A Canonical Differential Complex for Jacobi Manifolds

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1. Introduction

Jacobi structures were independently introduced by Lichnerowicz [27; 28] and Kirillov [21], and they are a combined generalization of symplectic or Poisson structures and of contact structures.

A Jacobi structure on an n-dimensional manifold M is a pair (Λ, E), where Λ is a skew-symmetric tensor field of type $(2,0)$ and E a vector field on M verifying $[\Lambda,\Lambda] = 2E \wedge \Lambda$ and $[E,\Lambda] = 0$. The manifold M endowed with a Jacobi structure is called a Jacobi manifold. A bracket of functions (called Jacobi bracket) is then defined by $\{f, g\} = \Lambda(df, dg) + fE(g) - gE(f)$. Thus, the algebra $C^\infty(M,\mathbb{R})$ of C^∞ functions on M, endowed with the Jacobi bracket, is a local Lie algebra in the sense of Kirillov (see [21]). Conversely, a structure of local Lie algebra on $C^\infty(M,\mathbb{R})$ defines a Jacobi structure on M (see [16; 21]). When E identically vanishes, we recover the notion of Poisson manifold. Another link between Jacobi and Poisson manifolds is the following. Take a regular Jacobi manifold, that is, the vector field E defines a regular foliation; thus, the quotient manifold inherits a Poisson structure.

The purpose of this paper is to extend to Jacobi manifolds the construction of the canonical double complex for Poisson manifolds due to Koszul [23] and Brylinski [6]. The first step is to define an appropiate differential operator $\delta = [i(\Lambda), d]$ that extends the one introduced by Koszul [23] and Brylinski [6]. The restriction of δ to the complex of basic differential forms $\Omega^*_B(M)$ is a homology operator, and the resultant homology groups will be called canonical. Motivated by Brylinski, we propose the following problem.

Problem A-J. Give conditions on a compact Jacobi manifold which ensure that any basic cohomology class in $H^*_B(M)$ has a harmonic representative α, that is, $d\alpha = 0$ and $\delta\alpha = 0$.

Moreover, the relation $\delta d + d\delta = 0$ allows us to introduce a double complex. Associated with it, there exist two spectral sequences. The second spectral sequence always degenerates at the first term; however, this is not true for the first one. Hence we propose the following problem.

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Problem B-J. Give conditions on a compact Jacobi manifold that ensure the degeneracy at the first term of the first spectral sequence.

Both problems were proposed by Brylinski in [6] and solved in [10; 11; 12; 13; 30; 42] in the context of Poisson manifolds. In this paper we study these problems for Jacobi manifolds.

The paper is structured as follows. In Section 2, we introduce the notation and preliminary results that are necessary for the rest of the paper. We present symplectic, almost cosymplectic, contact, and locally conformal symplectic manifolds as examples of Jacobi manifolds. In fact, a Jacobi manifold possesses a generalized foliation, with even-dimensional leaves being locally conformal symplectic manifolds and odd-dimensional leaves being contact manifolds (see [9; 28; 29]). We can say that symplectic manifolds are the bricks used to construct Poisson structures; however, Jacobi structures are more involved and so we need symplectic, locally conformal symplectic, and contact bricks.

The extension of the Koszul–Brylinski operator δ is given in Section 3. It is the commutator of the contraction by the 2-vector Λ and the exterior differential. Acting on basic forms, we have $\delta^2 = 0$, and thus δ defines a canonical homology. We also have $\delta d + d\delta = 0$; we can then define a canonical double complex $(\mathcal{E}^{\text{per}}_{p,q} = \Omega_B^{q-p}(M), d, \delta)$, where $\Omega_B^k(M)$ denotes the space of basic k-forms with respect to the vector field E. We prove that the second spectral sequence always degenerates at the first term by using a master formula that generalizes the one obtained in [13]. For the first spectral sequence, we prove that for contact manifolds it degenerates at the first term (Section 4). Notice that the double complex $(\mathcal{E}^{\text{per}}_{p,q}, d, \delta)$ coincides with that defined by Brylinski [6] for Poisson manifolds, and in this case the first spectral sequence for a symplectic manifold degenerates at the first term. Our result thus holds for both symplectic and contact manifolds. However, it is no longer true for arbitrary Jacobi manifolds. In [13] we have shown a counterexample in the context of Poisson manifolds. Here, we exhibit a non-Poisson Jacobi counterexample—more precisely, a locally conformal symplectic (l.c.s.) manifold that is obtained as a circle bundle over an almost cosymplectic manifold.

With respect to Problem A-J, we prove that any basic cohomology class on a compact contact manifold has a harmonic representative if and only if it satisfies a hard Lefschetz theorem. This result is the analog in odd dimension to Mathieu's result for symplectic manifolds [30]. Thus, there is a natural parallelism for the odd- and even-dimensional cases. We also exhibit a strict Jacobi counterexample, a circle bundle over the Kodaira–Thurston manifold. With respect to the finiteness of the canonical homology groups, we prove that, for a contact manifold, they are isomorphic to the basic de Rham cohomology groups. Therefore, they have finite dimension if, for instance, the manifold is K-contact or Sasakian. Of course, the finiteness of the canonical homology groups is guaranteed for compact symplectic manifolds [6].

Section 5 is devoted to the study of the canonical homology of a particular kind of Jacobi manifolds—namely, the locally conformal symplectic manifolds. A very interesting case are the so-called locally conformal symplectic manifolds of the first kind (see Vaisman [39]). In contrast with the symplectic case, we prove that, in general, the canonical homology groups of an arbitrary l.c.s. manifold are not finite-dimensional. Moreover, we exhibit a 6-dimensional compact l.c.s. nilmanifold for which the first spectral sequence does not degenerate at the first term.

All the manifolds considered throughout this paper are assumed to be connected.

2. Jacobi Manifolds

Let M be a C^∞ manifold. Denote by $\mathfrak{X}(M)$ the Lie algebra of the vector fields on M and by $C^\infty(M,\mathbb{R})$ the algebra of C^∞ real-valued functions on M. A *Jacobi structure* on M is a pair (Λ, E), where Λ is a skew-symmetric tensor field of type $(2,0)$ and E a vector field on M verifying

$$[\Lambda,\Lambda] = 2E\wedge\Lambda, \qquad \mathcal{L}_E\Lambda = [E,\Lambda] = 0. \tag{1}$$

Here [,] is the Schouten–Nijenhuis bracket and $\mathcal{L}$ is the Lie derivative. The manifold M endowed with a Jacobi structure is called a *Jacobi manifold*. If (M,Λ,E) is a Jacobi manifold, we can define a bracket of functions (called *Jacobi bracket*) as follows:

$$\{f,g\} = \Lambda(df,dg) + fE(g) - gE(f) \quad \text{for all } f,g\in C^\infty(M,\mathbb{R}). \tag{2}$$

The mapping $\{\ ,\ \}\colon C^\infty(M,\mathbb{R})\times C^\infty(M,\mathbb{R})\to C^\infty(M,\mathbb{R})$ is bilinear and verifies

(i) $\operatorname{support}\{f,g\}\subset \operatorname{support} f\cap \operatorname{support} g$,
(ii) $\{f,g\} = -\{g,f\}$, and
(iii) $\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\} = 0$ (Jacobi's identity)

for $f,g,h\in C^\infty(M,\mathbb{R})$.

Thus, the space $C^\infty(M,\mathbb{R})$ endowed with the Jacobi bracket is a *local Lie algebra* in the sense of Kirillov (see [21]). Conversely, a structure of local Lie algebra on the space $C^\infty(M,\mathbb{R})$ of real-valued functions on a manifold M determines a Jacobi structure on M (see [16; 21]).

If the vector field E vanishes, then { , } is a derivation in each argument, that is, { , } defines a *Poisson bracket* on M. In this case, (1) reduces to $[\Lambda,\Lambda]=0$, and (M,Λ) is a *Poisson manifold*. The Poisson and Jacobi manifolds were introduced by Lichnerowicz (see [26; 28]; see also [3; 17; 25; 40; 41]).

The main examples of Poisson manifolds are symplectic and almost cosymplectic manifolds. A *symplectic manifold* is a pair (M,Ω), where M is an even-dimensional manifold and Ω is a closed nondegenerate 2-form on M. We define a skew-symmetric tensor field Λ of type $(2,0)$ on M given by

$$\Lambda(\alpha,\beta) = \Omega(\flat^{-1}(\alpha),\flat^{-1}(\beta))$$

for all $\alpha,\beta\in\Omega^1(M)$, where $\Omega^1(M)$ is the space of all 1-forms on M and $\flat\colon \mathfrak{X}(M)\to\Omega^1(M)$ is the isomorphism of $C^\infty(M,\mathbb{R})$-modules defined by $\flat(X) = i_X\Omega$. If we choose canonical coordinates (q^i,p_i) on M, we have

$$\Omega = \sum_{i=1}^{m} dq^i \wedge dp_i, \qquad \Lambda = \sum_{i=1}^{m} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i},$$

with $\dim M = 2m$.

An *almost cosymplectic manifold* (see Blair [4]) is a triple (M, Φ, η), where M is an odd-dimensional manifold, Φ is a closed 2-form, and η is a closed 1-form on M such that $\eta \wedge \Phi^m$ is a volume form with $\dim M = 2m+1$. If $\flat\colon \mathfrak{X}(M) \to \Omega^1(M)$ is the isomorphism of $C^\infty(M, \mathbb{R})$-modules defined by $\flat(X) = i_X\Phi + (i_X\eta)\eta$, then the vector field $\xi = \flat^{-1}(\eta)$ is called the *Reeb vector field* of M. The vector field ξ is characterized by the relations $i_\xi\Phi = 0$ and $i_\xi\eta = 1$. It should be noticed that an almost cosymplectic manifold was called "cosymplectic" by Libermann [24]. A skew-symmetric tensor field Λ of type $(2, 0)$ on M is defined by

$$\Lambda(\alpha, \beta) = \Phi(\flat^{-1}(\alpha), \flat^{-1}(\beta)) = \Phi(\flat^{-1}(\alpha - \alpha(\xi)\eta), \flat^{-1}(\beta - \beta(\xi)\eta))$$

for all $\alpha, \beta \in \Omega^1(M)$. Thus, (M, Λ) becomes a Poisson manifold. In canonical coordinates $(q^1, \dots, q^m, p_1, \dots, p_m, z)$, $\dim M = 2m + 1$, we have

$$\Phi = \sum_{i=1}^{m} dq^i \wedge dp_i, \quad \eta = dz, \quad \xi = \frac{\partial}{\partial z}, \quad \Lambda = \sum_{i=1}^{m} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}$$

(see [12; 13]). Other very interesting examples of Jacobi manifolds that are *not* Poisson manifolds are the contact manifolds and the locally conformal symplectic manifolds that we will next describe.

Let M be a $(2m+1)$-dimensional manifold and η a 1-form on M. We say that η is a *contact* 1-form if $\eta \wedge (d\eta)^m \neq 0$ at every point. In such a case (M, η) is termed a *contact manifold* (see e.g. [4]). Using the classical theorem of Darboux, around every point of M there exist canonical coordinates $(t, q^1, \dots, q^m, p_1, \dots, p_m)$ such that

$$\eta = dt - \sum_i p_i dq^i.$$

A contact manifold (M, η) is a Jacobi manifold. In fact, we define the skew-symmetric tensor field Λ of type $(2, 0)$ on M given by

$$\Lambda(\alpha, \beta) = d\eta(\flat^{-1}(\alpha), \flat^{-1}(\beta))$$

for all $\alpha, \beta \in \Omega^1(M)$, where $\flat\colon \mathfrak{X}(M) \to \Omega^1(M)$ is the isomorphism of $C^\infty(M, \mathbb{R})$-modules defined by $\flat(X) = i_X d\eta + \eta(X)\eta$. The vector field E is just the Reeb vector field $\xi = \flat^{-1}(\eta)$ of (M, η). Using canonical coordinates, we have

$$\Lambda = \sum_i \left(\frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial t} \right) \wedge \frac{\partial}{\partial p_i}, \qquad E = \frac{\partial}{\partial t}.$$

We remark that $i_\xi\eta = 1$ and $i_\xi d\eta = 0$.

On the other hand, let us recall that an *almost symplectic manifold* is a pair (M, Ω), where M is an even-dimensional manifold and Ω is a nondegenerate 2-form on M. An almost symplectic manifold is said to be *locally conformal symplectic* (l.c.s.) if, for each point $x \in M$, there is an open neighborhood U such that $d(e^{-\sigma}\Omega) = 0$ for some function $\sigma : U \to \mathbb{R}$ (see e.g. [16; 39]). Equivalently, (M, Ω) is a l.c.s. manifold if there exists a closed 1-form ω such that

$$d\Omega = \omega \wedge \Omega. \tag{3}$$

The 1-form ω is called the *Lee 1-form* of M. It is obvious that the l.c.s. manifolds with Lee 1-form identically zero are just symplectic manifolds.

In a similar way as for contact manifolds, we define a skew-symmetric tensor field Λ of type $(2, 0)$ and a vector field E on M by

$$\Lambda(\alpha, \beta) = \Omega(\flat^{-1}(\alpha), \flat^{-1}(\beta)) \quad \text{and} \quad E = \flat^{-1}\omega \tag{4}$$

for all 1-forms α and β, where $\flat\colon \mathfrak{X}(M) \to \Omega^1(M)$ is the isomorphism of $C^\infty(M, \mathbb{R})$-modules defined by $\flat(X) = i_X\Omega$. Then (M, Λ, E) is a Jacobi manifold. Notice that

$$\omega(E) = 0, \quad \mathcal{L}_E\omega = 0, \quad \mathcal{L}_E\Omega = 0. \tag{5}$$

The contact manifolds and the locally conformal symplectic manifolds are a particular class of Jacobi manifolds known as transitive Jacobi manifolds.

Take a Jacobi manifold (M, Λ, E) and define a linear mapping

$$\Lambda_x^\#\colon T_x^*M \to T_xM$$

given by $\langle \Lambda_x^\#(\alpha), \beta\rangle = \Lambda_x(\alpha, \beta)$ for all $\alpha, \beta \in T_x^*M$ and $x \in M$. For a contact manifold (M, η) with Reeb vector field ξ, we have that $\Lambda_x^\#(\alpha) = -\flat_x^{-1}(\alpha) + \alpha(\xi_x)\xi_x$ for all $\alpha \in T_x^*M$. For a l.c.s. manifold (M, Ω), we obtain $\Lambda^\# = -\flat^{-1}$.

The Jacobi manifold (M, Λ, E) is said to be *transitive* if, for all $x \in M$, the tangent space T_xM is generated by $\Lambda_x^\#(T_x^*M)$ and E_x [9]. Let (M, Λ, E) be a transitive Jacobi manifold. Then we have the following statements (see [9] and the references therein).

(a) If $\dim M = 2m + 1$ then, for every $x \in M$, it follows that

$$T_xM = \Lambda_x^\#(T_x^*M) \oplus \langle E_x\rangle.$$

Therefore, the 1-form η defined by $\eta_x(u + \lambda E_x) = \lambda$ for $u \in \Lambda_x^\#(T_x^*M)$ and $\lambda \in \mathbb{R}$ is a contact 1-form.

(b) If $\dim M = 2m$ then we deduce that $\Lambda_x^\#\colon T_x^*M \to T_xM$ is an isomorphism. Thus, if we put

$$\Omega_x(X, Y) = \Lambda_x((\Lambda_x^\#)^{-1}X, (\Lambda_x^\#)^{-1}Y) \quad \text{for all } X, Y \in T_xM$$

and if $\omega_x = i_{E_x}\Omega_x$, we get that (M, Ω) is a l.c.s. manifold with Lee 1-form ω.

Therefore, a transitive Jacobi manifold (M, Λ, E) is a contact or a l.c.s. manifold.

Next, we will prove that an arbitrary Jacobi manifold is foliated by leaves that are contact or l.c.s. manifolds. Roughly speaking, a Jacobi manifold is made of contact or l.c.s. pieces.

Let (M, Λ, E) be a Jacobi manifold. If $f \in C^\infty(M, \mathbb{R})$, then the vector field X_f defined by $X_f = \Lambda^\#(df) + fE$ is called the *Hamiltonian vector field* associated with f. It should be noticed that the Hamiltonian vector field associated with the constant function 1 is just E. A direct computation shows that $[X_f, X_g] = X_{\{f,g\}}$ [28]. Denote by D_x the subspace of T_xM generated by all the Hamiltonian vector fields evaluated at the point $x \in M$. In other words, $D_x = \Lambda_x^\#(T_x^*M) + \langle E_x\rangle$. Since D is involutive, one easily deduces that D defines a generalized foliation in the sense of Sussmann [36]. This foliation is termed the *characteristic foliation* in [9]. Moreover, if L is a leaf of D then the Jacobi structure (Λ, E) on M induces a transitive Jacobi structure (Λ_L, E_L) on L. Thus, we deduce that the leaves of D are contact or l.c.s. manifolds (for a detailed study we refer to [9; 16]).

Next, we explain the local structure of Jacobi manifolds. Let (M, Λ, E) be a Jacobi manifold with Jacobi bracket $\{\ ,\ \}$. Given a nonzero function a on M, we construct a new Jacobi structure on M by putting

$$\Lambda_a = a\Lambda, \qquad E_a = aE + [\Lambda, a].$$

We say that (Λ, E) and (Λ_a, E_a) are *conformally equivalent*. The Jacobi bracket arising from (Λ_a, E_a) becomes

$$\{f, g\}_a = \frac{1}{a}\{af, ag\} \quad \text{for all } f, g \in C^\infty(M, \mathbb{R}).$$

The following result was proved in [9].

THEOREM 2.1. *Let (M, Λ, E) be an n-dimensional Jacobi manifold with Jacobi bracket $\{\ ,\ \}$, x_0 a point in M, and S the leaf passing through x_0.*

(i) *If S is a contact leaf with odd dimension $2m+1$, then there exist local coordinates $(t, q^1, \dots, q^m, p_1, \dots, p_m, z^1, \dots, z^{n-2m-1})$ centered at x_0 such that*

$$E = \frac{\partial}{\partial t},$$

$$\Lambda = \sum_{i=1}^{m}\left(\frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial t}\right) \wedge \frac{\partial}{\partial p_i} + \sum_{\alpha=1}^{n-2m-1} \Lambda^\alpha \frac{\partial}{\partial t} \wedge \frac{\partial}{\partial z^\alpha} + \sum_{1 \le \alpha < \beta \le n-2m-1} \Lambda^{\alpha\beta} \frac{\partial}{\partial z^\alpha} \wedge \frac{\partial}{\partial z^\beta},$$

and the functions Λ^α and $\Lambda^{\alpha\beta}$ do not depend on the coordinates t, q^i, p_i and vanish at x_0. Thus, the Jacobi bracket is given by

$$\{t, q^i\} = -q^i, \quad \{t, z^\alpha\} = \Lambda^\alpha - q^\alpha, \quad \{q^i, p_j\} = \delta^i_j, \quad \{z^\alpha, z^\beta\} = \Lambda^{\alpha\beta},$$

the other brackets of coordinate functions being zero.

(ii) *If S is a locally conformal symplectic leaf with even dimension $2m$, then there exists a nonzero function a defined on a neighborhood of x_0 as well as local coordinates $(q^1, \dots, q^m, p_1, \dots, p_m, z^1, \dots, z^{n-2m})$ centered at x_0 such that the Jacobi structure (Λ_a, E_a), locally conformal to (Λ, E), is given by*

$$E_a = \sum_{\alpha=1}^{n-2m} (E_a)^\alpha \frac{\partial}{\partial z^\alpha},$$

$$\Lambda_a = \sum_{i=1}^{m} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} + \sum_{i=1}^{m}\sum_{\alpha=1}^{n-m} p_i (E_a)^\alpha \frac{\partial}{\partial z^\alpha} \wedge \frac{\partial}{\partial p_i} + \sum_{1 \le \alpha < \beta \le n-2m} (\Lambda_a)^{\alpha\beta} \frac{\partial}{\partial z^\alpha} \wedge \frac{\partial}{\partial z^\beta},$$

and the functions $(E_a)^\alpha$ and $(\Lambda_a)^{\alpha\beta}$ do not depend on the coordinates q^i, p_i and vanish at x_0. Thus, the Jacobi bracket is given by

$$\{q^i, p_j\}_a = \delta^i_j, \qquad \{q^i, z^\alpha\}_a = q^i (E_a)^\alpha,$$
$$\{z^\alpha, z^\beta\}_a = (\Lambda_a)^{\alpha\beta} + z^\alpha (E_a)^\beta - z^\beta (E_a)^\alpha,$$

the other brackets of coordinate functions being zero.

REMARK 2.2. (i) A Jacobi manifold (M, Λ, E) is said to be *regular* if the vector field E is complete, $E \neq 0$ at every point, and the 1-dimensional foliation defined by E is regular in the sense of Palais [34]. In such a case, the space of leaves $\bar{M} = M/E$ has the structure of a differentiable manifold and the canonical projection $\pi : M \to \bar{M}$ is a fibration (surjective submersion). Moreover, we can define on $\bar{M}$ a 2-vector $\bar{\Lambda}$ as

$$\bar{\Lambda}(\bar{\alpha}, \bar{\beta}) \circ \pi = \Lambda(\pi^*\bar{\alpha}, \pi^*\bar{\beta})$$

for all $\bar{\alpha}, \bar{\beta} \in \Omega^1(\bar{M})$. Notice that, from (1), $\bar{\Lambda}$ is well-defined and $(\bar{M}, \bar{\Lambda})$ is a Poisson manifold (see [9]).

(ii) If (M, η) is a regular contact manifold with Reeb vector field ξ, then it is well known that the quotient Poisson manifold $\bar{M} = M/\xi$ is a symplectic manifold with symplectic form Ω such that $\pi^*\Omega = d\eta$ (see e.g. [4]). In fact, Ω is the dual 2-form of the bivector $\bar{\Lambda}$.

(iii) In Section 5, we will prove that if (M, Ω) is a regular l.c.s. manifold of the first kind then the quotient Poisson manifold is an almost cosymplectic manifold.

3. The Canonical Double Complex for Jacobi Manifolds

Let (M, Λ, E) be a Jacobi manifold, and denote by $\Omega^k(M)$ the space of differential k-forms on M. We introduce the differential operator $\delta : \Omega^k(M) \to \Omega^{k-1}(M)$ given by the commutator of $i(\Lambda)$ and the exterior differential d; that is,

$$\delta = [i(\Lambda), d] = i(\Lambda) \circ d - d \circ i(\Lambda) \tag{6}$$

(see [8]). We notice that if $E = 0$ (i.e., if (M, Λ) is a Poisson manifold) then δ is just the Koszul operator (see [6; 23]).

A direct computation gives the following explicit expression of δ.

PROPOSITION 3.1. *We have*

$$\begin{aligned}
&\delta(f_0\, df_1 \wedge \cdots \wedge df_k) \\
&\quad = \sum_{1 \le i \le k} (-1)^{i+1} (\{f_0, f_i\} - f_0 E(f_i) + f_i E(f_0)) df_1 \\
&\qquad \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge df_k \\
&\qquad + \sum_{1 \le i < j \le k} (-1)^{i+j} f_0\, d(\{f_i, f_j\} - f_i E(f_j) + f_j E(f_i)) \wedge df_1 \\
&\qquad \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge \widehat{df_j} \wedge \cdots \wedge df_k,
\end{aligned}$$

where the hat denotes missing arguments.

Proof. The result follows from a direct computation that takes (2) into account. □

PROPOSITION 3.2. *For a Jacobi manifold* (M, Λ, E) *we have*

$$\delta^2 = i(\Lambda)i_E d + di(\Lambda)i_E.$$

Proof. A straightforward computation, using (6), shows that

$$\delta^2 = i(\Lambda)di(\Lambda)d - di(\Lambda)i(\Lambda)d + di(\Lambda)di(\Lambda). \tag{7}$$

Using that $[[i(a), d], i(b)] = i([a, b])$ together with (1) and (6), it follows that

$$\delta i(\Lambda) - i(\Lambda)\delta = [\delta, i(\Lambda)] = [[i(\Lambda), d], i(\Lambda)] = i([\Lambda, \Lambda]) = 2i_E i(\Lambda). \tag{8}$$

Now, using again (6):

$$\delta i(\Lambda) - i(\Lambda)\delta = 2i(\Lambda)di(\Lambda) - di(\Lambda)i(\Lambda) - i(\Lambda)i(\Lambda)d. \tag{9}$$

Thus, from (8) and (9) we have

$$i(\Lambda)di(\Lambda) = i_E i(\Lambda) + \tfrac{1}{2}\{di(\Lambda)i(\Lambda) + i(\Lambda)i(\Lambda)d\}. \tag{10}$$

Finally, the proposition follows from (7) and (10). □

PROPOSITION 3.3. *For a Jacobi manifold* (M, Λ, E) *we have*:

(i) $i_E\delta = -\delta i_E$;
(ii) $\mathcal{L}_E\delta = \delta\mathcal{L}_E$.

Proof. (i) From (6) and the Cartan formula $\mathcal{L}_X = d \circ i_X + i_X \circ d$ it follows that

$$i_E \circ \delta = i_E(i(\Lambda)d - di(\Lambda)) = i(\Lambda)\mathcal{L}_E - i(\Lambda)di_E - \mathcal{L}_E i(\Lambda) + di_E i(\Lambda).$$

Now, using (1):

$$\mathcal{L}_E i(\Lambda) - i(\Lambda)\mathcal{L}_E = i(\mathcal{L}_E\Lambda) = i([E, \Lambda]) = 0. \tag{11}$$

Thus,

$$i_E \circ \delta = di(\Lambda)i_E - i(\Lambda)di_E = -\delta i_E.$$

(ii) Using again (11), and since $\mathcal{L}_E d = d\mathcal{L}_E$, it follows that

$$\begin{aligned}\mathcal{L}_E\delta &= \mathcal{L}_E(i(\Lambda)d - di(\Lambda)) = i(\Lambda)\mathcal{L}_E d - d\mathcal{L}_E i(\Lambda)\\ &= i(\Lambda)d\mathcal{L}_E - di(\Lambda)\mathcal{L}_E = \delta\mathcal{L}_E.\end{aligned}$$ □

For an integer k, we will denote by $\Omega^k_B(M)$ the subspace of *basic k-forms.* That is,

$$\begin{aligned}\Omega^k_B(M) &= \{\alpha \in \Omega^k(M) \mid i_E\alpha = 0,\ \mathcal{L}_E\alpha = 0\}\\ &= \{\alpha \in \Omega^k(M) \mid i_E\alpha = 0,\ i_E d\alpha = 0\}.\end{aligned}$$

If $C^\infty_B(M, \mathbb{R})$ is the space of *basic functions* on M, then $\Omega^k_B(M)$ is a $C^\infty_B(M, \mathbb{R})$-module.

The following corollary is a consequence of Propositions 3.2 and 3.3.

COROLLARY 3.4. *Let* $\alpha \in \Omega_B^k(M)$. *Then we have*:
(i) $\delta\alpha \in \Omega_B^{k-1}(M)$;
(ii) $\delta^2\alpha = 0$.

Using Proposition 3.1, we also deduce the following.

COROLLARY 3.5. *If* $f_0, f_1, \ldots, f_k$ *are basic functions, then the k-form* $f_0\, df_1 \wedge \cdots \wedge df_k$ *is basic and we have*

$$\begin{aligned}\delta(f_0\, df_1 \wedge \cdots \wedge df_k) = &\sum_{1\le i\le k} (-1)^{i+1}\{f_0, f_i\} df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge df_k \\ &+ \sum_{1\le i<j\le k} (-1)^{i+j} f_0\, d\{f_i, f_j\} \wedge df_1 \\ &\wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge \widehat{df_j} \wedge \cdots \wedge df_k.\end{aligned}$$

Let (M, Λ, E) be a Jacobi manifold. Corollary 3.4 allows us to introduce the differential complex

$$\cdots \longrightarrow \Omega_B^{k+1}(M) \xrightarrow{\delta_B} \Omega_B^k(M) \xrightarrow{\delta_B} \Omega_B^{k-1}(M) \longrightarrow \cdots,$$

where $\delta_B = \delta|_{\Omega_B^*(M)}$ and $\Omega_B^*(M) = \sum_k \Omega_B^k(M)$. This complex is called the *canonical complex* of M. If M is a Poisson manifold (i.e., if $E = 0$) then this complex is just the canonical complex introduced by Brylinski (see [6]). The homology of this complex is denoted by $H_*^{\mathrm{can}}(M)$ and is called the *canonical homology* of (M, Λ, E).

Also, we can consider the subcomplex of the de Rham complex given by the basic forms

$$\cdots \longrightarrow \Omega_B^{k-1}(M) \xrightarrow{d_B} \Omega_B^k(M) \xrightarrow{d_B} \Omega_B^{k+1}(M) \longrightarrow \cdots,$$

where $d_B = d|_{\Omega_B^*(M)}$. The cohomology of this complex is denoted by $H_B^*(M)$ and is called the *basic de Rham cohomology* of (M, Λ, E). A direct computation shows that $d\delta + \delta d = 0$. Thus we can introduce the *canonical double complex* $\mathcal{E}_{*,*}(M)$, defined by

$$\mathcal{E}_{p,q}(M) = \Omega_B^{q-p}(M) \quad \text{for } p, q \ge 0.$$

This double complex is concentrated on the first quadrant. Then we define the *periodic double complex* $\mathcal{E}_{*,*}^{\mathrm{per}}(M)$ by

$$\mathcal{E}_{p,q}^{\mathrm{per}}(M) = \Omega_B^{q-p}(M) \quad \text{for all } p, q \in \mathbb{Z}.$$

Thus (see e.g. [5]) there are two spectral sequences $\{E^r(M)\}$ and $\{'E^r(M)\}$ (of homological type) associated with the periodic double complex. Both of these spectral sequences converge to the total homology $H_*^D(M)$, that is, the homology of the total complex $(\mathcal{E}_k(M), D)$, where $\mathcal{E}_k(M) = \bigoplus_{p+q=k} \mathcal{E}_{p,q}(M)$ and $D = d + \delta$.

REMARK 3.6. (i) If M is a Poisson manifold (i.e., if $E = 0$) then the periodic double complex $\mathcal{E}_{*,*}^{\mathrm{per}}(M)$ of M coincides with the one previously defined by Brylinski [6].

(ii) Let (M, Λ, E) be a regular Jacobi manifold and $(\bar{M}, \bar{\Lambda})$ the corresponding quotient Poisson manifold (see Remark 2.2). We deduce that $\mathcal{E}^{\text{per}}_{*,*}(\bar{M})$ and $\mathcal{E}^{\text{per}}_{*,*}(M)$ are isomorphic in such a way that the behavior of the corresponding spectral sequences is just the same. In particular, the canonical homology group $H^{\text{can}}_p(M)$ is isomorphic to the canonical homology group $H^{\text{can}}_p(\bar{M})$, and the basic de Rham cohomology group $H^p_B(M)$ is isomorphic to the de Rham cohomology group $H^p(\bar{M})$.

Denote by δ_r the differential of bidegree $(-r, r-1)$, so that the groups $E^{r+1}_{p,q}(M)$ are isomorphic to the homology groups of the following sequence:

$$\cdots \longrightarrow E^r_{p+r,q-r+1}(M) \xrightarrow{\delta_r} E^r_{p,q}(M) \xrightarrow{\delta_r} E^r_{p-r,q+r-1}(M) \longrightarrow \cdots .$$

It should be noticed that a basic differential form $\alpha \in \mathcal{E}^{\text{per}}_{p,q}(M)$ lives to $E^r_{p,q}(M)$ if it satisfies

$$\delta\alpha = 0,\ d\alpha = \delta\alpha_1,\ d\alpha_1 = \delta\alpha_2, \dots, d\alpha_{r-3} = \delta\alpha_{r-2},\ d\alpha_{r-2} = \delta\alpha_{r-1} \qquad (12)$$

for some basic differential forms $\alpha_1, \dots, \alpha_{r-1}$. Denote by $[\alpha]_r$ the homology class defined by α in $E^r_{p,q}(M)$. The differential operator δ_r is then given by

$$\delta_r[\alpha]_r = [d\alpha_{r-1}]_r. \qquad (13)$$

In particular, for $r = 1$ the groups $E^1_{p,q}(M)$ of the first spectral sequence are isomorphic to the homology groups of the sequence

$$\cdots \longrightarrow \mathcal{E}^{\text{per}}_{p,q+1}(M) \xrightarrow{\delta_B} \mathcal{E}^{\text{per}}_{p,q}(M) \xrightarrow{\delta_B} \mathcal{E}^{\text{per}}_{p,q-1}(M) \longrightarrow \cdots .$$

Thus, we have

$$\begin{aligned} E^1_{p,q}(M) &\cong \frac{\{\alpha \in \mathcal{E}^{\text{per}}_{p,q}(M) \mid \delta_B\alpha = 0\}}{\delta_B(\mathcal{E}^{\text{per}}_{p,q+1}(M))} \\ &= \frac{\{\alpha \in \Omega^{q-p}_B(M) \mid \delta\alpha = 0\}}{\delta(\Omega^{q+1-p}_B(M))} \cong H^{\text{can}}_{q-p}(M). \end{aligned} \qquad (14)$$

For $r = 2$, the groups $E^2_{p,q}(M)$ are isomorphic to the homology groups of the sequence

$$\cdots \longrightarrow E^1_{p+1,q}(M) \xrightarrow{d} E^1_{p,q}(M) \xrightarrow{d} E^1_{p-1,q}(M) \longrightarrow \cdots .$$

From (14) we obtain

$$E^2_{p,q}(M) \cong \frac{\{\alpha \in \mathcal{E}^{\text{per}}_{p,q}(M) \mid \delta\alpha = 0 \text{ and } d\alpha = \delta\alpha_1 \text{ for some } \alpha_1 \in \mathcal{E}^{\text{per}}_{p-1,q+1}(M)\}}{d(H^{\text{can}}_{q-p-1}(M))}. \qquad (15)$$

Similar definitions can be given for the terms $E^r_{p,q}(M)$, $r \geq 3$.

Let $'\delta_r$ be the differential of bidegree $(r-1, -r)$, so that the groups $'E^{r+1}_{p,q}(M)$ are isomorphic to the homology groups of the sequence

$$\cdots \longrightarrow {}'E^r_{p-r+1,q+r}(M) \xrightarrow{'\delta_r} {}'E^r_{p,q}(M) \xrightarrow{'\delta_r} {}'E^r_{p+r-1,q-r}(M) \longrightarrow \cdots .$$

We notice that a basic differential form $\beta \in \mathcal{E}^{\text{per}}_{p,q}(M)$ lives to ${}'E^r_{p,q}(M)$ if it satisfies

$$d\beta = 0,\ \delta\beta = d\beta_1,\ \delta\beta_1 = d\beta_2, \ldots, \delta\beta_{r-3} = d\beta_{r-2},\ \delta\beta_{r-2} = d\beta_{r-1} \tag{16}$$

for some basic differential forms $\beta_1, \ldots, \beta_{r-1}$. Denote by ${}'[\beta]_r$ the homology class defined by β in ${}'E^r_{p,q}(M)$. The differential operator ${}'\delta_r$ is then defined by

$$'\delta'_r[\beta]_r = {}'[\delta\beta_{r-1}]_r. \tag{17}$$

For $r = 1$, the groups ${}'E^1_{p,q}(M)$ of the second spectral sequence are isomorphic to the homology groups of the sequence

$$\cdots \longrightarrow \mathcal{E}^{\text{per}}_{p+1,q}(M) \xrightarrow{d_B} \mathcal{E}^{\text{per}}_{p,q}(M) \xrightarrow{d_B} \mathcal{E}^{\text{per}}_{p-1,q}(M) \longrightarrow \cdots .$$

Thus, we have

$$\begin{aligned} {}'E^1_{p,q}(M) &\cong \frac{\{\alpha \in \mathcal{E}^{\text{per}}_{p,q}(M) \mid d_B\alpha = 0\}}{d_B(\mathcal{E}^{\text{per}}_{p+1,q}(M))} \\ &\cong \frac{\{\alpha \in \Omega^{q-p}_B(M) \mid d\alpha = 0\}}{d(\Omega^{q-p-1}_B(M))} \cong H^{q-p}_B(M). \end{aligned} \tag{18}$$

For $r = 2$, the groups ${}'E^2_{p,q}(M)$ are isomorphic to the homology groups of the sequence

$$\cdots \longrightarrow {}'E^1_{p,q+1}(M) \xrightarrow{\delta} {}'E^1_{p,q}(M) \xrightarrow{\delta} {}'E^1_{p,q-1}(M) \longrightarrow \cdots . \tag{19}$$

From (19), we obtain

$$\begin{aligned} &{}'E^2_{p,q}(M) \\ &\cong \frac{\{\alpha \in \mathcal{E}^{\text{per}}_{p,q}(M) \mid d\alpha = 0 \text{ and } \delta\alpha = d\alpha_1 \text{ for some } \alpha_1 \in \mathcal{E}^{\text{per}}_{p+1,q-1}(M)\}}{\delta(H^{q-p+1}_B(M))}. \end{aligned} \tag{20}$$

In order to study the second spectral sequence, we need the following master formula.

LEMMA 3.7. *For a Jacobi manifold* (M, Λ, E), *we have*

$$ki(\Lambda)di(\Lambda)^{k-1} = i(\Lambda)^k d + (k-1)di(\Lambda)^k + k(k-1)i(\Lambda)^{k-1}i_E \tag{21}$$

for all positive integers k.

Proof. We proceed by induction. For $k = 1$ the proof is trivial. From (8) we have

$$2i(\Lambda)di(\Lambda) = i(\Lambda)^2 d + di(\Lambda)^2 + 2i(\Lambda)i_E. \tag{22}$$

Thus, (21) holds for $k = 2$. Suppose that (21) is true for an arbitrary k:

$$ki(\Lambda)di(\Lambda)^{k-1} = i(\Lambda)^k d + (k-1)di(\Lambda)^k + k(k-1)i(\Lambda)^{k-1}i_E. \tag{23}$$

If we apply $i(\Lambda)^{k-1}$ (on the right) to both sides of (22), we obtain that

$$2i(\Lambda)di(\Lambda)^k = i(\Lambda)^2 di(\Lambda)^{k-1} + di(\Lambda)^{k+1} + 2i(\Lambda)^k i_E.$$

Thus,

$$2ki(\Lambda)di(\Lambda)^k = ki(\Lambda)^2 di(\Lambda)^{k-1} + kdi(\Lambda)^{k+1} + 2ki(\Lambda)^k i_E. \tag{24}$$

Now, if we apply $i(\Lambda)$ (on the left) to both sides of (23) then we deduce that

$$ki(\Lambda)^2 di(\Lambda)^{k-1} = i(\Lambda)^{k+1}d + (k-1)i(\Lambda)di(\Lambda)^k + k(k-1)i(\Lambda)^k i_E. \tag{25}$$

Adding (24) to (25), we finally have

$$(k+1)i(\Lambda)di(\Lambda)^k = i(\Lambda)^{k+1}d + kdi(\Lambda)^{k+1} + k(k+1)i(\Lambda)^k i_E. \qquad \square$$

THEOREM 3.8. *For a Jacobi manifold (M, Λ, E), the second spectral sequence of the double complex $\mathcal{E}^{\text{per}}_{p,q}(M)$ degenerates at $'E^1(M)$; that is, $'E^1(M) \cong {'E^\infty}(M)$.*

Proof. We will show that $'\delta_r = 0$ for all $r \geq 1$.

Consider $\beta \in \mathcal{E}^{\text{per}}_{p,q}(M)$ such that $'[\beta]_r \in {'E^r_{p,q}}(M)$. Hence, there exist basic differential forms $\beta_1, \dots, \beta_{r-1}$ that satisfy (16) and such that $'\delta'_r[\beta]_r = {'[\delta\beta_{r-1}]_r}$. In fact, since $d\beta = 0$, we have

$$\delta\beta = [i(\Lambda), d]\beta = d(-i(\Lambda)\beta).$$

Thus, we can take $\beta_1 = -i(\Lambda)\beta$. From (11) we deduce that

$$i_E\beta_1 = -i(\Lambda)i_E\beta = 0, \qquad \mathcal{L}_E\beta_1 = -i(\Lambda)\mathcal{L}_E\beta = 0,$$

which implies that $\beta_1 \in \mathcal{E}^{\text{per}}_{p+1,q-1}(M)$.

Now, using Lemma 3.7 yields

$$\delta\beta_1 = di(\Lambda)^2\beta - i(\Lambda)di(\Lambda)\beta = d(\tfrac{1}{2}i(\Lambda)^2\beta).$$

Thus, we can take $\beta_2 = \frac{1}{2}i(\Lambda)^2\beta$. It is clear that β_2 is also basic; that is, $i_E\beta_2 = 0$ and $\mathcal{L}_E\beta_2 = 0$. Proceeding further, we obtain that

$$\beta_s = \frac{(-1)^s}{s!}i(\Lambda)^s\beta \quad \text{for all } 1 \leq s \leq r-1$$

and moreover that β_s is basic. Then, a representative element of the class $'\delta'_r[\beta]_r = {'[\delta\beta_{r-1}]_r}$ is, using Lemma 3.7,

$$\delta\beta_{r-1} = \frac{(-1)^{r-1}}{(r-1)!}(i(\Lambda)di(\Lambda)^{r-1}\beta - di(\Lambda)^r\beta) = d\left(\frac{(-1)^r}{(r)!}i(\Lambda)^r\beta\right),$$

which implies that $\delta\beta_{r-1}$ defines the zero homology class in $'E^r_{p+r-1,q-r}(M)$. This completes the proof. $\square$

With regard to the first spectral sequence, Brylinski has proved that it degenerates at the first term for a compact symplectic manifold [6]. He also proposed the following problem.

PROBLEM B. Give conditions on a compact Poisson manifold that ensure the degeneracy at the first term of the first spectral sequence.

Fernández, Ibáñez, and de León [12; 13] have obtained a counterexample (a 5-dimensional compact almost cosymplectic manifold M^5) for which the first spectral sequence does not degenerate at the first term. Another example (a Poisson structure of rank 2 on the Kodaira–Thurston manifold KT) was given in [11]. Here, we propose the natural extension of Brylinski's Problem B.

PROBLEM B-J. Give conditions on a compact Jacobi manifold that ensure the degeneracy at the first term of the first spectral sequence.

In the next section we prove that the first spectral sequence degenerates for contact manifolds also.

4. Canonical Homology, Spectral Sequences, and Basic de Rham Cohomology on Contact Manifolds

In this section we study the double complex $\mathcal{E}^{\text{per}}_{*,*}(M)$ on a particular class of Jacobi manifolds: contact manifolds.

Let (M, η) be a $(2m+1)$-dimensional contact manifold and consider the isomorphism of $C^\infty(M, \mathbb{R})$-modules $\flat\colon \mathfrak{X}(M) \to \Omega^1(M)$ defined by $\flat(X) = i_X d\eta + \eta(X)\eta$. The mapping $\flat$ can be extended to a mapping from the space $\mathfrak{X}^k(M)$ of k-vectors onto the space of k-forms $\Omega^k(M)$ by putting $\flat(X_1 \wedge \cdots \wedge X_k) = \flat(X_1) \wedge \cdots \wedge \flat(X_k)$. Thus, $\flat$ is also an isomorphism of $C^\infty(M, \mathbb{R})$-modules.

Now, let $\mathfrak{X}^k_\eta(M)$ be the submodule of $\mathfrak{X}^k(M)$ defined by $\mathfrak{X}^k_\eta(M) = \{ K \in \mathfrak{X}^k(M) \mid i_\eta K = 0 \}$, where $(i_\eta K)(\alpha_1, \ldots, \alpha_{k-1}) = K(\eta, \alpha_1, \ldots, \alpha_{k-1})$, and let $\Omega^k_\xi(M)$ be the submodule of $\Omega^k(M)$ defined by $\Omega^k_\xi(M) = \{ \alpha \in \Omega^k(M) \mid i_\xi \alpha = 0 \}$. Then $\flat|_{\mathfrak{X}^k_\eta(M)}\colon \mathfrak{X}^k_\eta(M) \to \Omega^k_\xi(M)$ is an isomorphism of $C^\infty(M, \mathbb{R})$-modules.

We define the star operator $\tilde{*}_B\colon \Omega^k_\xi(M) \to \Omega^{2m-k}_\xi(M)$ by

$$\tilde{*}_B \alpha = i(\flat^{-1}|_{\mathfrak{X}^k_\eta(M)}(\alpha)) \frac{(d\eta)^m}{m!}. \tag{26}$$

Notice that, from $i_\xi d\eta = 0$, we have $i_\xi \tilde{*}_B \alpha = 0$. We will now prove some properties of this operator.

LEMMA 4.1. *Let (M, η) be a contact manifold. Then*:

(i) $\mathcal{L}_\xi \circ \tilde{*}_B = \tilde{*}_B \circ \mathcal{L}_\xi$;
(ii) *if α is a basic k-form then $\tilde{*}_B \alpha$ is a basic $(2m-k)$-form.*

Proof. Since $i_\xi \eta = 1$ and $i_\xi d\eta = 0$, we deduce that $\mathcal{L}_\xi \flat(X) = \flat([\xi, X])$ for all $X \in \mathfrak{X}(M)$. Thus,

$$\mathcal{L}_\xi \flat(K) = \flat(\mathcal{L}_\xi K) \tag{27}$$

for all $K \in \mathfrak{X}^k(M)$. Now, let $\alpha \in \Omega^k_\xi(M)$. Using (27) and that $[\mathcal{L}_\xi, i(K)] = i(\mathcal{L}_\xi K)$ for all $K \in \mathfrak{X}^k(M)$, we have

$$\mathcal{L}_\xi(\tilde{*}_B\alpha) = \mathcal{L}_\xi i(\flat^{-1}(\alpha))\frac{(d\eta)^m}{m!} = i(\flat^{-1}(\alpha))\mathcal{L}_\xi\left(\frac{(d\eta)^m}{m!}\right) + i(\mathcal{L}_\xi\flat^{-1}(\alpha))\frac{(d\eta)^m}{m!}$$
$$= i(\flat^{-1}(\mathcal{L}_\xi\alpha))\frac{(d\eta)^m}{m!} = \tilde{*}_B(\mathcal{L}_\xi\alpha).$$

This proves (i). Part (ii) is direct consequence of (i). □

PROPOSITION 4.2. *Let (M,η) be a contact manifold.*

(i) *If $\alpha \in \Omega^k_\xi(M)$, then $\tilde{*}^2_B\alpha = \alpha$.*
(ii) *If α is a basic k-form, then $\delta_B\alpha = (-1)^{k+1}\tilde{*}_B d_B\tilde{*}_B\alpha$.*

Proof. (i) For each point $x \in M$, let $\mathrm{Ann}(\eta_x)$ be the vector subspace of those tangent vectors in T_xM that are annihilated by η_x. Therefore, $\mathrm{Ann}(\eta_x)$ is a symplectic vector space with symplectic form $(d\eta)_x$, and $(\tilde{*}_B)_x$ is just the star isomorphism defined by the symplectic form $(d\eta)_x$ on $\mathrm{Ann}(\eta_x)$. Thus, the result follows from [24].

(ii) Let $\{I \times U, (t, q^i, p_i)\}$ be a system of canonical coordinates on M; that is,

$$\eta = dt - \sum_i p_i dq^i, \quad \Lambda = \sum_i\left(\frac{\partial}{\partial q^i} + p_i\frac{\partial}{\partial t}\right)\wedge\frac{\partial}{\partial p_i}, \quad E = \frac{\partial}{\partial t}.$$

We can consider in U the symplectic form $\Omega = \sum_i dq^i \wedge dp_i = d\eta$. Now, if β is a basic k-form then it is a k-form on U, and a direct computation shows that $\tilde{*}_B\beta = \tilde{*}_\Omega\beta$ and $i(\Lambda)\beta = i(\Lambda_\Omega)\beta$, where $\tilde{*}_\Omega$ is the star isomorphism defined on the space of forms in U by the symplectic form Ω and where $\Lambda_\Omega = \sum_i(\partial/\partial q^i)\wedge(\partial/\partial p_i)$ (see [24]). Thus, using Theorem 2.2.1 of [6], we conclude that $\delta_B\alpha = (-1)^{k+1}\tilde{*}_B d_B\tilde{*}_B\alpha$ for all basic k-forms α. □

The following corollary states that the canonical homology of a contact manifold is just its basic de Rham cohomology.

COROLLARY 4.3. *Let (M,η) be a contact manifold of dimension $2m+1$. Then the operator $\tilde{*}_B$ establishes an isomorphism of the canonical homology group $H^{\mathrm{can}}_k(M)$ with the basic de Rham cohomology group $H^{2m-k}_B(M)$.*

Corollary 4.3 permits us to obtain sufficient conditions to ensure the finiteness of the canonical homology groups of a compact contact manifold. In fact, we will prove that for a particular class of compact contact manifolds (the K-contact manifolds) the canonical homology groups have finite dimension.

Let (M,ϕ,ξ,η,g) be a $(2m+1)$-dimensional *almost contact metric manifold*; that is (see [4]), ϕ is a $(1,1)$ tensor field, η is a 1-form, ξ is a vector field, and g is a Riemannian metric on M such that

$$\phi^2 = -\mathrm{Id} + \eta\otimes\xi, \quad \eta(\xi) = 1, \quad \text{and} \quad g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y)$$

for $X, Y \in \mathfrak{X}(M)$, where Id is the identity transformation. Then we have $\phi(\xi) = 0$ and $\eta(X) = g(X,\xi)$ for all $X \in \mathfrak{X}(M)$. The *fundamental 2-form* Φ of M is defined by $\Phi(X, Y) = g(X, \phi Y)$, and the $(2m+1)$-form $\eta \wedge \Phi^m$ is a volume form on M. The almost contact metric manifold is said to be: *contact* if $d\eta = \Phi$; *K-contact* if it is contact and ξ is Killing; *normal* if $[\phi, \phi] + 2d\eta \otimes \xi = 0$; *Sasakian* if it is contact and normal; *almost cosymplectic* if Φ and η are closed; and *cosymplectic* if it is almost cosymplectic and normal (see [4]). If M is a Sasakian manifold then it is K-contact [4].

Next, let (M, ϕ, ξ, η, g) be a $(2m+1)$-dimensional compact K-contact manifold. Denote by $\mathcal{F}$ the foliation on M defined by ξ. It is clear that $\mathcal{F}$ is a transversally oriented foliation. In fact, $v = (d\eta)^m/m!$ is a transversal volume form associated with the foliation $\mathcal{F}$. If $\star$ is the Hodge star isomorphism, then the characteristic form $X_{\mathcal{F}} = \star v$ of $\mathcal{F}$ is just the 1-form η.

Since ξ is Killing, we deduce that $\mathcal{F}$ is a Riemannian foliation and that g is a bundle-like metric. Moreover, we have

$$g(\nabla_X \xi, Y) = d\eta(X, Y) = \Phi(X, Y) = -g(\phi X, Y)$$

for $X, Y \in \mathfrak{X}(M)$, where ∇ is the Riemannian connection of g. This implies that $\nabla_X \xi = -\phi X$. In particular, $\nabla_\xi \xi = 0$ and thus the mean curvature 1-form associated with $\mathcal{F}$ is null.

We can define a star operator $\bar{\star}_B \colon \Omega_B^k(M) \to \Omega_B^{2m-k}(M)$ by

$$\alpha \wedge \bar{\star}_B \alpha = v$$

for $\alpha \in \Omega_B^k(M)$ (see [19]). The relations between $\star$ and $\bar{\star}_B$ are characterized by the formulas

$$\bar{\star}_B \alpha = (-1)^k i_\xi \star \alpha \quad \text{and} \quad \star\alpha = \bar{\star}_B \alpha \wedge \eta \tag{28}$$

for $\alpha \in \Omega_B^k(M)$. Thus, we obtain

$$\bar{\star}_B^2 \alpha = (-1)^k \alpha \tag{29}$$

for $\alpha \in \Omega_B^k(M)$. The global scalar product $\langle\ ,\ \rangle$ on $\Omega^k(M)$ restricts on basic forms to the expression

$$\langle \alpha, \alpha' \rangle = \int_M \alpha \wedge \bar{\star}_B \alpha' \wedge \eta.$$

With respect to this scalar product, the adjoint operator $\bar{\delta}_B \colon \Omega_B^k(M) \to \Omega_B^{k-1}(M)$ of d_B is given by

$$\bar{\delta}_B = -\bar{\star}_B d_B \bar{\star}_B \tag{30}$$

(see [19]). A direct computation, using (28) and the fact that $i_\xi \alpha = (-1)^{k+1}(\eta \wedge \star\alpha)$ for $\alpha \in \Omega_B^k(M)$ (see [14]), shows that

$$\bar{\delta}\alpha = \bar{\delta}_B \alpha + \eta \wedge \mathcal{A}(\alpha), \tag{31}$$

where $\bar{\delta}$ is the codifferential on M, $\mathcal{A} = \star[d\eta]\star$, and $[d\eta]$ is the operator defined by $[d\eta]\alpha = \alpha \wedge d\eta$.

Now, let $\bar{\Delta}_B \colon \Omega_B^k(M) \to \Omega_B^k(M)$ be the basic Laplacian (i.e., $\bar{\Delta}_B = d_B \bar{\delta}_B + \bar{\delta}_B d_B$), and let $\Omega_{BH}^k(M)$ be the space of transversally harmonic basic k-forms (i.e., the kernel of $\bar{\Delta}_B$ on $\Omega^k_B(M)$). We have that a basic k-form α is transversally harmonic if

$$d_B\alpha = d\alpha = 0 \quad \text{and} \quad \bar{\delta}_B\alpha = \bar{\delta}\alpha - \eta \wedge \mathcal{A}\alpha = 0. \tag{32}$$

In [1] and [2], the authors obtained a basic de Rham–Hodge decomposition for a transversally oriented Riemannian foliation on a compact orientable Riemannian manifold with bundle-like metric and with basic mean curvature 1-form (for a different proof of this result see [20]; we also refer to [38] for a general reference). Using this result, it follows that there is a decomposition into mutually orthogonal subspaces

$$\Omega^k_B(M) = \bar{\Delta}_B(\Omega^k_B(M)) \oplus \Omega^k_{BH}(M) = \operatorname{im} d_B \oplus \operatorname{im} \bar{\delta}_B \oplus \Omega^k_{BH}(M)$$

and, moreover, the space $\Omega^k_{BH}(M)$ has finite dimension. Therefore, since the basic de Rham cohomology group $H^k_B(M)$ is isomorphic to the space $\Omega^k_{BH}(M)$, we conclude that $H^k_B(M)$ has finite dimension. From Corollary 4.3, this then implies our next result.

COROLLARY 4.4. *Let (M, η) be a compact K-contact manifold. Then the canonical homology groups $H^{\text{can}}_*(M)$ have finite dimension.*

REMARK 4.5. For a compact symplectic manifold, the canonical homology groups have finite dimension [6; 13]. For compact contact manifolds, we only are able to assure the finiteness for K-contact or Sasakian manifolds.

We return to the general case of an arbitrary contact manifold (M, η). Since $\delta_B\alpha = (-1)^{k+1}\tilde{*}_B d_B \tilde{*}_B\alpha$ for all basic k-form α, we can introduce a harmonic theory on contact manifolds.

DEFINITION 4.6. A basic k-form α on a contact manifold (M, η) is said to be *harmonic* if $d\alpha = 0$ and $\delta_B\alpha = 0$.

A similar definition was introduced by Brylinski in [6] for symplectic and (more generally) Poisson manifolds by using the Koszul operator. Here, we also extend Definition 4.6 for arbitrary Jacobi manifolds in such a way that it would be consistent with Brylinski's definition.

In [6], Brylinski proposed the following problem.

PROBLEM A. Give conditions on a compact Poisson manifold which ensure that any cohomology class in $H^*(M)$ has a harmonic representative α, that is, $d\alpha = 0$ and $\delta\alpha = 0$.

Brylinski proved that this holds for compact Kähler manifolds and cotangent bundles, and he conjectured that this would be true for any compact symplectic manifold. This conjecture was recently disproved by Fernández, Ibáñez, and de León [10] by exhibiting a counterexample. More generally, Mathieu [30] has proved that a compact symplectic manifold (N, Ω) verifies Brylinski's conjecture if and only if it satisfies the strong Lefschetz theorem: for any $n \leq m$, $\dim N = 2m$, the cup product $[\Omega]^n \colon H^{m-n}(N) \to H^{m+n}(N)$ is an isomorphism. A simpler proof of this fact was made by Yang [42]. As a consequence, the odd Betti numbers of a compact symplectic manifold verifying Brylinski's conjecture are even. In the odd-dimensional setting, it was recently proved by Ibáñez [18] that, for a compact cosymplectic manifold, any de Rham cohomology class has a harmonic representative.

The following problem is a natural extension of the previous one.

PROBLEM A-J. Give conditions on a compact Jacobi manifold which ensure that any basic cohomology class in $H_B^*(M)$ has a harmonic representative α, that is, $d\alpha = 0$ and $\delta_B \alpha = 0$.

The result obtained by Mathieu gives an answer for Problem A-J in the symplectic setting. We will give a similar answer for Problem A-J in the contact setting.

Let (M, η) be a contact manifold of dimension $2m+1$, let $\Omega_B(M) = \sum_k \Omega_B^k(M)$ be the real vector space of basic forms on M, and consider on $\Omega_B(M)$ the restrictions d_B and δ_B of the differential operators d and δ (see Section 3). We also introduce the operators $[d\eta]$, f, and h, which are given by $[d\eta](\alpha) = d\eta \wedge \alpha$, $f(\alpha) = i(\Lambda)\alpha$, and $h(\alpha) = (k-m)\alpha$ for all $\alpha \in \Omega_B^k(M)$. It is clear that $[d\eta]$, f, and h are endomorphisms of $\Omega_B(M)$. In fact, if $\alpha \in \Omega_B^k(M)$ then $[d\eta](\alpha) \in \Omega_B^{k+2}(M)$, $f(\alpha) \in \Omega_B^{k-2}(M)$, and $h(\alpha) \in \Omega_B^k(M)$. Moreover, from a direct computation using (6), (8), and the local expressions of η and Λ in canonical coordinates, we obtain the following lemma.

LEMMA 4.7. *Let (M, η) be a contact manifold of dimension $2m+1$. Then the operators d_B, δ_B, $[d\eta]$, f, and h verify*:

(i) $[h, [d\eta]] = 2[d\eta]$, $[h, f] = -2f$, $[[d\eta], f] = h$;
(ii) $[[d\eta], d_B] = 0$, $[h, d_B] = d_B$, $[f, d_B] = \delta_B$, $[[d\eta], \delta_B] = d_B$, $[h, \delta_B] = -\delta_B$, $[f, \delta_B] = 0$.

From (i) we deduce that $\{[d\eta], f, h\}$ spans a Lie algebra isomorphic to sl(2), the Lie algebra of all 2×2-matrices of trace 0. From (ii) and since $d_B^2 = \delta_B^2 = d_B\delta_B + \delta_B d_B = 0$ we have that the operators $[d\eta]$, f, h, d_B, and δ_B span the Lie super-algebra $\mathrm{sl}(2) \times \mathbb{R}^2$. Therefore, the space $\Omega_B(M)$ viewed as a $(\mathrm{sl}(2) \times \mathbb{R}^2)$-module belongs to the category $\mathcal{V}$ of all $(\mathrm{sl}(2) \times \mathbb{R}^2)$-modules on which h acts diagonally with only finitely many different eigenvalues (this is studied in [30]). We denote by $H^*_{\mathrm{har}\, B}(M)$ the subspace of all the cohomology classes in $H_B^*(M)$ that contain at least one harmonic form:

$$H^k_{\mathrm{har}\, B}(M) = \{\, a \in H_B^k(M) \mid \exists \alpha \in \Omega_B^k(M) \text{ with } d_B\alpha = 0,\ \delta_B\alpha = 0, \text{ and } a = [\alpha]\,\}.$$

Consequently, using Theorem 1 and Lemma 5 of [30], we obtain the following result.

THEOREM 4.8. *Let (M, η) be a (not necessarily compact) contact manifold of dimension $2m+1$. Then $H_B^*(M) = H_{\text{har }B}^*(M)$ if and only if, for any $n \leq m$, the cup product $[d\eta]^n \colon H_B^{m-n}(M) \to H_B^{m+n}(M)$ is onto.*

If (M, ϕ, ξ, η, g) is a compact K-contact manifold of dimension $2m+1$, we deduce that the basic de Rham cohomology group $H_B^k(M)$ is isomorphic to the space $\Omega_{BH}^k(M)$ of transversally harmonic basic k-forms and that $H_B^k(M)$ has finite dimension. Furthermore, from (29) and (30) we obtain that the spaces $\Omega_{BH}^{m-n}(M)$ and $\Omega_{BH}^{m+n}(M)$ are isomorphic, which implies that the dimension of $H_B^{m-n}(M)$ is equal to the dimension of $H_B^{m+n}(M)$. Therefore, using Theorem 4.8, we have the following corollary.

COROLLARY 4.9. *Let (M, ϕ, ξ, η, g) be a compact K-contact manifold of dimension $2m+1$. Then the following two assertions are equivalent.*

(i) $H_B^*(M) = H_{\text{har }B}^*(M)$.
(ii) *For any $n \leq m$, the cup product $[d\eta]^n \colon H_B^{m-n}(M) \to H_B^{m+n}(M)$ is an isomorphism.*

Sasakian manifolds may be considered as an odd-dimensional counterpart of Kähler manifolds. This leads to our next result.

COROLLARY 4.10. *Let (M, ϕ, ξ, η, g) be a compact Sasakian manifold. Then*

$$H_B^*(M) = H_{\text{har }B}^*(M).$$

Proof. We will use the fact that a Sasakian manifold is K-contact. Suppose that $\dim M = 2m+1$. A (not necessarily basic) k-form α on M is called *C-harmonic* by Ogawa [33] if

$$d\alpha = 0 \quad \text{and} \quad \bar{\delta}\alpha = \eta \wedge \mathcal{A}\alpha,$$

where $\bar{\delta}$ is the codifferential on M and $\mathcal{A} = \star[d\eta]\star$, with $\star$ the Hodge star isomorphism. Notice that the C-harmonic basic forms are just the transversally harmonic basic forms (see (32)).

If $k \leq m$ and α is a C-harmonic k-form, then Ogawa [33] proved that α is basic and, as a consequence, the $(k+2)$-form $[d\eta]\alpha = \alpha \wedge d\eta$ is also C-harmonic. Now, if α is a transversally harmonic basic k-form (with k arbitrary) then, proceeding as in [33], we also can prove that $[d\eta]\alpha = \alpha \wedge d\eta$ is a transversally harmonic basic $(k+2)$-form.

Using these results, we will deduce that the cup product $[d\eta]^n \colon H_B^{m-n}(M) \to H_B^{m+n}(M)$ is an isomorphism for $n \leq m$. The basic de Rham cohomology group $H_B^{m-n}(M)$ (respectively, $H_B^{m+n}(M)$) has finite dimension, and it can be identified with the space $\Omega_{BH}^{m-n}(M)$ (resp. $\Omega_{BH}^{m+n}(M)$) of transversally harmonic basic $(m-n)$-forms (resp. $(m+n)$-forms). Under these identifications, the cup product $[d\eta]^n \colon H_B^{m-n}(M) \to H_B^{m+n}(M)$ is just the linear mapping

$$[d\eta]^n \colon \Omega_{BH}^{m-n}(M) \to \Omega_{BH}^{m+n}(M), \quad \alpha \mapsto \alpha \wedge (d\eta)^n.$$

On the other hand, from results for arbitrary almost contact metric manifolds (see [7, Lemma 6, Prop. 14]), we deduce that the cup product $[d\eta]^n\colon \Omega_{BH}^{m-n}(M) \to \Omega_{BH}^{m+n}(M)$ is injective. Finally, since the dimensions of the spaces $\Omega_{BH}^{m-n}(M)$ and $\Omega_{BH}^{m+n}(M)$ are equal, we conclude that the cup product is an isomorphism. This, in view of Corollary 4.9, ends the proof of our result. □

EXAMPLE 4.11. Consider the 3-dimensional torus $T^3 = \mathbb{R}^3/Z^3$ endowed with the contact 1-form

$$\eta = \cos(2\pi x^3)dx^1 + \sin(2\pi x^3)dx^2,$$

where (x^1, x^2, x^3) are the standard coordinates on R^3. The Reeb vector field is given by

$$\xi = \cos(2\pi x^3)\frac{\partial}{\partial x^1} + \sin(2\pi x^3)\frac{\partial}{\partial x^2}.$$

The contact manifold (T^3, η) is not regular, since ξ induces an irrational flow on the 2-dimensional torus $x^3 = 1/6$. In fact, the integral curve of ξ through $(0, 0, 1/6)$ is given by $(t/2, \sqrt{3}t/2, 1/6)$. The flow ξ is not Riemannian, since the union of the leaf closures whose dimension is maximal is not open in T^3 (see [31, Prop. 5.3, p. 157]). As we know, $H_B^0(T^3) = \mathbb{R}$ since T^3 is connected. Next, we will compute $H_B^2(T^3)$. We have that $[d\eta]$ defines a nontrivial class in $H_B^2(T^3)$, because our foliation (denoted by $\mathcal{F}$) is taut and transversally symplectic (see [38, Thm. 9.23, p. 125]). Moreover, since $\mathcal{F}$ is a subfoliation of the foliation defined by the canonical fibration $\pi\colon T^3 \to T^1$, $\pi(x^1, x^2, x^3) = x^3$, it follows that the filtration defined by π on the de Rham complex $(\Omega^*(T^3), d)$ induces a filtration on the basic complex $(\Omega_B^*(T^3), d_B)$ such that the corresponding spectral sequence (E_i, d_i) converges to $H_B^*(T^3)$. Using standard arguments, we obtain that

$$H_B^2(T^3) \cong E_2^{1,1} \cong H^1(T^1, \mathcal{H}^1),$$

where $\mathcal{H}^1$ is the presheaf defined by $\mathcal{H}^1(U) = H_B^1(\pi^{-1}(U))$ with U an open set in T^1. Using that the foliation $\mathcal{F}$ restricted to each fiber of π is linear, we deduce that $H_B^2(T^3) \cong \mathbb{R}$. Finally, from Theorem 4.8 we conclude that $H_B^*(T^3) = H_{\text{har }B}^*(T^3)$.

It should be noticed that T^3 admits neither Sasakian nor regular contact structures (see e.g. [4, pp. 71, 77]).

We now consider the particular case of regular contact structures in order to obtain an example of contact manifold M for which $H_B^*(M) \ncong H_{\text{har }B}^*(M)$. Theorem 4.8 yields the following corollary.

COROLLARY 4.12. *Let (M, η) be a (not necessarily compact) regular contact manifold of dimension $2m+1$ with Reeb vector field ξ, and suppose that $(M/\xi, \Omega)$ is the induced symplectic quotient manifold. Then $H_B^*(M) = H_{\text{har }B}^*(M)$ if and only if, for any $n \leq m$, the cup product $[\Omega]^n\colon H^{m-n}(M/\xi) \to H^{m+n}(M/\xi)$ is onto, where $H^*(M/\xi)$ is the de Rham cohomology of M/ξ.*

From Corollary 4.12 we obtain the following.

COROLLARY 4.13. *Let (M, η) be a compact regular contact manifold with Reeb vector field ξ, and suppose that $(M/\xi, \Omega)$ is the induced symplectic quotient manifold. Then $H_B^*(M) = H_{\text{har }B}^*(M)$ if and only if M/ξ verifies the strong Lefschetz theorem.*

EXAMPLE 4.14. Let H be the *Heisenberg group* consisting of real matrices of the form

$$H = \left\{ \begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x_1, x_2, x_3 \in \mathbb{R} \right\};$$

H is a 3-dimensional connected, simply connected, and nilpotent Lie group. A standard computation shows that a basis for the left invariant 1-forms on H is given by $\{dx_1, dx_2, dx_3 - x_1 dx_2\}$. Now, we take the compact quotient $\Gamma\backslash H$, where Γ is the uniform subgroup of H consisting of those matrices whose entries are integers. Hence $\Gamma\backslash H$ is a 3-dimensional compact nilmanifold, and the 1-forms $dx_1, dx_2, dx_3 - x_1 dx_2$ all descend to 1-forms $\alpha_1, \alpha_2, \alpha_3$ on $\Gamma\backslash H$.

The *Kodaira–Thurston manifold* KT [37] is

$$KT = (\Gamma\backslash H) \times S^1.$$

Denote by α_4 the canonical 1-form on S^1. Then $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is a basis for the 1-forms on KT such that

$$d\alpha_1 = d\alpha_2 = d\alpha_4 = 0, \qquad d\alpha_3 = -\alpha_1 \wedge \alpha_2.$$

We recall that there exists a one-to-one correspondence between equivalence classes of principal circle bundles over a manifold N and the cohomology group $H^2(N, \mathbb{Z})$. Furthermore, given an integral closed 2-form Ω on N, there is a principal circle bundle $\pi: M \to N$ with connection form η such that $\pi^*\Omega = d\eta$; that is, Ω is the curvature form of the connection [22].

The 2-form $\Omega = 2\alpha_2 \wedge \alpha_3 + \alpha_1 \wedge \alpha_4$ on KT is symplectic and defines an integer class. Thus, there exists a principal circle bundle $\pi: M \to KT$ with connection form α_5 such that $\pi^*\Omega = d\alpha_5$. We denote by the same symbols the lifted 1-forms α_i $(1 \le i \le 4)$ to M. It should be noticed that M is also a compact nilmanifold, with structure equations

$$d\alpha_1 = d\alpha_2 = d\alpha_4 = 0, \quad d\alpha_3 = -\alpha_1 \wedge \alpha_2, \quad d\alpha_5 = 2\alpha_2 \wedge \alpha_3 + \alpha_1 \wedge \alpha_4. \quad (33)$$

Moreover, (M, α_5) is a regular contact manifold, and the induced symplectic quotient manifold is just KT. Since KT is a compact nilmanifold, its de Rham cohomology can be easily computed by using Nomizu's theorem [32; 35]. In fact, $b_1(KT) = 3$, and we deduce that KT does not verify the strong Lefschetz theorem. Thus, we conclude that

$$H_B^*(M) \not\cong H_{\text{har }B}^*(M).$$

To end the example, if we integrate the structure equations (33) we can realize M as the nilmanifold $\bar{\Gamma}\backslash G$, where G is the group consisting of the matrices

$$G = \left\{ \begin{pmatrix} 1 & x_1 & x_2 & 2x_1x_2 & x_4 & x_3 & x_5 \\ 0 & 1 & 0 & 2x_2 & 0 & x_2 & -(x_2)^2 - x_4 \\ 0 & 0 & 1 & 2x_1 & 0 & 0 & -2x_3 \\ 0 & 0 & 0 & 1 & 0 & 0 & -x_2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \;\middle|\; x_1, x_2, x_3, x_4, x_5 \in \mathbb{R} \right\}$$

and $\bar{\Gamma}$ is the subgroup of G consisting of the matrices with integer entries.

In the remainder of this section we study the behavior of the first spectral sequence for contact manifolds.

THEOREM 4.15. *Let (M, η) be a contact manifold of dimension $2m+1$. Then, for all $r \geq 0$, the homomorphism*

$$f_r \colon E^r_{p,q}(M) \to {}'E^r_{q,2m+p}(M)$$

given by

$$f_r[\alpha]_r = {}'[\tilde{*}_B \alpha]_r$$

is an isomorphism of homology groups. Moreover, f_r commutes with the differential; that is,

$$(f_r \circ \delta_r)[\alpha]_r = (-1)^{q-p+1}({}'\delta_r \circ f_r)[\alpha]_r$$

for all $[\alpha]_r \in E^r_{p,q}(M)$.

Proof. Let $\alpha \in \mathcal{E}^{\mathrm{per}}_{p,q}(M)$. Then, from Lemma 4.1, $\tilde{*}_B \alpha \in \mathcal{E}^{\mathrm{per}}_{q,2m+p}(M)$. Moreover, if α lives to $E^r_{p,q}(M)$ then there exist basic forms $\alpha_i \in \mathcal{E}^{\mathrm{per}}_{p-i,q+i}(M)$ $(i = 1, \dots, r-1)$ that satisfy conditions (12); that is,

$$\delta\alpha = 0,\ d\alpha = \delta\alpha_1,\ d\alpha_1 = \delta\alpha_2, \dots, d\alpha_{r-3} = \delta\alpha_{r-2},\ d\alpha_{r-2} = \delta\alpha_{r-1}.$$

We will show that $\beta = \tilde{*}_B \alpha$ lives to ${}'E^r_{q,2m+p}(M)$. To do this, we consider the differential forms $\beta_i = \tilde{*}_B \alpha_i$ $(i = 1, \dots, r-1)$. Then, using Proposition 4.2, it follows that

$$d\beta = 0,\ \delta\beta = d\beta_1,\ \delta\beta_1 = d\beta_2, \dots, \delta\beta_{r-3} = d\beta_{r-2},\ \delta\beta_{r-2} = d\beta_{r-1}$$

and so $\tilde{*}_B \alpha$ lives to ${}'E^r_{q,2m+p}(M)$. Moreover, using again Proposition 4.2, we deduce that f_r is an isomorphism.

On the other hand, we have that $\alpha_{r-1} \in \Omega_B^{q-p+2(r-1)}(M)$. As a result, $\tilde{*}_B \alpha_{r-1} \in \Omega_B^{2m-q+p-2(r-1)}(M)$. Then, using (13) and (17), we have

$$(f_r \circ \delta_r)[\alpha]_r = f_r[d\alpha_{r-1}]_r = {}'[\tilde{*}_B d\alpha_{r-1}]_r,$$
$$({}'\delta_r \circ f_r)[\alpha]_r = {}'\delta_r {}'[\tilde{*}_B \alpha]_r = {}'[\delta_B \tilde{*}_B \alpha_{r-1}]_r.$$

Now, from Proposition 4.2, we have $(\delta_B \tilde{*}_B)\alpha_{r-1} = (-1)^{q-p+1} \tilde{*}_B d\alpha_{r-1}$; we consequently obtain that

$$(f_r \circ \delta_r)[\alpha]_r = (-1)^{q-p+1}({}'\delta_r \circ f_r)[\alpha]_r. \qquad \square$$

Using Theorems 3.8 and 4.15, we conclude as follows.

THEOREM 4.16. *Let (M, η) be a contact manifold of dimension $2m+1$. Then the first spectral sequence of the double complex $\mathcal{E}^{\text{per}}_{p,q}(M)$ degenerates at $E^1(M)$; that is, $E^1(M) \cong E^\infty(M)$.*

5. Canonical Homology and Spectral Sequences of Locally Conformal Symplectic Manifolds

In this section we study the canonical homology and the behavior of the first spectral sequence for l.c.s. manifolds. Particularly, we will study the case of l.c.s. manifolds of the first kind according to Vaisman's classification [39].

Let (M, Ω) be a l.c.s. manifold with Lee 1-form ω. A vector field X on M is said to be an *infinitesimal automorphism* of (M, Ω) if $\mathcal{L}_X\Omega = 0$. We denote by $\mathfrak{X}_\Omega(M)$ the space of the infinitesimal automorphisms of (M, Ω). If $X \in \mathfrak{X}_\Omega(M)$ then, using (3), we deduce that $\mathcal{L}_X\omega = d(\omega(X)) = 0$, which implies that $\omega(X)$ is constant. Moreover, if $X, Y \in \mathfrak{X}_\Omega(M)$ then $[X, Y] \in \mathfrak{X}_\Omega(M)$. Thus, $\mathfrak{X}_\Omega(M)$ is a Lie subalgebra of the Lie algebra $\mathfrak{X}(M)$ of the vector fields on M (see [39]).

Consider now the homomorphism $l : \mathfrak{X}_\Omega(M) \to \mathbb{R}$ defined by

$$l(X) = \omega(X)$$

for $X \in \mathfrak{X}_\Omega(M)$. We call l the *Lee homomorphism* of $\mathfrak{X}_\Omega(M)$ (see [39]). Since ω is closed, l is a Lie algebra homomorphism for the commutative Lie algebra structure of $\mathbb{R}$, and it is clear that the homomorphism l is either trivial or an epimorphism.

DEFINITION 5.1 [39]. A l.c.s. manifold M is said to be *of the first kind* if the Lee homomorphism l is an epimorphism.

We remark that a l.c.s. manifold (M, Ω) is of the first kind if and only if there exists $X \in \mathfrak{X}_\Omega(M)$ such that $l(X) \neq 0$. In fact, the following theorem gives the structure of a l.c.s. manifold of the first kind.

THEOREM 5.2 [39]. *Let (M, Ω) be a $2m$-dimensional l.c.s. manifold of the first kind with Lee 1-form ω, and suppose that (Λ, E) is the associated Jacobi structure on M. Then there exists $U \in \mathfrak{X}_\Omega(M)$ such that $l(U) = \omega(U) = 1$ and, if θ is the 1-form on M given by $\theta = -i_U\Omega$, we have:*

$$\Omega = d\theta - w \wedge \theta$$

and

$$\theta(E) = 1, \quad i_U d\theta = i_E d\theta = 0, \quad [E, U] = 0.$$

Moreover, $\omega \wedge \theta \wedge (d\theta)^{m-1}$ is a volume form on M.

If (M, Ω) is a l.c.s. manifold of the first kind and $U \in \mathfrak{X}_\Omega(M)$ is such that $\omega(U) = 1$, then U is said to be a *basic infinitesimal automorphism* of (M, Ω).

Next, we study the canonical homology of a l.c.s. manifold of the first kind.

PROPOSITION 5.3. *Let (M, Ω) be a l.c.s. manifold of the first kind with Lee 1-form ω, and let U be a basic infinitesimal automorphism of (M, Ω). If (Λ, E) is the associated Jacobi structure on M and δ is the canonical operator, then*:

(i) $i_U\delta + \delta i_U = 0$;
(ii) $\delta(\omega \wedge \alpha) = -\omega \wedge \delta\alpha + \mathcal{L}_E\alpha$ *for all* α.

Proof. Denote by $\flat$ the canonical isomorphism $\flat\colon \mathfrak{X}(M) \to \Omega^1(M)$, $\flat(X) = i_X\Omega$. Since $U \in \mathfrak{X}_\Omega(M)$, we deduce that

$$\mathcal{L}_U\flat(X) = \flat\mathcal{L}_U X = \flat[U, X].$$

Thus, $\mathcal{L}_U\flat^{-1}\alpha = \flat^{-1}\mathcal{L}_U\alpha$ for all $\alpha \in \Omega^1(M)$; from (4), this implies that

$$(\mathcal{L}_U\Lambda)(\alpha, \beta) = (\mathcal{L}_U\Omega)(\flat^{-1}\alpha, \flat^{-1}\beta) = 0. \tag{34}$$

Therefore, from (6) and (34), we obtain that

$$\begin{aligned} i_U\delta\alpha &= i(\Lambda)\mathcal{L}_U\alpha - \mathcal{L}_U i(\Lambda)\alpha - \delta i_U\alpha \\ &= -i(\mathcal{L}_U\Lambda)\alpha - \delta i_U\alpha = -\delta i_U\alpha, \end{aligned}$$

which proves (i).

Now suppose that $\tilde{\Lambda}$ is the 2-vector on M given by

$$\tilde{\Lambda} = \Lambda - E \wedge U.$$

Since $i_\theta\Lambda = U$ and $i_w\Lambda = -E$ (see Theorem 5.2), we have that $i_\theta\tilde{\Lambda} = i_\omega\tilde{\Lambda} = 0$. Consequently, from Theorem 5.2 we get that

$$i(\Lambda)(\omega \wedge \alpha) = \omega \wedge i(\tilde{\Lambda})\alpha - i_E\alpha + \omega \wedge i_U i_E\alpha = \omega \wedge i(\Lambda)\alpha - i_E\alpha \tag{35}$$

and therefore, using that ω is closed, we finally obtain

$$\delta(\omega\wedge\alpha) = \omega\wedge di(\Lambda)\alpha - \omega\wedge i(\Lambda)d\alpha + di_E\alpha + i_E d\alpha = -\omega\wedge\delta\alpha + \mathcal{L}_E\alpha. \tag{36}$$

□

We next consider the submodule $\Omega^k_U(M)$ of $\Omega^k(M)$ defined by

$$\Omega^k_U(M) = \{\,\alpha \in \Omega^k(M) \mid i_U\alpha = 0\,\}$$

and the subspace $\Omega^k_{BU}(M)$ defined by

$$\Omega^k_{BU}(M) = \{\,\alpha \in \Omega^k_B(M) \mid i_U\alpha = 0\,\}.$$

Proposition 5.3 allows us to introduce the following subcomplex of the canonical complex of M:

$$\cdots \longrightarrow \Omega^{k+1}_{BU}(M) \xrightarrow{\delta_B} \Omega^k_{BU}(M) \xrightarrow{\delta_B} \Omega^{k-1}_{BU}(M) \longrightarrow \cdots.$$

We denote by $H^{\mathrm{can}}_{*U}(M)$ the homology of this complex, that is,

$$H^{\mathrm{can}}_{kU}(M) = \frac{\mathrm{Ker}\{\delta_B\colon \Omega^k_{BU}(M) \to \Omega^{k-1}_{BU}(M)\}}{\delta_B(\Omega^{k+1}_{BU}(M))}.$$

THEOREM 5.4. *Let (M, Ω) be a l.c.s. manifold of the first kind with Lee 1-form ω, and let U be a basic infinitesimal automorphism. Let $\tilde{F}_k\colon \Omega^k_B(M) \to \Omega^k_{BU}(M) \oplus \Omega^{k-1}_{BU}(M)$ be the isomorphism of $C^\infty_B(M, \mathbb{R})$-modules defined by*

$$\tilde{F}_k(\alpha) = (\alpha - \omega \wedge i_U\alpha, i_U\alpha).$$

Then $\tilde{F}_k$ induces an isomorphism $F_k\colon H^{\mathrm{can}}_k(M) \to H^{\mathrm{can}}_{kU}(M) \oplus H^{\mathrm{can}}_{(k-1)U}(M)$.

Proof. If (Λ, E) is the associated Jacobi structure on M, it is easy to prove (see (5)) that

$$i_E(\alpha - \omega \wedge i_U\alpha) = i_U(\alpha - \omega \wedge i_U\alpha) = 0, \quad i_E i_U\alpha = 0,$$

for $\alpha \in \Omega^k_B(M)$. Furthermore, from (5) and Theorem 5.2, we have that

$$\mathcal{L}_E(\alpha - \omega \wedge i_U\alpha) = -\omega \wedge \mathcal{L}_E i_U\alpha = -\omega \wedge i_U\mathcal{L}_E\alpha = 0$$

and

$$\mathcal{L}_E(i_U\alpha) = i_U\mathcal{L}_E\alpha = 0.$$

Thus, $\alpha - \omega \wedge i_U\alpha \in \Omega^k_{BU}(M)$, $i_U\alpha \in \Omega^{k-1}_{BU}(M)$, and $\tilde{F}_k$ is an isomorphism. In fact, the inverse homomorphism is defined by

$$\tilde{F}^{-1}_k\colon \Omega^k_{BU}(M) \oplus \Omega^{k-1}_{BU}(M) \to \Omega^k_B(M), \quad (\alpha, \beta) \mapsto \alpha + \omega \wedge \beta.$$

On the other hand, from Proposition 5.3 we deduce that

$$\tilde{F}_k(\delta_B\alpha) = (\delta_B(\alpha - \omega \wedge i_U\alpha), -\delta_B i_U\alpha),$$

which implies that $\tilde{F}_k$ induces a homomorphism $F_k\colon H^{\mathrm{can}}_k(M) \to H^{\mathrm{can}}_{kU}(M) \oplus H^{\mathrm{can}}_{(k-1)U}(M)$. In a similar way, the homomorphism $\tilde{F}^{-1}_k$ induces a homomorphism $G_k\colon H^{\mathrm{can}}_{kU}(M) \oplus H^{\mathrm{can}}_{(k-1)U}(M) \to H^{\mathrm{can}}_k(M)$, and it is obvious that $G_k \circ F_k = \mathrm{Id}_{H^{\mathrm{can}}_k(M)}$ and $F_k \circ G_k = \mathrm{Id}_{H^{\mathrm{can}}_{kU}(M) \oplus H^{\mathrm{can}}_{(k-1)U}(M)}$. This ends the proof of our result. □

Now we define an operator $d_U\colon \Omega^k_U(M) \to \Omega^{k+1}_U(M)$ given by

$$d_U\alpha = d\alpha - \omega \wedge i_U d\alpha \quad \text{for all } \alpha \in \Omega^k_U(M). \tag{37}$$

The following properties will be useful in the sequel.

PROPOSITION 5.5. *Let M be a l.c.s. manifold of the first kind, U a basic infinitesimal automorphism, and (Λ, E) the associated Jacobi structure on M. If α is a form on M such that $i_U\alpha = 0$, we have*:

(i) $d^2_U\alpha = 0$;
(ii) $(i_E d_U + d_U i_E)\alpha = \mathcal{L}_E\alpha$;
(iii) $(\mathcal{L}_E d_U - d_U\mathcal{L}_E)\alpha = 0$.

Proof. A direct computation, using (37) and the fact that ω is closed, shows (i). Next, from (5) and Theorem 5.2, we deduce that

$$
\begin{aligned}
i_E d_U \alpha &= \mathcal{L}_E \alpha - \omega \wedge i_U \mathcal{L}_E \alpha - d_U i_E \alpha \\
&= \mathcal{L}_E \alpha - \omega \wedge \mathcal{L}_E i_U \alpha - d_U i_E \alpha = \mathcal{L}_E \alpha - d_U i_E \alpha
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{L}_E d_U \alpha &= d\mathcal{L}_E \alpha - \omega \wedge \mathcal{L}_E i_U d\alpha \\
&= d\mathcal{L}_E \alpha - \omega \wedge i_U d\mathcal{L}_E \alpha = d_U \mathcal{L}_E \alpha,
\end{aligned}
$$

which proves (ii) and (iii). □

Proposition 5.5 allows us to introduce the following differential complex:

$$\cdots \longrightarrow \Omega^{k-1}_{BU}(M) \xrightarrow{d_U} \Omega^k_{BU}(M) \xrightarrow{d_U} \Omega^{k+1}_{BU}(M) \longrightarrow \cdots .$$

Its cohomology is denoted by $H^*_{BU}(M)$, that is,

$$H^k_{BU}(M) = \frac{\operatorname{Ker}\{d_U : \Omega^k_{BU}(M) \to \Omega^{k+1}_{BU}(M)\}}{d_U(\Omega^{k-1}_{BU}(M))}.$$

Next, we study the relationship between the cohomology groups $H^*_{BU}(M)$ and the canonical homology groups $H^{\mathrm{can}}_*(M)$ of M. For this purpose, we introduce a star operator as follows. Let $\flat : \mathfrak{X}(M) \to \Omega^1(M)$ be the canonical isomorphism

$$X \in \mathfrak{X}(M) \longrightarrow \flat(X) = i_X \Omega.$$

The mapping $\flat$ can be extended to a mapping from the space $\mathfrak{X}^k(M)$ of k-vectors into the space $\Omega^k(M)$ by putting $\flat(X_1 \wedge \cdots \wedge X_k) = \flat(X_1) \wedge \cdots \wedge \flat(X_k)$. This extension is an isomorphism of $C^\infty(M, \mathbb{R})$-modules.

Now, denote by $\mathfrak{X}^k_{(\theta,\omega)}(M)$ the submodule of $\mathfrak{X}^k(M)$ defined by

$$\mathfrak{X}^k_{(\theta,\omega)}(M) = \{\, K \in \mathfrak{X}^k(M) \mid i_\theta K = i_\omega K = 0 \,\}$$

and by $\Omega^k_{(E,U)}(M)$ the submodule of $\Omega^k(M)$ defined by

$$\Omega^k_{(E,U)}(M) = \{\, \alpha \in \Omega^k(M) \mid i_E \alpha = i_U \alpha = 0 \,\},$$

where θ is the 1-form on M given by $\theta = -i_U \Omega$. Hence, the mapping $\flat|_{\mathfrak{X}^k_{(\theta,\omega)}(M)} : \mathfrak{X}^k_{(\theta,\omega)}(M) \to \Omega^k_{(E,U)}(M)$ is an isomorphism of $C^\infty(M, \mathbb{R})$-modules.

We define a star operator $\tilde{*}_B : \Omega^k_{(E,U)}(M) \to \Omega^{2m-2-k}_{(E,U)}(M)$ given by

$$\tilde{*}_B \alpha = i((\flat|_{\mathfrak{X}^k_{(\theta,\omega)}(M)})^{-1}(\alpha)) \frac{(d\theta)^{m-1}}{(m-1)!} \tag{38}$$

for $0 \le k \le 2m-2$, where $\dim M = 2m$. Notice that $i_U(\tilde{*}_B \alpha) = i_E(\tilde{*}_B \alpha) = 0$ for all $\alpha \in \Omega^k_{(E,U)}(M)$, since $i_U d\theta = i_E d\theta = 0$ (see Theorem 5.2). We next state some properties of this operator.

Lemma 5.6. *We have*:

(i) $\mathcal{L}_E \circ \tilde{*}_B = \tilde{*}_B \circ \mathcal{L}_E$;

(ii) *if α is a basic k-form and $i_U \alpha = 0$, then $\tilde{*}_B \alpha$ is a basic $(2m-2-k)$-form.*

Proof. Since $\mathcal{L}_E\Omega = 0$, we deduce that

$$\mathcal{L}_E\flat(X) = \flat\mathcal{L}_E X = \flat[E, X]$$

for $X \in \mathfrak{X}(M)$. Thus,

$$\mathcal{L}_E\flat(K) = \flat\mathcal{L}_E K \tag{39}$$

for $K \in \mathfrak{X}^k(M)$. Therefore, if $\alpha \in \Omega^k_{(E,U)}(M)$ then from (39) and Theorem 5.2 it follows that

$$\begin{aligned}\mathcal{L}_E(\tilde{*}_B\alpha) &= \mathcal{L}_E i(\flat^{-1}(\alpha))\frac{(d\theta)^{m-1}}{(m-1)!}\\ &= i(\flat^{-1}(\alpha))\mathcal{L}_E\frac{(d\theta)^{m-1}}{(m-1)!} + i(\mathcal{L}_E\flat^{-1}(\alpha))\frac{(d\theta)^{m-1}}{(m-1)!}\\ &= i(\flat^{-1}(\mathcal{L}_E\alpha))\frac{(d\theta)^{m-1}}{(m-1)!} = \tilde{*}_B(\mathcal{L}_E\alpha).\end{aligned}$$

This proves (i). Part (ii) follows using (i). □

PROPOSITION 5.7. *Let (M, Ω) be a $2m$-dimensional l.c.s. manifold of the first kind, U a basic infinitesimal automorphism, and (Λ, E) the associated Jacobi structure on M. Suppose that $0 \le k \le 2m-2$.*

(i) *If $\alpha \in \Omega^k_{(E,U)}(M)$, then $\tilde{*}^2_B\alpha = \alpha$.*
(ii) *If α is a basic k-form such that $i_U\alpha = 0$, then $\delta_B\alpha = (-1)^{k+1}\tilde{*}_B d_U\tilde{*}_B\alpha$.*

Proof. For a point $x \in M$, consider the subspace S_x of T_xM given by

$$S_x = \{\, v \in T_xM \mid \theta_x(v) = \omega_x(v) = 0 \,\},$$

where θ is the 1-form on M defined by $\theta = -i_U\Omega$. Then S_x is a symplectic vector space with symplectic form $(d\theta)_x$ (see Theorem 5.2), and $(\tilde{*}_B)_x$ is the star isomorphism defined by the symplectic form $(d\theta)_x$ on S_x (see [24]). Thus, (i) follows using the results of [24].

Now suppose that $\dim M = 2m$ and that (V, ψ) is a local chart in M such that (see [39]) $\psi(V) = W \times I \times J$ with I and J open intervals of $\mathbb{R}$. Suppose also that $(q^1, \dots, q^{m-1}, p_1, \dots, p_{m-1}, t, s)$ are canonical coordinates on V such that

$$\theta = dt - \sum_{i=1}^{m-1} p_i dq^i, \quad \omega = ds. \tag{40}$$

From (4), (5), Theorem 5.2, and (40), we obtain

$$\begin{gathered} E = \frac{\partial}{\partial t}, \qquad U = \frac{\partial}{\partial s}, \\ \Lambda = \sum_i\left(\frac{\partial}{\partial q^i}\wedge\frac{\partial}{\partial p_i}\right) + \frac{\partial}{\partial t}\wedge\left(\frac{\partial}{\partial s} + \sum_i p_i\frac{\partial}{\partial p_i}\right). \end{gathered} \tag{41}$$

Therefore, if $\alpha \in \Omega^k_{BU}(M)$ then α can be viewed as a k-form on W along $W \times J$, that is,

$$\alpha\colon (x,s)\in W\times J\mapsto \alpha(x,s)\in(\Omega^k(W))_x.$$

Moreover, if we denote by d_W the exterior differential in W, then—using (37), (40), and (41)—we may deduce that

$$(d_W\alpha_s)_x = (d\alpha - ds\wedge i_{\partial/\partial s}d\alpha)_{(x,t,s)} = (d_U\alpha)_{(x,t,s)} \tag{42}$$

for $(x,t,s)\in W\times I\times J$, where α_s is the k-form in W given by

$$\alpha_s\colon y\in W\mapsto \alpha_s(y)=\alpha(y,s)\in(\Omega^k(W))_y.$$

On the other hand, from (38), (40), and (41) we have that

$$(\tilde{*}_B\alpha)_{(x,t,s)} = (\tilde{*}_{d\theta}\alpha_s)_x \quad\text{and}\quad (i(\Lambda)\alpha)_{(x,t,s)} = (i(\Lambda_{d\theta})\alpha_s)_x, \tag{43}$$

where $\tilde{*}_{d\theta}$ is the star isomorphism on W defined by the symplectic form $d\theta = \sum_i dq^i\wedge dp_i$ and where $\Lambda_{d\theta} = \sum_i \partial/\partial q^i\wedge\partial/\partial p_i$. Consequently, using (42), (43), and a result of Brylinski (see Theorem 2.2.1 of [6]), we deduce that

$$(i(\Lambda)d_U - d_U i(\Lambda))\alpha = (-1)^{k+1}(\tilde{*}_B\circ d_U\circ\tilde{*}_B)\alpha.$$

It is then sufficient to check that

$$\delta_B\alpha = (i(\Lambda)d_U - d_U i(\Lambda))\alpha. \tag{44}$$

However, from (35) we obtain that

$$(i(\Lambda)d_U - d_U i(\Lambda))\alpha = \delta_B\alpha - \omega\wedge i_U\delta_B\alpha,$$

which, from Proposition 5.3, implies (44). □

Using Theorem 5.4, Lemma 5.6, and Proposition 5.7 we conclude as follows.

COROLLARY 5.8. *Let M be a $2m$-dimensional l.c.s. manifold of the first kind, and let U be a basic infinitesimal automorphism. Then the star operator $\tilde{*}_B$ establishes an isomorphism of the cohomology group $H^k_{BU}(M)$ with the homology group $H^{\text{can}}_{(2m-2-k)U}(M)$ for $0\le k\le 2m-2$. Thus, there also are the following isomorphisms*:

$$H_k^{\text{can}}(M)\cong H_{BU}^{2m-2-k}(M)\oplus H_{BU}^{2m-1-k}(M)\quad(0\le k\le 2m-2),$$

$$H^{\text{can}}_{2m-1}(M)\cong H^0_{BU}(M),\qquad H^{\text{can}}_{2m}(M)=0.$$

In Section 4 we showed that the canonical homology groups of a compact K-contact manifold have finite dimension. Using Corollary 5.8, we will prove that the corresponding result does not hold for nonsymplectic l.c.s. manifolds. In fact, we will construct a counterexample. However, before exhibiting our counterexample, we will prove some useful general results.

PROPOSITION 5.9. *Let (M,Ω) be a regular l.c.s. manifold of the first kind with associated Jacobi structure (Λ,E). Then there exists an almost cosymplectic structure (Φ,η) on the quotient manifold $\bar{M}=M/E$ such that the induced Poisson structure of $\bar{M}$ is just the one given by (Φ,η).*

Proof. Let U be a basic infinitesimal automorphism of M, and let θ be the 1-form given by $\theta = -i_U\Omega$. From (5) and Theorem 5.2, it follows that there exists a unique 2-form Φ as well as a unique 1-form η on $\bar{M}$ such that

$$\pi^*\eta = \omega \quad \text{and} \quad \pi^*\Phi = d\theta, \tag{45}$$

where ω is the Lee 1-form of M and $\pi : M \to \bar{M}$ is the canonical projection. Thus, using (45) and Theorem 5.2, we obtain that the pair (Φ, η) is an almost cosymplectic structure on $\bar{M}$. Moreover, we also deduce that the vector field U is π-projectable, and its projection ξ is just the Reeb vector field of the almost cosymplectic manifold $(\bar{M}, \Phi, \eta)$.

Now, denote by $\flat\colon \mathfrak{X}(M) \to \Omega^1(M)$ and $\bar{\flat}\colon \mathfrak{X}(\bar{M}) \to \Omega^1(\bar{M})$ the isomorphisms defined by $\flat(X) = i_X\Omega$ and $\bar{\flat}(\bar{X}) = i_{\bar{X}}\Phi + \eta(\bar{X})\eta$, respectively.

If $\bar{\alpha}$ is a 1-form on $\bar{M}$, then by (4) we obtain that $\omega(\flat^{-1}\pi^*\bar{\alpha}) = 0$. Furthermore, $\bar{\alpha}(\xi) = 0$ if and only if $\theta(\flat^{-1}\pi^*\bar{\alpha}) = 0$. Thus, if $\bar{\alpha}(\xi) = 0$ we deduce that the vector field $\flat^{-1}(\pi^*\bar{\alpha})$ is π-projectable, and its projection is just the vector field $\bar{\flat}^{-1}\bar{\alpha}$ (see (45) and Theorem 5.2). Therefore, using (4), Theorem 5.2, and (45), we conclude that

$$\begin{aligned}
\Phi(\bar{\flat}^{-1}\bar{\alpha}, \bar{\flat}^{-1}\bar{\beta}) \circ \pi &= \Phi(\bar{\flat}^{-1}(\bar{\alpha} - \bar{\alpha}(\xi)\eta), \bar{\flat}^{-1}(\bar{\beta} - \bar{\beta}(\xi)\eta)) \circ \pi \\
&= d\theta(\flat^{-1}\pi^*\bar{\alpha} - (\bar{\alpha}(\xi) \circ \pi)E, \flat^{-1}\pi^*\bar{\beta} - (\bar{\beta}(\xi) \circ \pi)E) \\
&= d\theta(\flat^{-1}\pi^*\bar{\alpha}, \flat^{-1}\pi^*\bar{\beta}) \\
&= \Omega(\flat^{-1}\pi^*\bar{\alpha}, \flat^{-1}\pi^*\bar{\beta}) = \Lambda(\pi^*\bar{\alpha}, \pi^*\bar{\beta})
\end{aligned}$$

for $\bar{\alpha}, \bar{\beta} \in \Omega^1(\bar{M})$. This completes the proof. □

Let $(\bar{M}, \Phi, \eta)$ be an almost cosymplectic manifold with Reeb vector field ξ. We consider the submodule $\Omega^k_\xi(\bar{M})$ of $\Omega^k(\bar{M})$ given by

$$\Omega^k_\xi(\bar{M}) = \{\alpha \in \Omega^k(\bar{M}) \mid i_\xi\alpha = 0\},$$

and we define the operator $d_\xi : \Omega^k_\xi(\bar{M}) \to \Omega^{k+1}_\xi(\bar{M})$ by

$$d_\xi\alpha = d\alpha - \eta \wedge i_\xi(d\alpha) \quad \text{for all } \alpha \in \Omega^k_\xi(\bar{M}).$$

We have that $d_\xi^2 = 0$ (see [13]), so we can consider the corresponding differential complex

$$\cdots \longrightarrow \Omega^{k-1}_\xi(\bar{M}) \xrightarrow{d_\xi} \Omega^k_\xi(\bar{M}) \xrightarrow{d_\xi} \Omega^{k+1}_\xi(\bar{M}) \longrightarrow \cdots .$$

We denote by $H^*_\xi(\bar{M})$ the cohomology of this complex.

PROPOSITION 5.10. *Let $(\bar{M}, \Phi, \eta)$ be an almost cosymplectic manifold with Reeb vector field ξ. Suppose that $[\Phi] \in H^2(\bar{M}, \mathbb{Z})$. Then the following statements hold.*

(i) *There exists a principal circle bundle $\pi : M \to \bar{M}$ with connection form θ such that Φ is the curvature form of the connection; that is, $\pi^*\Phi = d\theta$.*

(ii) *M is a regular l.c.s. manifold of the first kind that induces the almost cosymplectic structure (Φ, η). Moreover, a basic infinitesimal automorphism U of M is the horizontal lift of ξ to M.*
(iii) $H^k_{BU}(M) \cong H^k_\xi(\bar{M})$ *for all k.*

Proof. (i) follows from [22].

(ii) Put $\Omega = d\theta - \pi^*\eta \wedge \theta$. A direct inspection shows that (M, Ω) is a l.c.s. manifold of the first kind with Lee 1-form $\omega = \pi^*\eta$. It is clear that a basic infinitesimal automorphism U of M is the horizontal lift ξ^H of ξ to M. Furthermore, the associated Jacobi structure on M is just (Λ, E), where E is the infinitesimal generator of the action of S^1. This implies that the corresponding quotient Poisson manifold M/E is the almost cosymplectic manifold $(\bar{M}, \Phi, \eta)$.

(iii) Using that $U = \xi^H$, we deduce that the isomorphism $\pi^*\colon \Omega^k(\bar{M}) \to \Omega^k_B(M)$ satisfies

$$\pi^* \circ i_\xi = i_U \circ \pi^*, \qquad \pi^* \circ d_\xi = d_U \circ \pi^*.$$

Therefore, $\pi^*\colon \Omega^k(\bar{M}) \to \Omega^k_B(M)$ induces an isomorphism between the cohomology groups $H^k_\xi(\bar{M})$ and $H^k_{BU}(M)$. □

REMARK 5.11. If $\bar{g}$ is a Riemannian metric on $\bar{M}$, then $g = \bar{g} + \theta \otimes \theta$ is a Riemannian metric on M and E is Killing with respect to g.

EXAMPLE 5.12. Let $\bar{N}$ be a compact symplectic manifold with symplectic 2-form $\bar{\Phi}$. Consider the following almost cosymplectic structure (Φ, η) on $\bar{M} = \bar{N} \times S^1$:

$$\Phi = pr_1^*(\bar{\Phi}), \qquad \eta = pr_2^*(\theta),$$

where pr_1 and pr_2 are the canonical projections of $\bar{M}$ onto the first and second factor (respectively) and θ is the length element of S^1. Notice that the Reeb vector field ξ of $\bar{M}$ is the vector field ξ on S^1 characterized by the condition $\theta(\xi) = 1$.

Denote by $H^*(\bar{N})$ the de Rham cohomology of $\bar{N}$, and consider the $\mathbb{R}$-bilinear mapping

$$H^k(\bar{N}) \times C^\infty(S^1, \mathbb{R}) \to H^k_\xi(\bar{M})$$

defined by

$$([\alpha], f) \mapsto [(pr_2^*(f))pr_1^*(\alpha)].$$

Because $H^k(\bar{N})$ has finite dimension, we deduce that this mapping induces an isomorphism between the real vector spaces $H^k(\bar{N}) \otimes C^\infty(S^1, \mathbb{R})$ and $H^k_\xi(\bar{M})$. In particular, $H^k_\xi(\bar{M})$ has infinite dimension.

Now, suppose that $[\bar{\Phi}] \in H^2(\bar{N}, \mathbb{Z})$. Then it is clear that $[\Phi] \in H^2(\bar{M}, \mathbb{Z})$. Let $\pi : M \to \bar{M}$ be the principal circle bundle over $\bar{M}$ defined by $[\Phi]$ (see [22]). From Proposition 5.10 and Remark 5.11, we conclude that M is a compact l.c.s. manifold of the first kind and that there exists a Riemannian metric g on M such that E is a Killing vector field with respect to g, with (Λ, E) the associated Jacobi structure on M. However, the canonical homology groups of M have infinite dimension. In fact, using Corollary 5.8, Proposition 5.10 and the foregoing results, we have that

$$H_k^{\mathrm{can}}(M) \cong (H^{2m-2-k}(\bar{N}) \otimes C^\infty(S^1,\mathbb{R})) \oplus (H^{2m-1-k}(\bar{N}) \otimes C^\infty(S^1,\mathbb{R}))$$
$$\text{and} \quad H_{2m-1}^{\mathrm{can}}(M) \cong C^\infty(S^1,\mathbb{R})$$

for $0 \le k \le 2m-2$, where $\dim \bar{N} = 2m-2$.

In Section 4 we showed that the first spectral sequence of the canonical double complex of a contact manifold degenerates at the first term. This result does not hold for arbitrary l.c.s. manifolds as the next example will demonstrate. Before that, we have the following result.

PROPOSITION 5.13. *Under the same hypotheses as in Proposition 5.10, the first spectral sequence of the canonical double complex of M degenerates at the first term if and only if the first spectral sequence of the canonical double complex of $\bar{M}$ does so also.*

EXAMPLE 5.14. Let K be the 5-dimensional connected, simply connected, nilpotent Lie group consisting of the real matrices of the form

$$K = \left\{ \begin{pmatrix} 1 & x_1 & x_2 & x_5 & x_3 & x_4 \\ 0 & 1 & 0 & 0 & 0 & -x_2 \\ 0 & 0 & 1 & 0 & -x_5 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \;\middle|\; x_i \in \mathbb{R} \right\}$$

(see [12; 13]). A basis for the left invariant 1-forms is given by $\{dx_1, dx_2, dx_3 + x_2dx_5, dx_4 + x_1dx_2, dx_5\}$. We take the compact quotient $\bar{M} = \Gamma\backslash K$, where Γ is the uniform subgroup of K consisting of those matrices with integer entries. Thus, $\bar{M}$ is a 5-dimensional compact nilmanifold, and the 1-forms $\{dx_1, dx_2, dx_3 + x_2dx_5, dx_4 + x_1dx_2, dx_5\}$ all descend to a basis of 1-forms $\{\alpha_1, \dots, \alpha_5\}$ such that

$$d\alpha_1 = d\alpha_2 = d\alpha_5 = 0,$$
$$d\alpha_3 = \alpha_2 \wedge \alpha_5, \qquad d\alpha_4 = \alpha_1 \wedge \alpha_2.$$

Define a 2-form $\Phi = \alpha_1 \wedge \alpha_4 + \alpha_2 \wedge \alpha_3$ and a 1-form $\eta = \alpha_5$ on $\bar{M}$. The pair (Φ, η) is an almost cosymplectic structure on $\bar{M}$ (see [12; 13]). Moreover, the 2-form Φ defines an integer class, say, $[\Phi] \in H^2(\bar{M}, \mathbb{Z})$, and hence there exists a principal circle bundle $\pi: M \to \bar{M}$ with connection form α_6 such that $\pi^*\Phi = d\alpha_6$. We denote by the same symbols the forms on $\bar{M}$ and their pull-backs to M. Thus, we deduce that M is a compact nilmanifold with structure equations

$$\begin{gathered} d\alpha_1 = d\alpha_2 = d\alpha_5 = 0, \\ d\alpha_3 = \alpha_2 \wedge \alpha_5, \qquad d\alpha_4 = \alpha_1 \wedge \alpha_2, \\ d\alpha_6 = \alpha_1 \wedge \alpha_4 + \alpha_2 \wedge \alpha_3. \end{gathered} \tag{46}$$

Furthermore, from Proposition 5.10 we get that (M, Ω) is a l.c.s. manifold of the first kind, with

$$\Omega = \alpha_1 \wedge \alpha_4 + \alpha_2 \wedge \alpha_3 - \alpha_5 \wedge \alpha_6.$$

Since the first spectral sequence of $\bar{M}$ does not degenerate at the first term (see [12; 13]), the same holds for M.

To end the example, if we integrate the structure equations (46) we can realize M as the nilmanifold $\bar{\Gamma}\backslash G$, where G is the group consisting of the matrices of the form

$$\begin{pmatrix} 1 & x_1 & x_2 & (x_1)^2 & (x_2)^2 & x_5 & x_3 & x_4 & x_6 \\ 0 & 1 & 0 & 2x_1 & 0 & 0 & 0 & -x_2 & -2x_4 \\ 0 & 0 & 1 & 0 & 2x_2 & 0 & -x_5 & 0 & -2x_3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & x_2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & x_5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(with $x_1, x_2, x_3, x_4, x_5, x_6 \in \mathbb{R}$) and $\bar{\Gamma}$ is the subgroup of G consisting of the matrices with integer entries.

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