0}^{\infty}\cdots\sum_{m_k=0}^{\infty}\sum_{\beta+\bar{\mathbf{1}}\in\mathcal{U}(\mathcal{A})} \frac{(m_1+1)\cdots(m_k+1)\cdot\left|z_0^{\beta+\sum_{j=1}^k m_j\alpha_j}\right|^2}{\left|z_{\mathcal{F}}^{(\beta+\bar{\mathbf{1}})+\sum_{j=1}^k m_j\alpha_j}\right|^2\cdot A^{n-k}(\mathcal{F})} \\
&\quad \approx \prod_{j=1}^k\left(\sum_{m=0}^{\infty}(m+1)e^{2m\alpha_j\cdot(w_0-w_{\mathcal{F}})}\right)\cdot\sum_{\beta+\bar{\mathbf{1}}\in\mathcal{U}(\mathcal{A})}\frac{|z_0^\beta|^2}{\|z^\beta\|^2} \\
&\quad = \prod_{j=1}^k\frac{1}{(1-e^{2\alpha_j\cdot(w_0-w_{\mathcal{F}})})^2}\cdot\sum_{\beta+\bar{\mathbf{1}}\in\mathcal{U}(\mathcal{A})}\frac{|z_0^\beta|^2}{\|z^\beta\|^2} \\
&\quad = \prod_{j=1}^k\frac{1}{(1-|(z_0/z_{\mathcal{F}})^{\alpha_j}|^2)^2}\cdot\sum_{\beta+\bar{\mathbf{1}}\in\mathcal{U}(\mathcal{A})}\frac{|z_0^\beta|^2}{\|z^\beta\|^2} \\
&\quad \approx \prod_{j=1}^k\frac{1}{(1-|(z_0/z_{\mathcal{F}})^{\alpha_j}|)^2}\cdot\sum_{\beta+\bar{\mathbf{1}}\in\mathcal{U}(\mathcal{A})}\frac{|z_0^\beta|^2}{\|z^\beta\|^2}.
\end{aligned}$$

Finally by (3.1), we have the following result.

THEOREM 3.1. *Let P be an (M,ε)-nondegenerate bounded monomial polyhedron, and let $\zeta_\beta(z) = z^\beta$. Then for $\mathcal{A} = \{\alpha_1,\ldots,\alpha_k\}$ we have*

$$K_P(z_0,z_0) \approx \sum_{\mathcal{A}\in\mathbb{A}}\left(\prod_{j=1}^k\frac{1}{(1-|(z_0/z_{\mathcal{F}})^{\alpha_j}|)^2}\cdot\sum_{\beta+\bar{\mathbf{1}}\in\mathcal{U}(\mathcal{A})}\frac{|z_0^\beta|^2}{\|\zeta_\beta\|^2}\right), \tag{3.2}$$

with the ratio depending only on M, ε, and n. That is, there exist constants $C = C(M,\varepsilon,n)$ and $c = c(M,\varepsilon,n)$ such that $C > c > 0$ and the Bergman kernel for P can be estimated as

$$\begin{aligned}
c\cdot\sum_{\mathcal{A}\in\mathbb{A}}&\left(\prod_{j=1}^k\frac{1}{(1-|(z_0/z_{\mathcal{F}})^{\alpha_j}|)^2}\cdot\sum_{\beta+\bar{\mathbf{1}}\in\mathcal{U}(\mathcal{A})}\frac{|z_0^\beta|^2}{\|\zeta_\beta\|^2}\right) \\
&\le K_P(z_0,z_0) \\
&\le C\cdot\sum_{\mathcal{A}\in\mathbb{A}}\left(\prod_{j=1}^k\frac{1}{(1-|(z_0/z_{\mathcal{F}})^{\alpha_j}|)^2}\cdot\sum_{\beta+\bar{\mathbf{1}}\in\mathcal{U}(\mathcal{A})}\frac{|z_0^\beta|^2}{\|\zeta_\beta\|^2}\right),
\end{aligned}$$

where $\log|z_{\mathcal{F}}| \in \mathcal{F} = \mathcal{F}(\mathcal{A})$.

REMARK 3.1. $|z_{\mathcal{F}}^{\alpha_j}|$ does not depend on the choice of $z_{\mathcal{F}}$ as long as $\log|z_{\mathcal{F}}| \in \mathcal{F}(\mathcal{A})$ and $\alpha_j \in \mathcal{A}$.

References

[1] J. Bruna, A. Nagel, and S. Wainger, *Convex hypersurfaces and Fourier transforms,* Ann. of Math. (2) 127 (1988), 333–365.
[2] D. Catlin, *Boundary invariants of pseudoconvex domain*, Ann. of Math. (2) 120 (1984), 529–586.
[3] ———, *Estimates of invariant metrics on pseudoconvex domains of dimension two,* Math. Z. 200 (1989), 429–466.
[4] D. C. Chang, A. Nagel, and E. M. Stein, *Estimates for the* $\bar\partial$*-Neumann problem in pseudoconvex domains of finite type in* $\mathbb{C}^2$, Acta Math. 169 (1992), 153–228.
[5] J. D'Angelo, *Real hypersurfaces, orders of contact, and applications,* Ann. of Math. (2) 115 (1982), 615–637.
[6] L. Hörmander, *An introduction to complex analysis in several variables,* 3rd ed., North-Holland, Amsterdam, 1990.
[7] J. McNeal, *Boundary behavior of the Bergman kernel function in* $\mathbb{C}^2$, Duke Math. J. 58 (1989), 499–512.
[8] ———, *Estimates on the Bergman kernels of convex domains,* Adv. Math. 109 (1994), 108–139.
[9] A. Nagel, J. P. Rosay, E. M. Stein, and S. Wainger, *Estimates for the Bergman and Szegö kernels in* $\mathbb{C}^2$, Ann. of Math. (2) 129 (1989), 113–149.

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