Injectivity and the Pre-Schwarzian Derivative

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Many basic theorems about conformal mapping involve the pre-Schwarzian derivative f''/f'. This paper studies the inner radius of injectivity $\tau(D)$ of a simply connected domain D in the complex plane, other than the plane itself, with respect to that operator. In answer to questions posed by Gehring [9], we show that $\tau(D)$ never exceeds $1/2$ and that it equals $1/2$ for some domains other than disks and half-planes. We also show that every such domain is convex.

Let $\rho_D|dz|$ be the hyperbolic metric of D. When D is the unit disk, for example, $\rho_D(z)$ equals $2/(1-|z|^2)$, and when D is the right half-plane $\rho_D(x+iy)$ equals $1/x$. The inner radius of injectivity $\tau(D)$ is defined as the supremum of all numbers $c \geq 0$ such that every analytic function f in D satisfying the bound $|f''/f'| \leq c\rho_D$ is injective.

In the case of a disk or half-plane, τ is known to equal $1/2$. One part of the argument is due to Becker [4], who proves that $\tau \geq 1/2$ for the unit disk B. In fact, he proves a stronger result: An analytic function f in B is injective if $f'(0) \neq 0$ and

$$\left| z \cdot \frac{f''}{f'}(z) \right| \leq \frac{1}{1-|z|^2}, \quad z \in B.$$

A second ingredient is due to Becker and Pommerenke [5], who show that $\tau \leq 1/2$ for the right half-plane H. Citing an observation by Gehring, those authors conclude that equality holds in both instances. Indeed, the general formula

$$\frac{(f \circ h)''}{(f \circ h)'}(z) = \frac{h''}{h'}(z) + h'(z) \cdot \frac{f''}{f'}(h(z))$$

implies that τ is invariant under affine transformations from one domain onto another. Since any two points in H are contained in a disk that is in turn contained in H, it follows from the Schwarz lemma that $\tau(B) \leq \tau(H)$. Both quantities therefore equal $1/2$, and the conclusion extends to any disk or half-plane.

Gehring points out many parallels between $\tau(D)$ and the inner radius of injectivity $\sigma(D)$ with respect to the Schwarzian derivative $S(f) = (f''/f')' - (f''/f')^2/2$. The latter is defined as the supremum of all numbers $c \geq 0$ such that every analytic function f in D satisfying $|S(f)| \leq c\rho_D^2$ is injective. Both quantities are positive for quasidisks and zero otherwise; Martio and Sarvas [14] and Astala and Gehring [3] prove that result for τ, and Ahlfors [1] and Gehring [8] prove it

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for σ. Furthermore, both equal $1/2$ for a disk or half-plane, for σ is invariant under Möbius transformations and Nehari [15] and Hille [10] show that $\sigma(B)$ equals $1/2$. The present paper establishes yet another parallel—that τ, like σ, is bounded by $1/2$ (cf. Lehto [12, p. 127]). However, the extremal domains differ; whereas Lehtinen [11] proves that disks and half-planes are the only domains for which σ equals $1/2$, we demonstrate the following.

THEOREM 1. *If h is an analytic function in the unit disk B such that $h'(0) \neq 0$ and $|z \cdot h''(z)/h'(z)| \leq 1/2$ for all $z \in B$, then $\tau(h(B)) \geq 1/2$.*

The hypotheses imply that h is injective and that the image $h(B)$ is convex (Theorem 2.11 in [7]). On the other hand, there exist convex domains for which τ is less than $1/2$. Consider the strip $S = \{\, x + iy : |y| < \pi/2 \,\}$, for example. The function $f_t : z \mapsto e^{itz}$ is noninjective in S when $t > 0$, and $|f_t''/f_t'| = t$. Since $\rho_S(x + iy) = \sec y \geq 1$, it follows that $\tau(S) \leq t$ for all $t > 0$ and hence that $\tau(S)$ vanishes. Using the same functions in a domain $D \subseteq S$, and using the inequality $\rho_D \geq \rho_S$ obtained from the Schwarz lemma, one sees that $\tau = 0$ for a semi-infinite strip and that $\tau \leq 2/\ell$ for a rectangle of size $(\ell\pi) \times \pi$.

Proof of Theorem 1. Let f be an analytic function in the image $D = h(B)$ such that $|f''/f'| \leq (1/2)\rho_D$, and let g be the composite $f \circ h$. Since $|h'(z)|\rho_D(hz) = 2/(1 - |z|^2)$,

$$\left| z \cdot \frac{g''}{g'}(z) \right| = \left| z \cdot \frac{h''}{h'}(z) + zh'(z) \cdot \frac{f''}{f'}(hz) \right| \leq \frac{1}{2} + \frac{|z|}{1 - |z|^2} < \frac{1}{1 - |z|^2}, \quad z \in B.$$

By Becker's theorem, g is injective. Therefore f is injective, and Theorem 1 follows. □

Becker proves his theorem by a Löwner argument, deforming f to the identity through a family of mappings in which injectivity of any member implies injectivity of its predecessors. Ahlfors [2] uses a direct method to show that a locally injective analytic function f in B is injective if there exist a complex number c and a real number k such that $|c| \leq k < 1$ and

$$\left| z \cdot \frac{f''}{f'}(z) + \frac{c|z|^2}{1 - |z|^2} \right| \leq \frac{k}{1 - |z|^2}, \quad z \in B.$$

Moreover, he proves that f admits a $(1 + k)/(1 - k)$-quasiconformal extension to the Riemann sphere. One obtains Becker's result as a corollary by taking $c = 0$ and considering the functions $z \mapsto f(rz)$ for $r < 1$. Chuaqui [6] proves Becker's theorem in one step by applying a generalization of Nehari's univalence criterion, which involves the *Schwarzian* derivative, to the metric $|f'|\rho_B|dz|$ in B. The same method also yields the sharp criterion $|(f''/f')(x + iy)| \leq (1/2)/x$ for univalence in the right half-plane. Indeed, it applies to any (round) disk D in the Riemann sphere and yields the following criterion: If f is meromorphic and locally injective in D, and if $f^{-1}\{\infty\} = \{\infty\} \cap D$, then f is injective if

$$|(\rho_z/\rho) \cdot f''/f'| \leq (1/4)\rho^2, \quad \rho = \rho_D.$$

The functions $w(z)$ appearing in the proofs that follow are extremal functions for this criterion. Becker and Pommerenke's function, used in the proof of Theorem 2, is extremal for the right half-plane, and the functions $w(z)$ in the proof of Theorem 3 are extremal for the domain $|z| > 1$ in the sphere. Chuaqui's paper provided the motivation for considering such functions.

The remainder of this paper consists of proofs of the following theorems.

THEOREM 2. *If D is convex, then $\tau(D) \le 1/2$.*

THEOREM 3. *If D is not convex, then $\tau(D) < 1/2$.*

We begin with the proof of Theorem 2. Consider the function $w \mapsto w+\log(w-1)$ in $\mathbf{C} - (-\infty, 1]$, the branch of the logarithm being chosen so that $|\arg(w-1)| < \pi$. This function, introduced by Becker and Pommerenke, maps its domain conformally onto the plane less $\{\, x \pm i\pi : x \le 0 \,\}$, taking the upper and lower halves of a disk about the origin onto slit neighborhoods of $i\pi$ and $-i\pi$, respectively. Let $z \mapsto w(z)$ be the inverse function, and for $h \in \mathbf{C}$ let $F_h(z) = 1 + (w(z) - 1)^{1+h}$.

LEMMA 4. *If $x + iy \in H$, then $x|(F_h''/F_h')(x + iy)| \le 1/2 + 4|h|/3$.*

Proof. One computes that

$$\frac{F_h''}{F_h'} = \frac{w''}{w'} + w' \cdot \frac{h}{w-1} = \frac{1}{w^2} + \frac{h}{w}, \quad w = w(z).$$

If $z = x + iy$ and $w(z) = u + iv$, then

$$x = u + \operatorname{Re}\{\log(w-1)\} = u + (1/2)\log(r^2 - 2u + 1), \quad u^2 + v^2 = r^2.$$

Consider x as a function of u, where r is fixed. When $r < 2$, the maximum value is $r^2/2$. It follows that if $|w(x + iy)| = r < 2$ then

$$x\left|\frac{F_h''}{F_h'}(x+iy)\right| \le \frac{r^2}{2}\left(\frac{1}{r^2} + \frac{|h|}{r}\right) \le 1/2 + |h|.$$

When $r \ge 2$, the maximum value is $r + \log(r-1)$, which is less than $4r/3$. Hence, if $|w(x + iy)| = r \ge 2$, then

$$x\left|\frac{F_h''}{F_h'}(x+iy)\right| \le \frac{r + \log(r-1)}{r^2} + \frac{(4r/3)|h|}{r} \le 1/2 + 4|h|/3.$$

The lemma follows. □

For distinct points $z^+, z^- \in H$, let $h = h(z^+, z^-)$ be the solution of

$$(1 + h)\{\log(w(z^+) - 1) - \log(w(z^-) - 1)\} = 2\pi i.$$

Thus $F_h(z^+) = F_h(z^-)$, and h approaches zero as $z^{\pm} \to \pm i\pi$. Consider a convex domain D in the plane other than the plane itself. By means of an affine transformation that maps a chosen point $z_0 \in D$ to the positive real axis and maps a nearest point $z' \in \partial D$ to the origin, one sees that D is affinely equivalent to a convex, open set D' that omits the origin but includes a disk $\{\, z : |z - r| < r \,\}$. Since the rays through the origin that emanate from points in that disk exhaust the left half-plane, D' is contained in the right half-plane H. Inflating by a positive scalar multiplication if necessary, one can further assure that D' contains distinct points z^+, z^- such that the modulus of $h = h(z^+, z^-)$ is less than a prescribed number ε. But then

$$\tau(D) = \tau(D') \leq \sup_{D'} \frac{|F_h''/F_h'|}{\rho_{D'}} \leq \sup_{D'} \frac{|F_h''/F_h'|}{\rho_H} \leq 1/2 + 4\varepsilon/3.$$

Since ε was arbitrary, $\tau(D) \leq 1/2$. This argument proves Theorem 2.

The foregoing arguments apply to some nonconvex domains as well, but one can only conclude that $\tau \leq 1/2$. To obtain the stronger conclusion of Theorem 3, we use a family of mappings parameterized by a number $a \geq 1$, which will ultimately be chosen to match a given nonconvex domain. For now, let a be fixed, and consider the function

$$w \mapsto z = a^{-a/(a+1)}(w+a)^{a/(a+1)}(w-1)^{1/(a+1)}, \quad w \in \mathbf{C} - [-a, 1].$$

Here the arguments of $w+a$ and $w-1$ are to be chosen so as to differ by less than π; the result is then well-defined. By examining behavior on either side of the slit $[-a, 1]$, one sees that the mapping $w \mapsto z$ takes $\mathbf{C} - [-a, 1]$ conformally onto the plane less the radial segments $[0, e^{\pm i\pi/(a+1)}]$, mapping the upper and lower halves of a disk about the origin to slit neighborhoods of $e^{i\pi/(a+1)}$ and $e^{-i\pi/(a+1)}$, respectively. The mappings z_a so defined are related to the one used to prove Theorem 2 in that

$$\lim_{a\to\infty} (a+1)(z_a(w) - 1) = w + \log(w-1), \quad w \in \mathbf{C} - (-\infty, 1],$$

the convergence being uniform on compact sets.

Let $z \mapsto w(z)$ be the inverse function, and let E be the planar domain $|z| > 1$. The following lemma is the key to Theorem 3.

LEMMA 5. *If $z \in E$, then*

$$\left| z \cdot \frac{w''}{w'}(z) \right| \leq \frac{1}{|z|^2 - 1}.$$

Proof. A computation shows that

$$zw' = \frac{(w+a)(w-1)}{w}, \qquad z \cdot \frac{w''}{w'} = \frac{a}{w^2}. \tag{1}$$

Viewing z as a function of $u = \mathrm{Re}(w)$ on a circle $|w| = r$, one has

$$\frac{1}{|z|^2} \cdot \frac{d|z|^2}{du} = \frac{2(a-1)r^2 - 4au}{(r^2 + 2au + a^2)(r^2 - 2u + 1)}.$$

If $a = 1$, or if $a > 1$ and $r < 2a/(a-1)$, then $|z|^2$ attains a maximum at $u = (a-1)r^2/(2a)$, and the maximum value is $1 + r^2/a$. If $r \geq 2a/(a-1)$, then the maximum occurs at $u = r$ and the maximum value $z(r)^2$ is bounded by $1 + r^2/a$, for those two quantities are equal when $r = 2a/(a-1)$ and their ratio decreases thereafter. This analysis shows that

$$|z|^2 \le 1 + |w(z)|^2/a, \quad z \in \mathbf{C} - [0, e^{\pm i\pi/(a+1)}]. \tag{2}$$

The lemma then follows from the second equation in (1). □

The presence of the factor z in Lemma 5 will allow us to establish our next result.

THEOREM 6. *If $z^+, z^- \in E$ are sufficiently close to $e^{\pm i\pi/(a+1)}$, respectively, then there is an analytic function F in E such that $F(z^+) = F(z^-)$ and*

$$\sup_{z\in E}\left|\frac{F''}{F'}(z)\right|(|z|^2 - 1) < 1.$$

Before proving this result, we deduce Theorem 3. Let D be a nonconvex domain in the plane. If D is dense, then $\tau(D) = 0$ by Astala and Gehring's theorem. Assume, then, that D is not dense. As noted by Martin and Osgood (Lemma 3.14 in [13]), the complement of D contains a disk whose boundary intersects ∂D in at least two points. It follows that D is affinely equivalent to a domain $D' \subseteq E$ whose closure includes the points $e^{\pm i\pi/(a+1)}$ for some $a \ge 1$. By the Schwarz lemma,

$$\rho_{D'}(z) \ge \rho_E(z) = \frac{1}{|z| \cdot \log|z|} > \frac{2}{|z|^2 - 1}, \quad z \in D'.$$

Theorem 6 then provides a noninjective function F in D' such that the supremum of $|F''/F'|/\rho_{D'}$ is less than $1/2$. Therefore $\tau(D) = \tau(D') < 1/2$, and the proof of Theorem 3 is complete.

Let $\Omega = w(E)$; this is the exterior of a figure eight that crosses itself at the origin. We prove Theorem 6 by deforming the inclusion of Ω into the plane in two independent ways. The result is a family $\{f_{t,\beta}\}$ of analytic functions in Ω, parameterized by complex pairs (t, β) near $(0, 1)$; the functions $F_{t,\beta}\colon z \mapsto f_{t,\beta}(w(z))$ in E constitute a two-parameter deformation of w. When t is small and positive, there is a value $\beta(t)$ such that $F_{t,\beta(t)}$ maps E onto the exterior of another figure eight; furthermore, this mapping, and all mappings obtained from nearby parameter values, satisfy better bounds than did w. One fulfills the conditions of Theorem 6 by choosing F from among those nearby functions.

Let $m > 0$ be the infimum of $|(w + a)(w - 1)|$ in Ω. For $t \in \mathbf{C}$ such that $(a + 1)|t| < m/2$, let

$$f_t(w) = w + t \log\left(\frac{w + a}{w - 1}\right), \quad w \in \Omega,$$

the branch of the logarithm being chosen so that the second term vanishes at infinity. One then has

$$f_t'(w) = 1 - \frac{t(a+1)}{(w+a)(w-1)}, \qquad f_t''(w) = \frac{t(a+1)(2w+a-1)}{(w+a)^2(w-1)^2},$$

and the restriction on t implies that $|f_t' - 1| < 1/2$.

For $\beta \in \mathbf{C}$ such that $|\beta - 1| < 1$, define $f_{t,\beta}\colon \Omega \to \mathbf{C}$ and $F_{t,\beta}\colon E \to \mathbf{C}$ by

$$f_{t,\beta}(w) = w + \int_\infty^w \bigl(f_t'(\zeta)^\beta \cdot (1 + a/\zeta)^{(\beta-1)/(a+1)}(1 - 1/\zeta)^{a(\beta-1)/(a+1)} - 1\bigr)\,d\zeta,$$

$$F_{t,\beta}(z) = f_{t,\beta}(w(z)).$$

Here the path of integration is to lie in Ω, and in each exponential expression the logarithm of the base is that which vanishes at infinity. Because the integrand is $O(|\zeta|^{-2})$ as $\zeta \to \infty$, the integral is well defined, and $f_{t,1}$ equals f_t. As in the proof of Lemma 9 to follow, one also sees that $f_{t,\beta}(w)$ depends holomorphically upon its three arguments. Note that

$$\begin{aligned} f_{t,\beta}'(w) &= f_t'(w)^\beta \left(\frac{(w+a)^{1/(a+1)}(w-1)^{a/(a+1)}}{w}\right)^{\beta-1} \\ &= f_t'(w)^\beta \left(\frac{(w+a)(w-1)}{a^{a/(a+1)} z w}\right)^{\beta-1}. \end{aligned}$$

Therefore, by equation (1),

$$F_{t,\beta}'(z) = f_{t,\beta}'(w) \cdot w' = a^{-a(\beta-1)/(a+1)} f_t'(w)^\beta (w')^\beta = a^{-a(\beta-1)/(a+1)} \cdot F_{t,1}'(z)^\beta.$$

It follows that the pre-Schwarzian derivative of $F_{t,\beta}$ is β times that of $F_{t,1}$.

The conditions $(a+1)|t| < m/2$ and $|\beta - 1| < 1$ are implicit in all that follows.

LEMMA 7. (a) *There is a number M, independent of t and β, such that*

$$\left| z \cdot \frac{F_{t,\beta}''}{F_{t,\beta}'}(z) \right| \le |\beta| \cdot \frac{a}{|w|^2}(1 + M|t|), \quad z \in E, \ w = w(z).$$

(b) *If* $(a+1)|t| < \min\{m/2, a/6\}$, *then*

$$\left| z \cdot \frac{F_{t,\beta}''}{F_{t,\beta}'}(z) \right| \le |\beta| \cdot \frac{a}{|w|^2}\left(1 + \frac{a+1}{4a}|t|\right), \quad z \in E, \ |w| = |w(z)| < 1/6.$$

Proof. Consider the equation

$$\begin{aligned} z \cdot \frac{F_{t,\beta}''}{F_{t,\beta}'}(z) &= z \cdot \beta \frac{F_{t,1}''}{F_{t,1}'}(z) = \beta\left(z\frac{w''}{w'} + z w' \cdot \frac{f_t''}{f_t'}(w)\right) \\ &= \frac{\beta}{w^2}\left(a + \frac{t(a+1)(2w + a - 1)w}{(w+a)(w-1) - t(a+1)}\right). \end{aligned} \tag{3}$$

In view of the definition of m, the second term in the final expression is bounded by a constant times $|t|$, and assertion (a) follows.

Suppose that $(a+1)|t| < a/6$. If w is any complex number of modulus less than $1/6$, then straightforward estimates show that

$$|(2w + a - 1)w| < a/6, \qquad |(w+a)(w-1) - t(a+1)| > 2a/3.$$

Assertion (b) follows directly from equation (3) and these bounds. □

LEMMA 8. *If $|t|$ is sufficiently small and $|\beta| < 1-(a+1)|t|/(2a)$, then*

$$\sup_{z\in E}\left|\frac{F''_{t,\beta}}{F'_{t,\beta}}(z)\right|(|z|^2-1) \le 1-\frac{a+1}{4a}|t|.$$

Proof. Let $\varphi(z)$ be the ratio of $|z|^2-1$ to $|z|\cdot|w(z)|^2/a$ for $z\in E$. By equation (2), that ratio is less than unity, and it approaches zero as $z\to\infty$. It also approaches zero as z approaches any point in the unit circle other than $e^{\pm i\pi/(a+1)}$. Since neither of the latter points is in the closure of the set $S=\{z\in E : |w(z)|\ge 1/6\}$, the supremum of φ in S is a number $s<1$.

Suppose that $(a+1)|t| < \min\{m/2, a/6\}$ and $|\beta| < 1-(a+1)|t|/(2a)$. By part (a) of Lemma 7,

$$\sup_{z\in S}\left|\frac{F''_{t,\beta}}{F'_{t,\beta}}(z)\right|(|z|^2-1) = \sup_{z\in S}\varphi(z)\left|z\cdot\frac{F''_{t,\beta}}{F'_{t,\beta}}(z)\right|\cdot\frac{|w(z)|^2}{a} \le s(1+M|t|).$$

The latter, in turn, is less than $1-(a+1)|t|/(4a)$ when $|t|$ is small. If $S'=E-S$, then equation (2) and part (b) of Lemma 7 imply that

$$\sup_{z\in S'}\left|\frac{F''_{t,\beta}}{F'_{t,\beta}}(z)\right|(|z|^2-1) \le \sup_{z\in S'}\left|\frac{F''_{t,\beta}}{F'_{t,\beta}}(z)\right|\cdot\frac{|w(z)|^2}{a} \le \sup_{z\in S'}\left|z\cdot\frac{F''_{t,\beta}}{F'_{t,\beta}}(z)\right|\cdot\frac{|w(z)|^2}{a}$$
$$\le |\beta|\left(1+\frac{a+1}{4a}|t|\right) \le 1-\frac{a+1}{4a}|t|.$$

The lemma follows. □

LEMMA 9. *Suppose that $(a+1)|t_0| < m/2$ and $|\beta_0-1| < 1$. As $t\to t_0$ and $\beta\to\beta_0$, and as $w\in\Omega$ approaches the origin through either the upper or lower half-plane, $f_{t,\beta}(w)$ approaches limits $g^{\pm}(t_0,\beta_0)$, respectively. The functions $g^{\pm}$ are holomorphic, and*

$$\frac{\partial g^{\pm}}{\partial t}(0,1) = \log a \mp i\pi, \qquad \frac{\partial g^{\pm}}{\partial \beta}(0,1) = \frac{a}{a+1}(\log a \mp i\pi).$$

Proof. Because $f_{t,\beta}(\bar w) = \overline{f_{\bar t,\bar\beta}(w)}$, it is enough to prove the assertions about g^+.

The main step in the proof is to bound the integrand $I_{t,\beta}$ in the definition of $f_{t,\beta}$. We show that there are positive numbers C and C' such that, whenever $(a+1)|t| < m/2$, $|\beta-1| < 1$, and $w\in\Omega$,

$$|I_{t,\beta}(w)| \le C|w|^{-2} \qquad \text{if } |w|\ge 2a, \tag{4}$$
$$|I_{t,\beta}(w)+1| \le C'|w|^{1-\mathrm{Re}(\beta)} \quad \text{if } |w|\le 2a. \tag{5}$$

In the derivation that follows, any assertion about bounds means that the bounds are uniform: they hold for all such t, β, and w as long as w satisfies certain explicit restrictions.

Recall that $I_{t,\beta}(w)$ equals $f_t'(w)^{\beta} b(w)^{\beta-1}-1$, where

$$b(w) = (1+a/w)^{1/(a+1)}(1-1/w)^{a/(a+1)}.$$

From the formula for f_t', one sees that $|f_t'(w)-1|$ is bounded by a constant times $|w|^{-2}$ when $|w|\geq 2a$. Since $|f_t'(w)-1|$ is always less than $1/2$, Taylor's theorem then implies that the logarithm of $f_t'(w)$ is also bounded by a constant times $|w|^{-2}$ in that domain. In turn, since

$$|\beta\log(f_t'(w))|\leq 2|\log(1/2)|=\log 4,\quad w\in\Omega,$$

another application of Taylor's theorem yields a bound

$$|f_t'(w)^\beta-1|=|e^{\beta\log(f_t'(w))}-1|\leq C_1|w|^{-2},\quad |w|\geq 2a. \tag{6}$$

For the same values of w, Taylor's theorem provides bounds

$$|\log(1+a/w)-a/w|\leq C_2|w|^{-2},\qquad |\log(1-1/w)+1/w|\leq C_3|w|^{-2}.$$

It follows that $|\log b(w)|$ is no greater than a constant times $|w|^{-2}$, and hence that

$$|b(w)^{\beta-1}-1|\leq C_4|w|^{-2},\quad |w|\geq 2a.$$

Inequality (4) is a consequence of this bound and (6).

To obtain (5), one need only bound $b(w)^{\beta-1}$ by a constant times $|w|^{1-\mathrm{Re}(\beta)}$ when $|w|\leq 2a$, for $|f_t'(w)^\beta|\leq 4$. In that domain, $|w\cdot b(w)|$ is bounded above and below by positive constants. Since the arguments of $1+a/w$ and $1-1/w$ are between $\pm\pi$, so too is the argument of $b(w)$. Therefore

$$|b(w)^{\beta-1}|<e^{\pi|\mathrm{Im}(\beta)|}\cdot|b(w)|^{\mathrm{Re}(\beta)-1}\leq C_5|w|^{1-\mathrm{Re}(\beta)},\quad |w|\leq 2a,$$

and (5) follows.

The positive imaginary axis is contained in Ω. Integrating along that axis, let

$$g(t,\beta)=\int_\infty^0\big(f_t'(\zeta)^\beta(1+a/\zeta)^{(\beta-1)/(a+1)}(1-1/\zeta)^{a(\beta-1)/(a+1)}-1\big)\,d\zeta.$$

Inequalities (4) and (5) imply that the integral exists. In fact, for each positive number ε, they provide an integrable function $M(\zeta)$ that bounds the integrand of $g(t,\beta)$ whenever $\mathrm{Re}(\beta)<2-\varepsilon$. By the dominated convergence theorem, it follows that g is holomorphic and that differentiation under the integral sign is valid, for Cauchy's integral formula shows that a bound $|\varphi|\leq M$ on an analytic function in a disk $|z-z_0|\leq r$ implies a bound

$$\left|\frac{\varphi(z)-\varphi(z_0)}{z-z_0}-\varphi'(z_0)\right|\leq 2M|z-z_0|/r^2,\quad |z-z_0|<r/2.$$

Differentiating under the integral sign yields the values

$$\frac{\partial g}{\partial t}(0,1)=\log a-i\pi,\qquad \frac{\partial g}{\partial\beta}(0,1)=\frac{a}{a+1}(\log a-i\pi).$$

It remains to show that $f_{t,\beta}(w)$ converges to $g(t_0,\beta_0)$ as $t\to t_0$, $\beta\to\beta_0$, and $w\to 0$ through the upper half-plane. For $\delta\in(0,1]$, the intersection of Ω with the upper half of the circle $|w|=\delta$ is a single arc. By integrating $f_{t,\beta}'$ along a subarc and applying the bound (5), one finds that

$$|f_{t,\beta}(w)-f_{t,\beta}(i\delta)|<(\pi/2)C'\delta^{2-\mathrm{Re}(\beta)},\quad w\in\Omega,\ \mathrm{Im}(w)>0,\ |w|=\delta\in(0,1].$$

One also has

$$|f_{t,\beta}(i\delta) - g(t,\beta)| = \left| \int_0^{i\delta} f'_{t,\beta}(\zeta)\, d\zeta \right| \le \int_0^{\delta} C' t^{1-\mathrm{Re}(\beta)}\, dt = \frac{C'\delta^{2-\mathrm{Re}(\beta)}}{2-\mathrm{Re}(\beta)}.$$

Let ε be a positive number less than $2 - \mathrm{Re}(\beta_0)$. If β is near enough to β_0 that $\varepsilon < 2 - \mathrm{Re}(\beta)$, then the previous estimates imply that

$$|f_{t,\beta}(w) - g(t,\beta)| \le \left(\frac{\pi}{2} + \frac{1}{\varepsilon}\right) C'\delta^{\varepsilon}, \quad w \in \Omega,\ \mathrm{Im}(w) > 0,\ |w| = \delta \in (0,1].$$

By the continuity of g, it follows that $f_{t,\beta}(w)$ approaches $g(t_0,\beta_0)$ as $(t,\beta,w) \to (t_0,\beta_0,0)$. This completes the proof of Lemma 9. □

Since $g^+ - g^-$ vanishes at $(0,1)$ and its partial derivative with respect to β does not, the implicit function theorem provides an analytic function $t \mapsto \beta(t)$, defined for t near zero, such that $\beta(0) = 1$ and $(g^+ - g^-)(t,\beta(t)) = 0$. Using the formulas from Lemma 9, one sees that $\beta'(0) = -(a+1)/a$. It follows that $|\beta(t)| < 1 - (a+1)t/(2a)$ when t is small and positive. Fix such a value t, first reducing it if necessary so that Lemma 8 applies and so that the function $h\colon \beta \mapsto g^+(t,\beta) - g^-(t,\beta)$ is not constant. By Lemma 9, h is a locally uniform limit of the functions

$$\beta \mapsto F_{t,\beta}(z^+) - F_{t,\beta}(z^-)$$

as $z^+, z^- \in E$ approach $e^{\pm i\pi/(a+1)}$, respectively. It follows that the displayed function has a zero $\beta(z^+,z^-) \approx \beta(t)$ when $z^\pm$ are near $e^{\pm i\pi/(a+1)}$; thus the function $F = F_{t,\beta(z^+,z^-)}$ maps z^+ and z^- to the same image. When those points are sufficiently near $e^{\pm i\pi/(a+1)}$, Lemma 8 implies that

$$\sup_{z\in E} \left| \frac{F''}{F'}(z) \right| (|z|^2 - 1) \le 1 - \frac{a+1}{4a} t < 1.$$

This argument proves Theorem 6.

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