A Pure Power Product Version of the Hilbert Nullstellensatz

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I. Background

Let k be a field and $R = k[x_0, \dots, x_n]$. Then what one might call the *radical version* of Hilbert's Nullstellensatz states that, for any homogeneous ideal $\mathfrak{A} = (f_1, \dots, f_m)$ with radical $\mathfrak{R}$, some power of $\mathfrak{R}$ lies in $\mathfrak{A}$:

$$\mathfrak{R}^e \subset \mathfrak{A}.$$

From now on, let us denote by e the minimum such exponent for this $\mathfrak{A}$.

Rabinowitsch [Ra] showed that this formulation is equivalent to the following (apparently weaker) assertion, which has been called the *Bezout version* of the Nullstellensatz: If $g_1, \dots, g_m$ in $S = k[x_1, \dots, x_n]$ have no common zeros (say, in an algebraic closure of k), then there exist $a_1, \dots, a_m$ in S such that

$$1 = a_1 g_1 + \cdots + a_m g_m.$$

Denote by a the minimal value of $\max \deg a_i$ of all choices of a_i in S satisfying this identity.

If $d_i = \deg f_i = \deg g_i > 0$ $(i = 1, \dots, m)$, then general upper bounds for e and a are intimately related, and here we regard them as equivalent. Nearly optimal bounds for a and e were achieved almost a decade ago. To discuss these bounds, for the remainder of the paper let us order the degrees so that

$$D = d_2 \geq d_3 \geq \cdots \geq d_m \geq d_1.$$

The classical work of Hermann [He] was taken up again by Masser and Wüstholz [MW] to establish the first effective version of the Nullstellensatz. They showed that, in the Bezout form,

$$a \leq 2(2D)^{2n-1}.$$

Masser and Philippon gave a family of examples, which was refined a bit in [B2] to show that, in certain cases,

$$a = D^n - D;$$

correspondingly, $e \geq D^n$. Another family of examples was devised by Kollár [Ko].

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The present author used analytic considerations suggested by Berenstein and Yger and an inequality of [B1] developed for algebraic independence proofs to show [B2; B3] that, for characteristic zero,

$$a \leq n\mu D^{\mu} + \mu D,$$

where $\mu = \min\{m, n\}$. I remarked later [B4, p. 16] that the factor n is unnecessary, but never published the details, as the application was superceded by [Ko] and the underlying inequality by [JKS] (which in turn relies on the results of this paper).

The jump to arbitrary characteristic was made by Caniglia, Galligo, and Heintz, who proved [CGH] a radical form of the Nullstellensatz with

$$e \leq D^{n(n+3)/2}.$$

Very soon thereafter, Kollár employed local cohomology in an inspired way to establish [Ko] that, under the restriction that all but three $d_i \geq 3$,

$$e \leq d_1 \dots d_{\mu},$$

independent of the the characteristic of k. In light of the lower bound for a and therefore for e, Kollár's result is optimal of its form.

REPRISE. Since the basic work of this paper was completed, much additional interesting work on the Nullstellensatz has appeared. Shiffman [Sh] also uses cohomological methods to obtain bounds of roughly the strength of [CGH]. Moreover Philippon [P1] gives a very nice proof of the Bezout form—based on Kollár's proof but using homology of Koszul complexes—and he bounds the denominators over a certain class of fields including $\mathbb{Q}$ [P2]. Smietanski [Sm] takes up this approach in the case that the coefficients themselves come from a polynomial ring.

Berenstein and Yger have carried out an impressive program to obtain excellent arithmetic and geometric bounds in the Bezout form of the Nullstellensatz. In [BY1; BY2] they use Philippon's work and explicit integral identities to obtain excellent bounds for the sizes of the coefficients involved in the numerators and denominators when working over $\mathbb{Q}$. In [BGVY] they continue the surprising use of analytic tools and Grothendieck residues to obtain strong arithmetic information in the Bezout form. Finally, they develop [BY3] Lipman's algebraic theory of residues to obtain a powerful and purely algebraic approach while maintaining the overall strategy.

In [FG], Fitchas and Galligo give a detailed proof of Kollár's result using Ext. The excellent survey article [Te] reports on several of the developments which had taken place at that time. In particular, it gives the proof included in the first version of this paper, entitled "A Prime Power Product Version of the Nullstellensatz."

Heintz, Giusti, and co-workers have introduced straight-line programs arising from randomized arithmetic networks to determine whether polynomials have common zeros and, if not, to find coefficients in a Bezout identity [FGS; GHS].

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II. Statement of Results

In this note we adapt Kollár's cohomological technique somewhat to obtain a refinement of his radical Nullstellensatz, which we call a *pure power product version* of the Nullstellensatz. Moreover, properties of the classical Hilbert function provide a uniformly clean formulation as in [Ko] but without invoking excess intersection theory. For convenience we explicitly recall the notation of the introduction before stating the first result.

HYPOTHESES AND NOTATION. Let the ideal $\mathfrak{A}$ be generated by homogeneous polynomials $f_1, \ldots, f_m \in R = k[x_0, \ldots, x_n]$, with k a field. Assume that the polynomials f_i are indexed so that their degrees satisfy $d_2 \geq \cdots \geq d_m \geq d_1$. Let $\mathfrak{M}$ denote the maximal homogeneous ideal; that is, $\mathfrak{M} = (x_0, \ldots, x_n)$, the so-called *irrelevant* prime ideal. Finally make the (annoying) technical assumption that $d_{\mu-\rho+1} \geq 3$, where $\rho > 1$ denotes the height of $\mathfrak{A}$ and $\mu = \min\{m, n\}$.

THEOREM 1 (Prime Power Version). *There are relevant prime ideals* $\mathfrak{P}_1, \ldots, \mathfrak{P}_r$ *containing* $\mathfrak{A}$, *positive integers* $e_1, \ldots, e_r$, *and integral* $e_0 \geq 0$ *such that*

$$\mathfrak{M}^{e_0}\mathfrak{P}_1^{e_1} \ldots \mathfrak{P}_r^{e_r} \subset \mathfrak{A}, \tag{1}$$

where

$$e_0 + \sum e_i \deg \mathfrak{P}_i \leq d_1 \ldots d_\mu \tag{2}$$

when Card $k \geq d_1 \ldots d_\mu$.

CODA. *Regardless of* Card k, *there are always homogeneous prime ideals* $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ *in* $K[x_0, \ldots, x_n]$, $K = k(t)$, *with* $\mathfrak{P}_i = \mathfrak{p}_i \cap R$ *satisfying* (1) *and*

$$e_0 + \sum e_i \deg \mathfrak{p}_i \leq d_1 \ldots d_\mu.$$

REMARK 1. Note that e_0 receives special treatment only because $\deg \mathfrak{M}$ is null, and therefore we must either bound its contribution separately with regard to degree or else give $\mathfrak{M}$ "honorary" degree equal to 1.

REMARK 2. The proof furnishes such a product in which every *isolated* prime component $\mathfrak{P}$ of $\mathfrak{A}$ occurs with an exponent that is not less than the product of the exponent of its associated primary component in primary decompositions of $\mathfrak{A}$, multiplied by the factor $(1+3^\delta)/2$, where δ denotes the dimension of $\mathfrak{P}$ (i.e., $\delta = n - \text{height}\,\mathfrak{P}$).

Recall that $\rho(\mathfrak{P})$, the *height* of a prime ideal $\mathfrak{P}$, is defined to be the greatest integer s for which there is a strictly ascending chain of prime ideals

$$\mathfrak{P}_0 < \cdots < \mathfrak{P}_{s-1} < \mathfrak{P}_s = \mathfrak{P}.$$

There are now important contexts where $\mathfrak{P}$ already has another notion of height naturally ascribed to it, for example, via the Chow form when k is a number field. So in connivance with S. Lang, we propose the descriptive word "elevation" instead of "height". The notion has also been termed "rank" [No], which raises other dissonances. However, since no conflict of meaning occurs in this paper, we do not insist on the new coinage here. The height $\rho(\mathfrak{I})$ for any homogeneous ideal $\mathfrak{I}$ of R is the least height of any homogeneous prime ideal containing $\mathfrak{I}$. The ideal $\mathfrak{I}$ is said to be *unmixed* or *pure* if all its associated prime ideals have the same height.

We recover Kollár's sharp radical Nullstellensatz from the Coda, with $e \leq \sum e_i$, since for each $i = 1, \ldots, r$, $\deg \mathfrak{p}_i \geq 1$ and

$$\mathfrak{R} = \operatorname{Rad}(\mathfrak{A}) \subset \mathfrak{P}_i.$$

COROLLARY (Kollár). $e \leq d_1 \ldots d_\mu$.

REMARK 3. Another proof of Kollár's result is obtained from the theorem as follows. Let $e^* = \sum e_i$. If $\operatorname{Card} k \geq d_1 \ldots d_\mu$, then $\mathfrak{R} = \operatorname{Rad}(\mathfrak{A}) \subset \mathfrak{P}_i$ $(i = 1, \ldots, r)$ and, by the theorem, $\mathfrak{R}^{e^*} \subset \mathfrak{A}$.

If $\operatorname{Card} k < d_1 \ldots d_\mu$, then we can replace k by an algebraic extension K of sufficiently large degree to obtain prime ideals $\mathfrak{P}_1, \ldots, \mathfrak{P}_r$ of KR as in the theorem. Set $\mathfrak{p}_i = \mathfrak{P}_i \cap R$ $(i = 1, \ldots, r)$, so that $\mathfrak{R} \subset \mathfrak{p}_i$ and $\mathfrak{R}^{e^*} \subset \mathfrak{M}_0^{e_0}\mathfrak{p}_1^{e_1} \ldots \mathfrak{p}_r^{e_r} \subset K\mathfrak{A} \cap R$.

THEOREM 2 (Pure Power Version). *Assume that* $\operatorname{Card} k \geq d_1 \ldots d_\mu$. *For* $i = \rho = \rho(\mathfrak{A}), \ldots, n+1$, *there are exponents* $r_i \in \mathbb{Z}_{\geq 0}$ *and unmixed homogeneous ideals* K_i *satisfying the following conditions*:

(i) *if* $\mathfrak{K}_i \neq R$, *then* $\rho(\mathfrak{K}_i) = i$ *and* $\mathfrak{A} \subset \operatorname{Rad}(\mathfrak{K}_i)$;
(ii) $\mathfrak{K}_\rho^{r_\rho} \ldots \mathfrak{K}_n^{r_n} \mathfrak{M}^{r_{n+1}} \subset \mathfrak{A}$; *and*
(iii) $r_{n+1} + \sum_{\rho \leq j \leq n} r_j \deg \mathfrak{K}_j \leq d_1 \ldots d_\mu$.

REMARK 4. The proof of this theorem gives such a product with $\mathfrak{K}_i = R$ for any $i > m$. Moreover, to any isolated prime component $\mathfrak{P}$ of $\mathfrak{A}$ of height i there corresponds a $\mathfrak{P}$-primary component of $\mathfrak{K}_i$ of length not less than the length of the corresponding primary component $\mathfrak{Q}$ of $\mathfrak{A}$. For by construction (Lemma 0), $\mathfrak{P}$ will be an isolated prime component of $(h_1, \ldots, h_i)$, and in the local ring $R_{\mathfrak{P}}$,

$$\mathfrak{K}_i R_{\mathfrak{P}} = (h_1, \ldots, h_i) R_{\mathfrak{P}} \subset \mathfrak{Q} R_{\mathfrak{P}}.$$

REMARK 5. The assertions hold with no lower bounds assumed on the d_i if either

(i) $f_1, \ldots, f_m$ form a regular sequence (in which case no lower bounds on $\operatorname{Card} k$ are necessary either), or
(ii) $\rho = 1$.

The first claim follows from Bezout's theorem (Lemma 3). For the second claim, see Section VIII.

REMARK 6. Of course, if $\mathfrak{A}$ has no nontrivial zeros, then Macaulay's theorem (cf. Lemmas 0 and 4(2) with $\mathfrak{K}_n = R$ and $\mathfrak{I}_n = (h_1, \ldots, h_n)$) shows linear rather than multiplicative growth of the exponent with respect to the d_i:

$$\mathfrak{M}^e \subset \mathfrak{A}^d, \quad e := d_1 + \cdots + d_{n+1} - n.$$

REMARK 7. The prime power version of Nullstellensatz follows directly from the pure power version of Nullstellensatz. For then we write a primary decomposition for $\mathfrak{K}_i$ as $\mathfrak{K}_i = \bigcap_{j \in J_i} \mathfrak{Q}_j$, with each $\mathfrak{Q}_j$ a $\mathfrak{P}_j$-primary ideal and $\deg \mathfrak{K}_i = \sum(\operatorname{length} \mathfrak{Q}_j)(\deg \mathfrak{P}_j)$. (Compare, for example, [Gb, Satz V, p. 171].) However, for any $\mathfrak{P}$-primary ideal $\mathfrak{Q}$, the least exponent t such that $\mathfrak{P}^t \subset \mathfrak{Q}$, called the *exponent* of $\mathfrak{Q}$, satisfies $t \le \operatorname{length} \mathfrak{Q}$ (cf. [BM, Lemma 4]). Thus, by the pure power version, $\prod_i \big(\prod_j \big(\mathfrak{P}_j^{\operatorname{length} \mathfrak{Q}_j}\big)^{r_i}\big) \subset \mathfrak{A}$.

REMARK 8. Since Hilbert's intent in the Nullstellensatz is essentially to relate *powers* with containment, the pure power version may seem to be swimming against the current. For it somewhat schizophrenically veers toward the basic mind-set in the Lasker–Noether theory: intersection of primary ideals versus the more primitive powers of the radical ideal.

Since the response represents a personal point of view, the first person singular is appropriate here. I admit to not having a well-defined purpose in the pure power version. I see the prime power form as a natural extension of the point of view of the pure power version compared to the radical version. Its formulation is meant to evoke further questions and, possibly, to provoke further investigations before fixing a "true" balance point in the fruitful tension between the multiplicative point of view and that of Lasker–Noether theory.

For example, it would be interesting to know whether one can, in general, replace certain of the remaining powers in the pure power version by symbolic powers. It might be interesting to weaken the Bezout-type bound a bit and insist that the prime ideals involved in the prime power version be isolated. Some weakening of the degree bounds would be necessary, as is shown by the next example.

KOLLÁR'S EXAMPLE. Kollár has shown that one cannot in general use only isolated prime components of $\mathfrak{A}$ in a prime power product lying in $\mathfrak{A}$ and still satisfy the bound just given on the degrees. The pair of homogeneous polynomials

$$f_1 = (x_0^{d-1}x_1 - x_2^d)x_2^d \quad \text{and} \quad f_2 = (x_0^{d-1}x_1 - x_2^d)(x_0^{d-1}x_2 - x_1^d)$$

share a common factor and therefore define an ideal $\mathfrak{I}$ of dimension 1 whose unique isolated prime component $\mathfrak{P} = (x_0^{d-1}x_1 - x_2^d)$ has degree d in $k[x_0, x_1, x_2]$. But $\mathfrak{I}$ also has an embedded component of degree d^2, since

$$(f_1, f_2) = (x_0^{d-1}x_1 - x_2^d)(x_2^d, x_0^{d-1}x_2 - x_1^d).$$

The lowest power e of the isolated prime ideal $\mathfrak{P}$ lying in $\mathfrak{I}$ is $e = d^2 + 1$. Thus $e \cdot \deg \mathfrak{P} = (d^2 + 1)d > 4d^2 = (\deg f_1)(\deg f_2)$, as soon as $d \ge 4$.

Our approach to the pure product version is that of Kollár's, although the reader will find our point of view somewhat more algebraic and our exposition quite a bit more leisurely, in order to provide easier access for a more general audience. As in [BM; B1; B2], we take general homogeneous linear combinations of our generators so that we are reduced to treating ideals $(\mathfrak{J}, h) = \mathfrak{G} \cap \mathfrak{E}$, where h does not lie in any isolated prime component of the unmixed ideal $\mathfrak{J}$, and where $\mathfrak{G}$ involves only the isolated components of $(\mathfrak{J}, h)$ and $\mathfrak{E}$ involves only embedded ones. Bezout's theorem deals with $\mathfrak{G}$, but not much seems to be known about the possibilities for $\mathfrak{E}$. In [B1] and [B2], this problem was circumvented in characteristic 0 using complex analysis.

In [Ko], Kollár ingeniously links certain long exact sequences of local cohomology to inject the module $\mathfrak{G}/(\mathfrak{J}, h)$ into $H^1_{\mathfrak{E}^*}(R/\mathfrak{J})$, in arbitrary characteristic. Therefore any annihilator of this cohomology carries $\mathfrak{G}$ into $(\mathfrak{J}, h)$ and in this respect is a multiplicative replacement for $\mathfrak{E}$. When codim $\mathfrak{J} = 1$, the fundamental relation with depth shows that $H^i_{\mathfrak{E}^*} = 0$ for $i <$ height $\mathfrak{E}^*$, and Kollár's technique with long exact cohomology sequences furnishes annihilators for appropriate cohomology of $R/\mathfrak{G}$ in terms of those for $R/\mathfrak{J}$. In brief, the basic premise of the present paper is that Kollár's procedure naturally constructs certain annihilators in terms of products of unmixed ideals whose radicals contain $\mathfrak{A}$.

III. Rather Regular Sequences

To prove our main result, we reduce to the case where the generators $f_1, \dots, f_m$ form a sequence that is as regular as possible. We say that a sequence of homogeneous polynomials $h_1, \dots, h_l \in \mathfrak{A}$ is *rather regular* in $\mathfrak{A}$ if, for $i = 1, \dots, l-1$,

(a) h_{i+1} does not lie in any isolated prime component $\mathfrak{P}$ of $\mathfrak{B}_i = (h_1, \dots, h_i)$ unless $\mathfrak{P} \supset \mathfrak{A}$, and
(b) some isolated prime component of $\mathfrak{B}_i$ does not contain h_{i+1}.

LEMMA 0. *If* $\operatorname{Card} k \geq d_1 \dots d_\mu$, *then there is a rather regular sequence* $h_1(= f_1), h_2, \dots, h_l$ *in* $\mathfrak{A}$, *with each* $\deg h_i = \deg f_{j(i)}$, *for distinct indices* $j(1)(=1)$, $j(2) > \dots > j(l)$ *and such that* $\operatorname{Rad}\mathfrak{A} = \operatorname{Rad}(h_1, \dots, h_l)$. *Then* $l \geq \rho = \rho(\mathfrak{A})$, *while* $h_1, \dots, h_\rho$ *form a regular sequence, and* $\rho \geq$ *the vector space dimension of the* k*-span of the linear forms among* $f_1, \dots, f_m$.

Proof. The argument, but not quite the statement, of [BM, Lemma 5] would apply here. Instead, we give another simple approach, which gives a slightly stronger result. For $i = 1, \dots, l-1$, one selects the largest j such that not all of $f_j, \dots, f_m$ are contained in any isolated prime component of $\mathfrak{B}_i := (h_1, \dots, h_i)$ not containing $\mathfrak{A}$. Consider the linear map

$$\varepsilon\colon k^{N_j} \to k[x_0, \dots, x_n]_{\deg f_j}$$

given by

$$\bar{c} \mapsto c_0 f_j + \sum c_{\iota\lambda} M_{\iota\lambda},$$

where $M_{\iota\lambda}$ runs through the $N_j - 1$ polynomials of the form $f_\iota x_\lambda^{\deg f_j - \deg f_\iota}$ $(\iota > j)$, all having degree equal to $\deg f_j$. For each fixed isolated prime component

$\mathfrak{P}$ of $\mathfrak{B}_i$ not containing $\mathfrak{A}$, $\varepsilon^{-1}(\mathfrak{P})$ is a k-vector space satisfying $\dim \varepsilon^{-1}(\mathfrak{P}) \leq N_j - 1$, since f_j or at least one of the $M_{\iota\lambda}$ lies outside $\mathfrak{P}$. Consequently, as soon as $\operatorname{Card} k \geq \operatorname{Card}\{\,\mathfrak{P} : \mathfrak{P}$ prime component of $\mathfrak{B}_i$ not containing $\mathfrak{A}\,\}$, we see that

$$\bigcup_{\mathfrak{P}} \varepsilon^{-1}(\mathfrak{P}) \not\supseteq k^{N_j}.$$

In other words, there is a choice of $\bar{c} \in k^{N_j}$ with $\varepsilon(\bar{c})$ not lying in $\bigcup \mathfrak{P}$. By the definition of f_j it follows that $c_0 \neq 0$, and we may choose $c_0 = 1$. However, it follows from Lemma 3 that $\operatorname{Card}\{\,\mathfrak{P} : \mathfrak{P}$ prime component of $\mathfrak{B}_i\,\} \leq d_1 \ldots d_\mu$. Therefore, by our assumption on $\operatorname{Card} k$, the rather regular sequence $h_1, \ldots, h_i$ in $\mathfrak{A}$ can be extended as long as some isolated prime component of $\mathfrak{B}_i = (h_1, \ldots, h_i)$ does not contain all of $\mathfrak{A}$. Let $h_1, \ldots, h_l$ be a maximal rather regular sequence constructed in this way.

Now for any two distinct indices, say $i > i^*$, the corresponding j are also distinct. Otherwise, since $h_{i^*+1} \in \mathfrak{B}_i$, we see that $\mathfrak{B}_{i+1} = (\mathfrak{B}_i, h_{i+1}) = (\mathfrak{B}_i, h)$ with $h = h_{i+1} - h_{i^*+1}$, which does not involve f_j at all. Thus not all of $f_{j+1}, \ldots, f_m$ lie in any isolated prime component of $\mathfrak{B}_i$ not containing $\mathfrak{A}$, contrary to the maximality of our choice of j for i. This ensures that the degrees of the h_i correspond to the degrees of f_j with distinct indices j. In particular, $j > 1$ when $i > 1$.

Now $\mathfrak{B}_l \subset \mathfrak{A}$, so any isolated prime component of $\mathfrak{A}$ contains one of $\mathfrak{B}_l$'s isolated prime components. On the other hand, if l is maximal then every isolated prime component of $\mathfrak{B}_l$ contains an arbitrary linear combination of the form required for h_{l+1}, with $j \geq 2$. Since it also contains f_1, it also contains all of $\mathfrak{A}$ and one of $\mathfrak{A}$'s isolated prime components.

When $i < \rho$, no isolated prime components of $\mathfrak{B}_{i-1}$ contain $\mathfrak{A}$ and so none contain h_i. Therefore $h_1, \ldots, h_\rho$ form a regular sequence.

If we denote by ρ' the dimension of the k-vector space generated by those f_i that are linear and if $\rho' \geq 1$, then we may begin our rather regular sequence with $h_1, \ldots, h_{\rho'}$ equal to linear f_i that are k-linearly independent and therefore generate a regular sequence. Consequently, $\rho \geq \rho'$. $\square$

Thus, if $\operatorname{Card} k \geq d_1 \ldots d_\mu$, we can replace $f_2, \ldots, f_m$ by R-linear homogeneous combinations $h_2, \ldots, h_l$ of them to obtain a rather regular sequence in $\mathfrak{A}$ generating a subideal $\mathfrak{B} = \mathfrak{B}_l$ of $\mathfrak{A}$ with the same isolated prime components $\mathfrak{P}_j$. We have the same bounds on the smallest degree and on the largest $l - 1$ degrees of the generators as before. If one obtains a power product Π lying in $\mathfrak{B}$, then Π lies in $\mathfrak{A}$ as well.

IV. Products of Ideals: Annihilators of Cohomology and Inclusion

We need some notation to state the results of this section. Let $\mathfrak{I}_0 = (0)$ and, for $i = 1, \ldots, l$, define $\mathfrak{I}_i, \mathfrak{K}_i, \mathfrak{E}_i$ inductively by grouping the components of a primary decomposition of $(\mathfrak{I}_{i-1}, h_i)$ to obtain

$$(\mathfrak{I}_{i-1}, h_i) = \mathfrak{I}_i \cap \mathfrak{K}_i \cap \mathfrak{E}_i,$$

where

(i) $\mathfrak{E}_i$ is the (non-unique choice of) intersection of the embedded components of the left-hand side in some primary decomposition,

(ii) $\mathfrak{K}_i$ is the intersection of the isolated primary components of the left-hand side whose corresponding prime ideals contain h_{i+1} (i.e. all of $\mathfrak{A}$), and

(iii) $\mathfrak{I}_i$ is the intersection of the remaining primary components of the left-hand side (whose corresponding prime ideals do not contain h_{i+1}, i.e. $\mathfrak{A}$).

We consider $\mathfrak{E}_i$ or $\mathfrak{K}_i$ to be equal to R in the absence of the components described. Note inductively that, by the principal ideal theorem [No, p. 217], the prime components of $\mathfrak{I}_i \cap \mathfrak{K}_i$ have height exactly i, since, by assumption, h_i lies in no prime component of $\mathfrak{I}_{i-1}$. Since we are in the case $\rho > 1$, we know that $\mathfrak{I}_1 = (h_1)$ and $\mathfrak{K}_1 = \mathfrak{E}_1 = R$; to avoid the trivial case, we assume that $h_1 \notin k$.

PROPOSITION 1. *For $i = 1, \ldots, \lambda$ with $\lambda = \min\{l, n\}$,*

$$(\mathfrak{K}_1^{3^{i-2}} \mathfrak{K}_2^{3^{i-3}} \ldots \mathfrak{K}_{i-1}^{3^0})(\mathfrak{I}_i \cap \mathfrak{K}_i) \subset (\mathfrak{I}_{i-1}, h_i).$$

In order to establish this basic result, we adapt Kollár's use of cohomology. But we inductively construct ideals annihilating certain cohomology groups, rather than concentrating on obtaining the minimal power of the radical with this property. The heart of the proof is the following variant of Lemma 3.4 of [Ko]. If $\mathfrak{I}, \mathfrak{U}$ are unmixed ideals, then we set $\operatorname{codim}_{\mathfrak{I}} \mathfrak{U} = \operatorname{height} \mathfrak{U} - \operatorname{height} \mathfrak{I}$ when $\mathfrak{I} \subset \mathfrak{U}$ and $\operatorname{codim}_{\mathfrak{I}} \mathfrak{U} = 0$ otherwise. The basic properties of local cohomology are given in [Gt] and [Ha].

LEMMA 1. *Let a homogeneous polynomial $f \in R$ lie outside all prime components of an unmixed homogeneous ideal $\mathfrak{I}$ of R. Let $\mathfrak{I} + (f) = \mathfrak{G} \cap \mathfrak{E}$, where $\mathfrak{G}$ is the intersection of the isolated primary ideals and $\mathfrak{E}$ is an intersection of embedded primary ideals. Assume that, for ideals $\mathfrak{N}$ and $\mathfrak{N}^*$ and a radical ideal $\mathfrak{U} \supset \mathfrak{G}$,*

$$\mathfrak{N} \cdot H^i_{\mathfrak{U}}(R/\mathfrak{I}) = 0$$

for all $i <$ $\operatorname{codim}_{\mathfrak{I}} \mathfrak{U}$, and, for $\mathfrak{E}^* = \operatorname{Rad} \mathfrak{E}$,

$$\mathfrak{N}^* \cdot H^1_{\mathfrak{E}^*}(R/\mathfrak{I}) = 0.$$

Then, for all $i <$ $\operatorname{codim}_{\mathfrak{G}} \mathfrak{U}$,

$$\mathfrak{N}^2 \mathfrak{N}^* \cdot H^i_{\mathfrak{U}}(R/\mathfrak{G}) = 0.$$

Proof. The exact sequences (the first arising from multiplication by f)

$$0 \longrightarrow R/\mathfrak{I} \longrightarrow R/\mathfrak{I} \longrightarrow R/(\mathfrak{I}, f) \longrightarrow 0$$

and

$$0 \longrightarrow \mathfrak{G}/(\mathfrak{I}, f) \longrightarrow R/(\mathfrak{I}, f) \longrightarrow R/\mathfrak{G} \longrightarrow 0$$

give rise to the exact sequences on local cohomology [Gt]

$$\cdots \longrightarrow H^i_{\mathfrak{U}}(R/\mathfrak{J}) \longrightarrow H^i_{\mathfrak{U}}(R/(\mathfrak{J}, f)) \longrightarrow H^{i+1}_{\mathfrak{U}}(R/\mathfrak{J}) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow H^i_{\mathfrak{U}}(R/(\mathfrak{J}, f)) \longrightarrow H^i_{\mathfrak{U}}(R/\mathfrak{G}) \longrightarrow H^{i+1}_{\mathfrak{U}}(\mathfrak{G}/(\mathfrak{J}, f)) \longrightarrow \cdots .$$

Since by hypothesis $H^i_{\mathfrak{U}}(R/\mathfrak{J})$ ($i < \operatorname{codim}_{\mathfrak{J}} \mathfrak{U}$) is annihilated by $\mathfrak{N}$, we see from the first sequence that

$$\mathfrak{N}^2 \cdot H^i_{\mathfrak{U}}(R/(\mathfrak{J}, f)) = 0$$

for all $i+1 < \operatorname{codim}_{\mathfrak{J}} \mathfrak{U} = \operatorname{height} \mathfrak{U} - \operatorname{height} \mathfrak{J} = \operatorname{codim}_{\mathfrak{G}} \mathfrak{U} + 1$, by the principal ideal theorem.

Now if $\mathfrak{E} \neq R$ (i.e., if $(\mathfrak{J}, f)$ actually has embedded components), then

$$H^0_{\mathfrak{E}^*}(R/\mathfrak{J}) = \{\, g \in R : \mathfrak{E}^{*j} g = 0 \text{ in } R/\mathfrak{J}, \text{ some } j \,\}/\mathfrak{J}.$$

Since $\mathfrak{E}^* \supset \mathfrak{G}$ and $\mathfrak{J}$, $\mathfrak{G}$ are unmixed with height $\mathfrak{G} = 1 + \operatorname{height} \mathfrak{J}$, one can choose $u \in \mathfrak{E}^*$ but outside all the prime components of $\mathfrak{J}$. Then for g as in the preceding displayed line, $u^j g \in \mathfrak{J}$, and one definition of primary ideal guarantees that, since u lies in none of the prime components of $\mathfrak{J}$, g lies in each of the primary components of $\mathfrak{J}$—that is, $g \in \mathfrak{J}$. Thus $H^0_{\mathfrak{E}^*}(R/\mathfrak{J}) = 0$. Therefore, from parts of both cohomology sequences, we obtain the fundamental injection

$$\mathfrak{G}/(\mathfrak{J}, f) = H^0_{\mathfrak{E}^*}(\mathfrak{G}/(\mathfrak{J}, f)) \subset H^0_{\mathfrak{E}^*}(R/(\mathfrak{J}, f)) \hookrightarrow H^1_{\mathfrak{E}^*}(R/\mathfrak{J}). \tag{3}$$

Thus $\mathfrak{N}^*$ annihilates $\mathfrak{G}/(\mathfrak{J}, f)$ and all its cohomology. The lemma follows on applying the results of these two paragraphs in the second long exact sequence. □

LEMMA 2. *For $i \le \mu$, the ideal $\mathfrak{N}_i := \mathfrak{K}_1^{3^{i-1}} \mathfrak{K}_2^{3^{i-2}} \ldots \mathfrak{K}_i$ annihilates $H^j_{\mathfrak{U}}(R/\mathfrak{J}_i)$ for all radical $\mathfrak{U} \supset \mathfrak{J}_i$ and all $j < \operatorname{codim}_{\mathfrak{U}} \mathfrak{J}_i$.*

Proof. The proof proceeds by induction on i.

When $i = 1$ we have $\mathfrak{J}_1 = (f_1)$, and we can make f_1 the first term of a regular sequence in $\mathfrak{U}$ of length equal to $1 + \operatorname{depth}_{\mathfrak{U}} R/(f_1) = \operatorname{height} \mathfrak{U} = 1 + \operatorname{codim}_{\mathfrak{U}}(f_1)$. Now the basic Theorem 3.8 of [Gt] (cf. [Ha, Exer. 3.4, p. 217]) states that $H^j_{\mathfrak{U}}(R/\mathfrak{J}_1) = 0$ for all $j < \operatorname{codim}_{\mathfrak{U}} \mathfrak{J}_1$. Thus any ideal, in particular $\mathfrak{N}_1 = \mathfrak{K}_1$, annihilates the required cohomology modules.

To obtain the general case, we use Lemma 1 with $\mathfrak{J} = \mathfrak{J}_{i-1}$, $f = h_i$, and $\mathfrak{G} = \mathfrak{J}_i \cap \mathfrak{K}_i$. Consider the following short exact sequence:

$$0 \longrightarrow \mathfrak{J}_i/(\mathfrak{J}_i \cap \mathfrak{K}_i) \longrightarrow R/(\mathfrak{J}_i \cap \mathfrak{K}_i) \longrightarrow R/(\mathfrak{J}_i) \longrightarrow 0.$$

We know that multiplication by $\mathfrak{K}_i$ annihilates $\mathfrak{J}_i/(\mathfrak{J}_i \cap \mathfrak{K}_i)$ and consequently all its cohomology groups. Therefore, in any corresponding long exact cohomology sequence, multiplication by $\mathfrak{K}_i$ carries $H^j(R/(\mathfrak{J}_i))$ into the image of $H^j(R/(\mathfrak{J}_i \cap \mathfrak{K}_i))$.

According to the induction hypothesis,

$$\mathfrak{N}_{i-1} \cdot H^j_{\mathfrak{U}}(R/\mathfrak{J}_{i-1}) = 0 \quad \text{for } j < \operatorname{codim}_{\mathfrak{U}} \mathfrak{J}_{i-1}.$$

Therefore, by the preceding paragraph and Lemma 1, $\mathfrak{N}_{i-1}^3\mathfrak{K}_i \cdot H^j_{\mathfrak{U}}(R/\mathfrak{I}_i) = 0$ when $j < \operatorname{codim}_{\mathfrak{U}} \mathfrak{I}_i$, as claimed. □

Proof of Proposition 1. We establish the result by induction on i. With the definition that $\mathfrak{I}_0 = (0)$, the claim is true for $i = 1$. By (3), we see that, for $i > 1$, if there are embedded components $\mathfrak{E}_i$ then

$$(\mathfrak{K}_i \cap \mathfrak{I}_i)/(\mathfrak{I}_{i-1}, h_i) \subset H^1_{\mathfrak{E}_i^*}(R/\mathfrak{I}_{i-1});$$

by Lemma 2, $\mathfrak{N}_{i-1} \cdot H^1_{\mathfrak{E}_i^*}(R/\mathfrak{I}_{i-1}) = 0$. Thus $\mathfrak{N}_{i-1} \cdot (\mathfrak{K}_i \cap \mathfrak{I}_i)/(\mathfrak{I}_{i-1}, h_i) = 0$, or

$$(\mathfrak{K}_1^{3^{i-2}}\mathfrak{K}_2^{3^{i-3}} \dots \mathfrak{K}_{i-1}^{3^0})(\mathfrak{I}_i \cap \mathfrak{K}_i) \subset (\mathfrak{I}_{i-1}, h_i),$$

as claimed. If $(\mathfrak{I}_{i-1}, h_i)$ is unmixed then $\mathfrak{K}_i \cap \mathfrak{I}_i = (\mathfrak{I}_{i-1}, h_i)$, and the claim still remains true. □

As a corollary of Proposition 1, we can construct a product that lies inside the original ideal.

PROPOSITION 2. *For* $\lambda = \min\{l, n\}$,

$$(\mathfrak{K}_1^{(3^{\lambda-1}+1)/2}\mathfrak{K}_2^{(3^{\lambda-2}+1)/2} \dots \mathfrak{K}_\lambda) \cdot \mathfrak{I}_\lambda \subset (h_1, \dots, h_\lambda).$$

Proof. We show that, for decreasing $i = \lambda, \dots, 2$,

$$\mathfrak{N}_{i-1}\mathfrak{K}_i \cdots \mathfrak{N}_{\lambda-2}\mathfrak{K}_{\lambda-1} \cdot \mathfrak{N}_{\lambda-1}\mathfrak{K}_\lambda \cdot \mathfrak{I}_\lambda \subset (\mathfrak{I}_{i-1}, h_i, h_{i+1}, \dots, h_\lambda). \tag{4}$$

By Proposition 1, the claim is true for $i = \lambda$. In fact, for $i > 2$, Proposition 1 also implies the second inclusion in

$$\mathfrak{N}_{i-2}\mathfrak{K}_{i-1} \cdot \mathfrak{I}_{i-1} \subset \mathfrak{N}_{i-2}(\mathfrak{K}_{i-1} \cap \mathfrak{I}_{i-1}) \subset (\mathfrak{I}_{i-2}, h_{i-1}). \tag{5}$$

Thus, if we have the inclusion (4) for $i > 2$, then multiplying by $\mathfrak{N}_{i-2}\mathfrak{K}_{i-1}$ shows the claim (4) to be true for $i - 1$ as well. Rewriting claim (4) for $i = 2$ solely in terms of the ideals $\mathfrak{K}_j$ gives the result. □

It is this product and its twin in Proposition 3 that appear in Theorem 2. The rest of the paper is devoted to the bookkeeping that relates the degrees and exponents occurring here to $d_1 \dots d_\mu$. The main tool is Bezout's theorem, but when $l = n + 1$, a surprising amount of care is required to suppress extraneous terms.

In the proof of the theorem, we will use the following consequence (via a straightforward induction on i) of Bezout's theorem (see e.g. [Gb, Sec. 143.7]), where as usual we consider $\deg R = 0$ and the empty product $D_{i+1} \dots D_i = 1$.

LEMMA 3. *For* $i = 1, \dots, \lambda$, *if* $D_i = \deg h_i$ *then*

$$\deg \mathfrak{I}_i + \sum_{j=0}^{i} (\deg \mathfrak{K}_j) D_{j+1} \dots D_i = D_1 \dots D_i.$$

When $m = n+1$, we will need the following lemma, which is Lemma 2.6 of [Ko]. For ease of access to the reader, we include a complete proof, including the sharp part (2) from [P2]. Recall that $\mathfrak{M} = (x_0, \ldots, x_n)$.

LEMMA 4 (Kollár–Macaulay). *Let $D_{n+1} = \deg h_{n+1}$. If h_{n+1} vanishes on none of the (projective) zeros of $\mathfrak{I}_n$, then:*

(1) *for $d \geq \deg \mathfrak{I}_n - 1$, $\mathfrak{M}^{D_{n+1}+d} \subset (\mathfrak{I}_n, h_{n+1})$;*
(2) *if $h_1, \ldots, h_n$ is a regular sequence, then for $d \geq \sum_{i=1}^{n}(D_i - 1)$, we have*

$$\mathfrak{M}^{D_{n+1}+d} \cap (\mathfrak{K}_n, h_{n+1}) = \mathfrak{M}^{D_{n+1}+d} \cap (\mathfrak{I}_n \cap \mathfrak{K}_n, h_{n+1}).$$

Proof. First we determine in each case a degree t_0 such that, for $t \geq t_0$,

$$\dim[R/\mathfrak{I}_n]_t = \deg \mathfrak{I}_n,$$

where the subscript t indicates that we are considering residue classes of homogeneous polynomials of degree t.

Case (1). Since $\mathfrak{I} = \mathfrak{I}_n$ is unmixed of dimension 0, there is a linear form L not in any of the associated prime ideals of $\mathfrak{I}$. For every $t \geq 1$, multiplication by L gives an exact sequence

$$0 \longrightarrow [R/\mathfrak{I}]_{t-1} \longrightarrow [R/\mathfrak{I}]_t \longrightarrow [R/(\mathfrak{I}, L)]_t \longrightarrow 0,$$

which shows that $\dim[R/\mathfrak{I}]_t > \dim[R/\mathfrak{I}]_{t-1}$ if and only if $[R/(\mathfrak{I}, L)]_t \neq 0$. Once $[R/(\mathfrak{I}, L)]_t = 0$ for a certain $t = t_0$, the same is true for all larger t. In this way, the sequence of values $\dim[R/\mathfrak{I}]_t$ increases strictly with t until it stabilizes at $\deg \mathfrak{I}$. Since $\dim[R/\mathfrak{I}]_0 = 1$, we see that $\dim[R/\mathfrak{I}]_t = \deg \mathfrak{I}$ for all $t \geq t_0 = \deg \mathfrak{I} - 1$.

Case (2). The Hilbert functions of ideals generated by regular sequences are well understood. (see e.g. [Gb, p. 164, eq. (4a)]). In our case, $H(t; \mathfrak{I}_n \cap \mathfrak{K}_n) = D_1 \ldots D_n$, for all $t \geq \sum(D_i - 1)$. As we saw in the preceding paragraph, the values of the Hilbert functions of $\mathfrak{I}_n$ and $\mathfrak{K}_n$ grow with t until they stabilize at $\deg \mathfrak{I}_n$ and $\deg \mathfrak{K}_n$, respectively. However, for every t we have the canonical injection

$$0 \longrightarrow [R/(\mathfrak{I}_n \cap \mathfrak{K}_n)]_t \longrightarrow [R/\mathfrak{I}_n]_t \oplus [R/\mathfrak{K}_n]_t. \tag{6}$$

This injection becomes an isomorphism for all $t \geq \sum(D_i - 1)$, since then the image has maximal possible dimension $\deg \mathfrak{I}_n \cap \mathfrak{K}_n = \deg \mathfrak{I}_n + \deg \mathfrak{K}_n$. Thus, in case (2), $\dim[R/\mathfrak{I}_n]_t = \deg \mathfrak{I}_n$ for all $t \geq t_0 = \sum(D_i - 1)$.

Since h_{n+1} is not a zero divisor in $R/\mathfrak{I}_n$, multiplication by h_{n+1} gives an injection of $[R/\mathfrak{I}_n]_t$ into $[R/\mathfrak{I}_n]_{t+D_{n+1}}$, both being vector spaces of the same dimension for $t \geq t_0$. Thus the map is surjective, which shows that (mod $\mathfrak{I}_n$) all polynomials of degree $t + D_{n+1}$ are multiples of h_{n+1}; that is, $\mathfrak{M}^{t+D_{n+1}} \subset (\mathfrak{I}_n, h_{n+1})$. This completes the proof of part (1), and also furnishes useful information for part (2).

For Lemma 4(2), consider the following commutative diagram of horizontal short exact sequences for $t \geq D_{n+1} + \sum_{i \leq n}(D_i - 1) = D_{n+1} + t_0$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \left[\frac{(\mathfrak{I}_n\cap\mathfrak{K}_n,h_{n+1})}{(\mathfrak{I}_n\cap\mathfrak{K}_n)}\right]_t & \longrightarrow & \left[\frac{R}{(\mathfrak{I}_n\cap\mathfrak{K}_n)}\right]_t & \longrightarrow & \left[\frac{R}{(\mathfrak{I}_n\cap\mathfrak{K}_n,h_{n+1})}\right]_t & \longrightarrow & 0 \\
 & & \Big\| & & \Big\| & & & & \\
0 & \longrightarrow & \left[\frac{(\mathfrak{I}_n\cap\mathfrak{K}_n,\mathfrak{K}_n,h_{n+1})}{\mathfrak{K}_n}\right]_t & \longrightarrow & \left[\frac{R}{\mathfrak{K}_n}\right]_t & \longrightarrow & \left[\frac{R}{(\mathfrak{K}_n,h_{n+1})}\right]_t & \longrightarrow & 0 \\
 & & \oplus & & \oplus & & & & \\
0 & \longrightarrow & \left[\frac{(\mathfrak{I}_n\cap\mathfrak{K}_n,\mathfrak{I}_n,h_{n+1})}{\mathfrak{I}_n}\right]_t & \longrightarrow & \left[\frac{R}{\mathfrak{I}_n}\right]_t & \longrightarrow & \left[\frac{R}{(\mathfrak{I}_n,h_{n+1})}\right]_t & \longrightarrow & 0.
\end{array}$$

We have already established the second vertical equality for all $t \geq t_0$. From that equality, the first vertical equality can be seen for $t \geq D_{n+1} + t_0$ as follows. Given $a_1 \bmod \mathfrak{K}_n$ and $a_2 \bmod \mathfrak{I}_n$, both homogeneous of degree $t_1 = t - D_{n+1} \geq t_0$, by the inclusion of (6) there is a (unique) $\alpha \bmod \mathfrak{I}_n \cap \mathfrak{K}_n$ of degree t_1 such that $\alpha \equiv a_1 \bmod \mathfrak{K}_n$ and $\alpha \equiv a_2 \bmod \mathfrak{I}_n$. Thus $\alpha h_{n+1} \equiv a_1 h_{n+1} \bmod \mathfrak{K}_n$ and $\alpha h_{n+1} \equiv a_2 h_{n+1} \bmod \mathfrak{I}_n$, which is enough to establish the desired equality.

From the exact sequence

$$0 \longrightarrow [R/\mathfrak{I}_n]_t \longrightarrow [R/\mathfrak{I}_n]_t \longrightarrow [R/(\mathfrak{I}_n, h_{n+1})]_t \longrightarrow 0$$

(arising from multiplication by h_{n+1}), we see that $[R/(\mathfrak{I}_n, h_{n+1})]_t = 0$ since $t > t_0$. Comparing dimensions in the diagram shows that $\dim[R/(\mathfrak{I}_n \cap \mathfrak{K}_n, h_{n+1})]_t = \dim[R/(\mathfrak{K}_n, h_{n+1})]_t$, which implies that $\mathfrak{M}^t \cap (\mathfrak{I}_n \cap \mathfrak{K}_n, h_{n+1}) = \mathfrak{M}^t \cap (\mathfrak{K}_n, h_{n+1})$, as desired. □

V. Proof of Theorem 2

For the sequence $h_1, \ldots, h_l$ constructed in Lemma 0, we continue to use the notation $D_j := \deg h_j$ $(j = 1, \ldots, l)$.

A. CASE $l \leq n$. Here, by definition, $\mathfrak{I}_l = R$. According to Proposition 2,

$$\mathfrak{K}_1^{(3^{l-1}+1)/2} \ldots \mathfrak{K}_l \subset (h_1, \ldots, h_l),$$

as desired. Now recall that $h_1, \ldots, h_\rho$ form a regular sequence. Thus each $\mathfrak{B}_i = (h_1, \ldots, h_i)$ is unmixed for $i \leq \rho$, and in fact $\mathfrak{B}_i = \mathfrak{I}_i$ and $\mathfrak{K}_i = R$, so that $\deg \mathfrak{K}_1 = \cdots = \deg \mathfrak{K}_{\rho-1} = 0$. Notice now, according to our hypotheses, that $d_2 \geq \cdots \geq d_{l-\rho+1} \geq 3$ and, according to Lemma 0, that the D_i correspond to the degrees of f_j with distinct indices j. By Lemma 3, $\sum (\deg \mathfrak{K}_i) D_{i+1} \ldots D_l \leq D_1 \ldots D_l$, and thus

$$\begin{aligned}\sum (\deg \mathfrak{K}_i) 3^{l-i} &\leq \sum (\deg \mathfrak{K}_i) d_2 d_3 \ldots d_{l-i+1} \\ &\leq \sum (\deg \mathfrak{K}_i)(D_{i+1} \ldots D_l) \frac{d_1 \ldots d_l}{D_1 \ldots D_l} \leq d_1 \ldots d_l.\end{aligned}$$

This completes the demonstration when $l \leq n$.

B. CASE $l = n + 1$. We use the following analog of Proposition 2.

PROPOSITION 3. *If $l = n+1$ then, for $d = \deg \mathfrak{I}_n - 1$,*

$$\mathfrak{K}_1^{(3^{n-1}+1)/2}\mathfrak{K}_2^{(3^{n-2}+1)/2} \dots \mathfrak{K}_n \mathfrak{M}^{D_{n+1}+d} \subset (h_1, \dots, h_{n+1}).$$

Proof. The proof is that of Proposition 2, except that the downward induction begins with the information provided by Lemma 4 that

$$\mathfrak{M}^{D_{n+1}+d} \subset (\mathfrak{I}_n, h_{n+1}).$$

Then we use Proposition 1 repeatedly to show that, for $i = n, \dots, 1$,

$$(\mathfrak{N}_{i-1}\mathfrak{K}_i) \dots (\mathfrak{N}_{n-1}\mathfrak{K}_n)\mathfrak{M}^{D_{n+1}+d} \subset (\mathfrak{I}_{i-1}, h_i, \dots, h_{n+1}),$$

with $\mathfrak{I}_0 = (0)$, and rewrite everything for $i = 1$ in terms of the ideals $\mathfrak{K}_j$. □

In order to deduce Theorem 2 from Proposition 3, we must bound the quantity

$$C_n := D_{n+1} + \deg \mathfrak{I}_n - 1 + \sum (\deg \mathfrak{K}_i)(3^{n-i}+1)/2.$$

By Lemma 3,

$$\deg \mathfrak{I}_n + \sum (\deg \mathfrak{K}_i) D_{i+1} \dots D_n \le D_1 D_2 \dots D_n.$$

Thus we will have established the desired inequality if we can show that

$$C_n \le \left(\frac{d_1 \dots d_n}{D_1 \dots D_n}\right)(\deg \mathfrak{I}_n + \sum (\deg \mathfrak{K}_i) D_{i+1} \dots D_n) + \delta$$

with $\delta = 0$. In other words, since $\mathfrak{K}_i = R$ for $i \le \rho$, it is enough to show that $D_{n+1} - 1 \le B_\delta$, where

$$B_\delta = \delta + \left(\frac{d_1 \dots d_n}{D_1 \dots D_i} - 1\right)(\deg \mathfrak{I}_n + \deg \mathfrak{K}_n) \\ + \sum_{\rho \le i < n} (\deg \mathfrak{K}_i)\left(\frac{d_1 \dots d_n}{D_1 \dots D_n} - \frac{3^{n-i}+1}{2}\right).$$

We consider three subcases.

(i) $\rho < n-1$. The following two inequalities hold because the indices j with $d_j = 1$ on the right side can be considered as giving rise to distinct i with $D_i = 1$ on the left side and, according to our technical hypothesis, $d_2, \dots, d_{n-\rho+1} \ge 3$:

$$2D_2 \dots D_\rho D_{n+1} \le d_2 \dots d_n,$$
$$D_2 \dots D_\rho 3^{n-\rho} \le d_2 \dots d_n.$$

Adding these two inequalities and dividing by $2D_2 \dots D_\rho$ shows that $D_{n+1} - 1$ is dominated by the coefficient of $\deg \mathfrak{K}_\rho$ in B_δ:

$$D_{n+1} - 1 \le \frac{d_1 \dots d_n}{D_1 \dots D_\rho} - \frac{3^{n-\rho}+1}{2} - \frac{1}{2},$$

as desired (even with $\delta = -1/2$).

(ii) $\rho = n-1$. Now, by our technical hypothesis, $d_2 \ge 3$. So evidently, unless $\deg \mathfrak{K}_{n-1} = 1$,

$$0 \leq (\deg \mathfrak{K}_{n-1} - 1)\{d_2 - 2\} - 1.$$

Now $l = n + 1$, so for $i = 2, \ldots, \rho$ we have $D_i \leq d_{n+3-i}$. Therefore,

$$D_{n+1} - 1 \leq d_2 - 1 \leq (\deg \mathfrak{K}_{n-1})\left(\frac{d_1 \ldots d_n}{D_1 \ldots D_\rho} - 2\right) + \delta \leq B_\delta$$

with $\delta = 0$, as desired, unless $\deg \mathfrak{K}_{n-1} = 1$. Even then, by Lemma 4(2), $\mathfrak{M}^{g_0} \subset (\mathfrak{I}_n, h_{n+1})$, where $g_0 = D_{n+1} + \deg \mathfrak{I}_n - 1$. Multiplying by $\mathfrak{K}_{n-1}\mathfrak{K}_n$ shows (via Proposition 1) that

$$\mathfrak{K}_{n-1}\mathfrak{K}_n\mathfrak{M}^{g_0} \subset (\mathfrak{K}_{n-1}\mathfrak{I}_n\mathfrak{K}_n, h_{n+1}) \subset (\mathfrak{I}_{n-1}, h_n, h_{n+1}). \tag{7}$$

However, if $\deg \mathfrak{K}_{n-1} = 1$ then $\mathfrak{K}_{n-1}$ is a *prime* minimal primary component of $\mathfrak{A}$ and thus $h_n, h_{n+1} \in \mathfrak{K}_{n-1}$. Since the product on the left-hand side of (7) is also contained in $\mathfrak{K}_{n-1}$, we find that

$$\mathfrak{K}_{n-1}\mathfrak{K}_n\mathfrak{M}^{g_0} \subset (\mathfrak{I}_{n-1} \cap \mathfrak{K}_{n-1}, h_n, h_{n+1}) = (h_1, \ldots, h_{n+1}). \tag{8}$$

We want to let this inclusion play the role here analogous to that of Proposition 4 in subcase (i). So now we need to consider the redefined quantity

$$\begin{aligned} C_n &= g_0 + \deg \mathfrak{K}_n + \deg \mathfrak{K}_{n-1} \\ &< D_{n+1} + \deg \mathfrak{I}_n + \deg \mathfrak{K}_n + \deg \mathfrak{K}_{n-1} \\ &= D_{n+1} + (D_1 \ldots D_{n-1} - 1)D_n \leq D_1 \ldots D_{n-1}D_{n+1}, \end{aligned}$$

which implies the desired inequality $C_n \leq d_1 \ldots d_n$.

(iii) $\rho = n$. According to Lemma 4(2), we need only be sure that $D_1 + \cdots + D_{n+1} - n \leq d_1 \ldots d_n$. This is easy enough to verify by starting with $D_1 + D_{n+1} - 1 \leq d_1 d_2$ and recursively applying the inequality $x + y \leq (x + 1)y$, which is valid for $x \geq 0$ and $y \geq 1$. This completes the proof of Theorem 2. □

VI. Final Remarks

The proof demonstrates that our results still hold if we use the second case of Lemma 1 to remove all restrictions on the d_i and instead change the bound on $\sum(\deg \mathfrak{K}_i)(3^{n-i} + 1)/2$ to $(3/2)^\nu d_1 \ldots d_\mu$, where ν is the number of indices i such that $\mathfrak{C}_i \neq R$ while $D_i = 2$.

It is Philippon's sharpening as Lemma 4(2) that allows us to avoid excess intersection theory without introducing an extraneous factor $\mathfrak{M}$ into the product of prime ideals of the theorem in the case where $l = n + 1$, $\rho = n - 1$, and $\mathfrak{K}_{n-1}$ is a primary of length 2. Our proof could also have been given as easily in terms of the homology of Koszul complexes (as is [P2]) rather than local cohomology.

The sharp Bezout form follows from setting $x_0 = 1$ in a sharp homogeneous (radical, prime power product, or pure power product) form to obtain, as in [Ko], that

$$\deg a_i + \deg g_i \leq d_1 \ldots d_n.$$

VII. Open Questions

It is quite remarkable that the exponent in Lemma 2 does not depend on the polynomial f. On the other hand, it would be very interesting to know whether Lemma 2 holds with exponent involving 2 rather than 3, at least when f is quadratic. For a positive response would remove all restrictions on degree in the results.

The restriction on the cardinality of k is annoying. It would be interesting to remove it without increasing the bound on the exponents.

VIII. Appendix

Proof of Remark 5. Let f be the greatest common divisor of the f_i, with $d = \deg f$. If $d_1 = d$ then f_1 divides all f_i. Thus $\mathfrak{A} = (f_i)$, and the claim is obvious.

If $d_1 > d+1$, then there are pure ideals $\mathfrak{K}'_i$ as in Theorem 2 with

$$\mathfrak{M}^{c_0}\mathfrak{K}_1'^{c1} \dots \mathfrak{K}_n'^{c_n} \subset (f_1/f, \dots, f_m/f)$$

and

$$c_0 + \sum c_i \deg K'_i \leq \prod \max\{3, d_i - d\}.$$

Thus

$$(f)\mathfrak{M}^{c_0}\mathfrak{K}_1'^{c1} \dots \mathfrak{K}_n'^{c_n} \subset (f)(f_1/f, \dots, f_m/f) \subset \mathfrak{A}$$

and, since each $d_i \geq d+2 \geq 3$,

$$\begin{aligned} d + c_0 + \sum c_i \deg K'_i &\leq d + \prod \max\{3, d_i - d\} \\ &\leq d_1 \prod_{i>1} \max\{3, d_i - d\} \leq d_1 \dots d_\mu. \end{aligned}$$

When $d_1 = d+1$, a certain number of the f_i/f will be linear; set ρ' equal to the dimension of the k-vector space they generate. Then, after a linear change of variables, we may arrange these polynomials to be $x_{n-\rho'+1}, \dots, x_n$. Reducing modulo these variables, we consider the ideal $\mathfrak{A}'$ generated by the remaining (nonlinear) ratios

$$f_2/f = F_1, \dots, f_j/f = F_{j-1} \in R' = k[x_0, \dots, x_{n-\rho'}],$$

with $\mu' = \min\{n - \rho', j - 1\} < \mu$. In R' we obtain a product

$$\mathfrak{M}'^{c_0} \prod \mathfrak{K}_i'^{c_i} \subset \mathfrak{A}'$$

with

$$c_0 + \sum c_i \deg \mathfrak{K}'_i \leq \prod \max\{3, d'_i - d\}.$$

However, by our choice of F_i, each $d'_i - d \geq 2$. Hence $d'_i \geq 3$, and finally

$$(f)\mathfrak{M}'^{c_0} \prod \mathfrak{K}_i'^{c_i} \subset (f)\mathfrak{A}' \subset \mathfrak{A}$$

with

$$d + c_0 + \sum c_i \deg \mathfrak{K}_i' \le d + \prod_{i \le \mu'} \max\{3, d_i' - d\}$$
$$\le d_1 \cdot \prod_{i \le \mu'} \max\{3, d_i' - d\} \le d_1 \dots d_\mu.$$

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