A Counterexample Related to Hartogs' Phenomenon (A Question by E. Chirka)

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We will denote by U (resp. $\bar{U}$) the open (resp. closed) unit disk in $\mathbb{C}$. Chirka [1] (see also [2]) recently proved the following remarkable result.

THEOREM (Chirka). *Let f be a continuous function on $\bar{U}$ with values in U, and let S be its graph ($S = \{(\zeta, f(\zeta)) \in \mathbb{C}^2, \zeta \in \bar{U}\}$). Then every holomorphic function defined on a connected neighborhood of the set $(\partial U \times U) \cup S$ in $\mathbb{C} \times U$ extends holomorphically to the polydisk U^2.*

It is shown by a simple example in [1] that the condition $|f| < 1$ on U (not only on ∂U) is essential.

If f is holomorphic, the result is of course classical. Here, answering a question by Chirka, we show that surprisingly (?) the theorem just stated does not extend to higher dimensions.

Our result is as follows.

PROPOSITION. *There exist continuous functions φ_1, φ_2 defined on $\bar{U}$ and satisfying $|\varphi_1|, |\varphi_2| < 1$, and there exists a domain ω in $\mathbb{C}^3$ such that:*

(i) *ω contains $\partial U \times U^2$ and ω contains the graph of (φ_1, φ_2) (i.e., $(\zeta, \varphi_1(\zeta), \varphi_2(\zeta)) \in \omega$ for every $\zeta \in \bar{U}$); but*

(ii) *there exists a holomorphic function h on ω that does not extend holomorphically to U^3.*

REMARK. It may be worthwhile pointing out that, in the construction detailed next, the following is achieved: One can find an arbitrarily small neighborhood Z of $\partial U \times U^2$ and functions φ_1 and φ_2, as in the Proposition, such that the union of Z and of the graph of (φ_1, φ_2) has a basis of pseudoconvex neighborhoods.

An explicit example would still be desirable. The first and main step in the construction of the example is as follows. Find a strictly pseudoconvex domain $\Omega \subset \mathbb{C}^3$, as well as the graph Γ of a smooth function on the unit disk, $\Gamma = \{(\zeta, \varphi_1(\zeta), \varphi_2(\zeta)) \in \mathbb{C}^3, \zeta \in \bar{U}\}$, such that the following statements hold.

(a) Ω contains $\partial U \times U^2$.

(b) $|\varphi_1|$ and $|\varphi_2| < 1$, and $|\partial\varphi_1/\partial\bar{\zeta}| + |\partial\varphi_2/\partial\bar{\zeta}| \neq 0$.

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(c) There exists an $\varepsilon \in (0, 1)$ such that for $|\zeta| > 1-\varepsilon$, $(\zeta, \varphi_1(\zeta), \varphi_2(\zeta)) \in \Omega$, and for $|\zeta| < 1-\varepsilon$, $(\zeta, \varphi_1(\zeta), \varphi_2(\zeta)) \notin \bar{\Omega}$; Γ and Ω intersect transversally (along $|\zeta| = 1-\varepsilon$), and φ_1 and φ_2 are real analytic in a neighborhood of $|\zeta| = 1-\varepsilon$.

(d) The intersection curve γ of $\partial\Omega$ and Γ (given by $\gamma(\zeta) = (\zeta, \varphi_1(\zeta), \varphi_2(\zeta))$, $|\zeta| = 1-\varepsilon$) is a complex tangential curve in $\partial\Omega$; that is, $\dot{\gamma}(\zeta) \in T(\partial\Omega) \cap iT(\partial\Omega)$.

The construction of Ω and Γ is carried out in Section 1. It provides us with a strictly pseudoconvex domain Ω and a totally real disk ($\Gamma \cap \{|\zeta| \leq 1-\varepsilon\}$) attached to its boundary along a complex tangential curve.

Proof of Proposition. It is a well-known result of Eliashberg [3] (see especially Sec. 2), that the foregoing conditions, (d) being essential, ensure that $\bar{\Omega} \cup \Gamma$ has a basis of pseudoconvex neighborhoods. However, since the precise statement is somewhat hard to find in [3], we do provide a justification of this in Section 2.

Take a connected pseudoconvex neighborhood ω of $\bar{\Omega} \cup \Gamma$, not containing the unit polydisk U^3. Then any not holomorphically extendable function h defined on ω provides the desired example. □

1. Construction of Ω and Γ

In $\mathbb{C}^2$ we will use coordinates (ζ, z_1) and in $\mathbb{C}^3$ we will use (ζ, z_1, z_2).

1.1. Construction of Domain $V_r \subset \mathbb{C}^2$, with a Disk Attached

For $r > 0$ (to be taken small) we consider the strictly pseudoconvex domain

$$U_r = \{(\zeta, z_1) \in \mathbb{C}^2,\ (|\zeta|^2 - 1)^2 + r^2|z_1|^2 < r^2\}.$$

We claim that we can choose ε and $\kappa > 0$ such that the curve

$$\gamma_0 = \{((1-\varepsilon)e^{i\theta}, \kappa e^{i\theta}),\ \theta \in [0, 2\pi]\}$$

is a complex tangential curve in the strictly pseudoconvex hypersurface that bounds U_r. Indeed we get two conditions to satisfy:

$$[(1-\varepsilon)^2 - 1]^2 + r^2\kappa^2 = r^2, \tag{1}$$

$$2[(1-\varepsilon)^2 - 1](1-\varepsilon)^2 + r^2\kappa^2 = 0. \tag{2}$$

Equation (1) expresses that γ_0 is in ∂U_r, and (2) expresses complex tangentiality (details are left to the reader; just write $(\partial[(|\zeta|^2 - 1)^2 + r^2|z_1|^2],\ d\gamma_0/d\theta) = 0$). By subtraction, (1) and (2) give us

$$[(1-\varepsilon)^2 - 1]^2 + 2[1 - (1-\varepsilon)^2](1-\varepsilon^2) = r^2,$$

which allows us to get $\varepsilon \simeq r^2/4$. Then $\kappa = \sqrt{2(1-(1-\varepsilon)^2)(1-\varepsilon)^2}/r$; of course, $\kappa < 1$. (It is easy to immediately visualize what has been done: Take the intersection with $\mathbb{R}^2$, draw a line through 0 tangent to the boundary of U_r, and then take the $\mathbb{C}$ span.)

Now choose $\delta > 0$ such that $\kappa < 1 - \delta$, and set

$$V_r = \{(\zeta, z_1) \in \mathbb{C}^2,\ (\zeta, (1-\delta)z_1) \in U_r\}.$$

This is just a rescaling in the z_1 variable. Set

$$\varphi_1(\zeta) = \frac{\kappa}{1-\delta}\frac{\zeta}{|\zeta|}\chi(\zeta),$$

where χ is a smooth $\mathcal{C}^\infty$ function ($0 \le \chi \le 1$), $\chi \equiv 1$ out of a small neighborhood of 0, and $\chi \equiv 0$ on a smaller neighborhood of 0.

SUMMARY OF PROPERTIES. Here we summarize the useful properties of V_r and φ_1 just constructed.

(A) V_r is a strictly pseudoconvex domain defined by

$$V_r = \{(\zeta, z_1), (|\zeta|^2 - 1)^2 + r_1^2|z_1|^2 < r^2\} \quad \text{with } r_1 = r(1-\delta) < r.$$

So V_r contains $\partial U \times \bar{U}$.

(B) For $|\zeta| < 1 - \varepsilon$, $(\zeta, \varphi_1(\zeta)) \notin \bar{V}_r$.

(C) For $|\zeta| > 1 - \varepsilon$, $(\zeta, \varphi_1(\zeta)) \in V_r$ $(\zeta \in \bar{U})$.

(D) Along $|\zeta| = 1 - \varepsilon$, φ_1 is real analytic, and ∂V_r and the disk $(\zeta, \varphi_1(\zeta))$ intersect transversally along a curve that is a complex tangential curve in ∂V_r.

(E) Off a small neighborhood of 0 (support of $1 - \chi$), $\partial\varphi_1/\partial\bar{\zeta} \neq 0$.

(F) $|\varphi_1| < 1$.

1.2. Construction of Ω *and* Γ

Adding one dimension will allow us to strengthen (E) in order to obtain (b).

For N large enough and α small enough (to be chosen as stated hereafter), set

$$\Omega = \Bigg\{(\zeta, z_1, z_2) \in \mathbb{C}^3; \\ [(|\zeta|^2 - 1)^2 + r_1^2|z_1|^2]^N + \left|\frac{z_2}{N}\right|^{2N} \\ + \alpha[(|\zeta|^2-1)^2 + r_1^2|z_1|^2 + |z_2|^2] \le r^{2N} + \alpha r^2\Bigg\}.$$

One first chooses N large enough so that $r_1^{2N} + 1/N^{2N} < r^{2N}$, and then $\alpha > 0$ small enough so that $\partial U \times \bar{U}^2 \subset \Omega$. The only reason for not taking $\alpha = 0$ is in order to have strict pseudoconvexity. The idea here is of course based on the approximation of the polydisk by $|z_1|^N + |z_2|^N < 1$. Observe that $\Omega \cap \{z_2 = 0\} = V_r$ and that V_r is the (ζ, z_1) projection of Ω.

Take φ_1 as before, and let φ_2 be any smooth function (i) that is 0 on a neighborhood of $|\zeta| \ge 1 - \varepsilon$, (ii) that satisfies $\partial\varphi_2/\partial\bar{\zeta} \neq 0$ when $\partial\varphi_1/\partial\bar{\zeta} = 0$, and (iii) such that $|\varphi_2| < 1$ (notice how much flexibility there is). Then (a)–(d) are satisfied.

2

Our aim here is *not* to present original results. For this, the reader is referred to Eliashberg. Our goal is simply to present a rigorous and easily accessible (to an "ordinary complex analyst") proof of the fact that $\bar{\Omega} \cup \Gamma$, in Section 1, has a basis of pseudoconvex neighborhoods.

In Section 2.2, we use the hypotheses of real analyticity made previously. Although it is not clear that this is needed, at the very least it makes the proof easier.

In $\mathbb{C}^n$, coordinates will now be denoted by $(z_1, \ldots, z_n)$.

2.1. Holomorphically Convex Sets

DEFINITION. A compact set $K \subset \mathbb{C}^n$ will be said to be *holomorphically convex* if and only if there exists λ, a continuous plurisubharmonic function defined on a neighborhood of K, such that $\lambda \leq 0$ on K and $\lambda > 0$ off K.

By classical theorems, holomorphic convexity is equivalent to the following: There exists an open neighborhood W of K such that, for every $p \in W - K$, there exists h holomorphic on W such that $|h(p)| > 1$ but $\sup_K |h| < 1$. We then say that K is holomorphically convex in W.

Of course, λ provides us with a basis of pseudoconvex neighborhoods of K.

LEMMA 1. *Let K_1 and K_2 be holomorphically convex sets, and let W be a neighborhood of $K_1 \cap K_2$. Assume that, in W, $(K_1 \cup K_2) \cap W$ can be defined by $\mu \leq 0$, where μ is a continuous strictly plurisubharmonic function defined on W. Then $K_1 \cup K_2$ is holomorphically convex.*

Proof. Let λ_1 and λ_2 be plurisubharmonic functions defined, respectively, on neighborhoods of K_1 and K_2 ($\lambda_j \leq 0$ on K_j, $\lambda_j > 0$ off K_j). We can choose a convex increasing function χ—satisfying $\chi(t) = 0$ for $t \leq 0$ and $\chi(t) > 0$ for $t > 0$—which is so flat at zero that, in the neighborhood of $K_1 \cap K_2$, we have $0 \leq \chi \circ \lambda_j \leq \max(\mu, 0)$. Let H be a compact subset of W such that $K_1 \cap K_2$ is included in the interior of H.

Finally, consider the function λ defined in the following way:

$$
\begin{aligned}
\lambda &= \chi \circ \lambda_1 && \text{near } K_1 - W, \\
\lambda &= \chi \circ \lambda_2 && \text{near } K_2 - W, \\
\lambda &= \max(\mu, 0) && \text{near } H \cap (K_1 \cup K_2), \\
\lambda &= \max(\chi \circ \lambda_1, \mu - \alpha) && \text{near } K_1 \cap (W - H),
\end{aligned}
$$

where α is a small nonnegative ($\mathcal{C}^2$) function, $\alpha > 0$ near bW, and $\alpha = 0$ on a neighborhood of H. Similarly,

$$
\lambda = \max(\chi \circ \lambda_2, \mu - \beta) \quad \text{near } K_2 \cap (W - H).
$$

The preceding definitions all agree on the overlapping regions. □

2.2. Gluing of a Totally Real Surface to a Strictly Pseudoconvex Boundary along a Complex Tangential Curve

See [3] for more details.

We consider the following situation. We are given a real analytic simple closed curve γ (to take a closed curve simplifies exposition). We consider a strictly pseudoconvex boundary Σ defined near γ, containing γ, and we assume that γ is complex tangential ($\dot{\gamma} \in T\Sigma \cap iT\Sigma$). We consider S to be a real analytic (2-dimensional) surface with boundary γ, on the pseudoconcave side of Σ, and we assume that S and Σ meet transversally along γ. Furthermore we assume that S is totally real (along γ).

LEMMA 2. *There exists a neighborhood X of γ and a continuous plurisubharmonic function λ defined on X such that*

(i) $\lambda = 0$ *on* $(S \cup \Sigma) \cap X$ (*therefore* $\lambda \leq 0$ *on the pseudoconvex side of* Σ), *and*
(ii) $\lambda > 0$ *on the pseudoconcave side of* Σ, *off* S.

Proof. This is a local problem (on a small neighborhood of γ), so by using real analyticity and the fact that S is totally real we can make a *local* holomorphic change of variable (along γ) and assume that S is the unit disk in $\mathbb{R}^2 \times \{0\} \subset \mathbb{C}^2 \times \mathbb{C}^{n-2}$. After this change of variable we have the following situation.

Let $D' = \{(x_1, x_2, 0, \ldots, 0) \in \mathbb{C}^n, x_j \in \mathbb{R}, x_1^2 + x_2^2 \leq 1\}$ and $\gamma' = \partial D'$, the unit circle in $\mathbb{R}^2 \times \{0\}$. We have Σ' a strictly pseudoconvex boundary defined near γ', $D' \cap \Sigma' = \gamma'$. The curve γ' is a complex tangential curve in Σ', $D' - \gamma'$ lies on the pseudoconcave side of Σ', and D' and Σ' meet transversally.

We look for a continuous plurisubharmonic function λ' defined near γ' such that $\lambda' = 0$ on $\Sigma' \cup D'$ (near γ') and $\lambda' > 0$ on the pseudoconcave side of Σ', off D'.

We are now going to use strict pseudoconvexity and complex tangentiality. The complexification of the real curve γ' (i.e., the complex curve defined by $z_1^2 + z_2^2 = 1$, $z_3 = \cdots = z_n = 0$) intersects Σ' only along γ', and off γ' it stays on the pseudoconcave side of Σ' (near γ'). By using homotheties and transversality, the complex curves $\{z_1^2 + z_2^2 = t, z_3 = \cdots = z_n = 0\}$ are, for $t < 1$, entirely on the pseudoconcave side of Σ' (in a fixed neighborhood of γ'). We then fix a "small" strictly pseudoconvex bounded domain Ω' in $\mathbb{C}^n$ whose boundary coincides with Σ' near γ' and such that:

(∗) for any $t \leq 1$, the curve $(z_1^2 + z_2^2 = t, z_3 = \cdots = 0)$ does not intersect $\bar{\Omega}' - \gamma'$.

Let X' be an open neighborhood of $\bar{\Omega}'$ such that $\bar{\Omega}'$ is holomorphically convex in X'. Let $L' = D' \cap \{x_1^2 + x_2^2 \geq 1 - \eta\}$, where $\eta > 0$ is chosen small enough so that $L' \subset X'$. It is enough to prove that $\bar{\Omega}' \cup L'$ is holomorphically convex in X'.

(i) *Reduction to the case $n = 2$.* If $A \subset X'$ then the hull of A is the set of $p \in X'$ such that, for every holomorphic function h in X', $|h(p)| \leq \sup_A |h|$. We have to prove that $\bar{\Omega}' \cup L'$ is its own hull.

If a point $p \in X'$ belongs to the hull of $\bar{\Omega}' \cup L'$, then it belongs either to the hull of $\bar{\Omega}'$ (i.e. $\bar{\Omega}'$) or to $\mathbb{C}^2 \times \{0\}$. A possible argument for this is by using Jensen measures. So the nontrivial part of the hull of $\bar{\Omega}' \cup L'$, if it exists, lies entirely in $\mathbb{C}^2 \times \{0\}$. By the local maximum principle and the extension of functions, this reduces the problem to the slice $\mathbb{C}^2 \times \{0\} \cap X'$ and the domain $\Omega'' = \Omega' \cap (\mathbb{C}^2 \times \{0\})$.

(ii) *Case* $n = 2$. The function $k(z_1, z_2) = z_1^2 + z_2^2$ maps L' into $[0, 1]$ and, by $(*)$, $k(\bar{\Omega}'' - \gamma')$ does not intersect $(-\infty, +1]$; also, $k < 1$ on $L' - \gamma'$.

So, in some sense, k separates $\bar{\Omega}''$ and L'. By using polynomials in k it is then straightforward (see e.g. [4, Lemma 29.21]) to show that the hull of $\bar{\Omega}'' \cup L'$ is the union of the hulls of $\bar{\Omega}''$ and L'—that is, $\bar{\Omega}'' \cup L'$. □

2.3. *Application to Section 1*

With the notation of Section 1, we must find a continuous plurisubharmonic function λ defined on a neighborhood of $\bar{\Omega} \cup \Gamma$ such that $\bar{\Omega} \cup \Gamma = \{\lambda \le 0\}$. We decompose $\bar{\Omega} \cup \Gamma$ in the following way.

For $W_1 \subset\subset W_2$ conveniently chosen neighborhoods of γ (the curve along which Γ is attached to $\bar{\Omega}$), with W_2 strictly pseudoconvex, we consider Ω_1 to be a strictly pseudoconvex domain such that $\Omega - W_2 \subset \Omega_1 \subset \Omega - W_1$. Although this is not essential, Ω_1 is introduced in order to apply Lemma 1 as stated. We set $K_1 = \bar{\Omega}_1 \cup (\Gamma - W_1)$ and $K_2 = (\bar{\Omega} \cup \Gamma) \cap \bar{W}_2$. The set K_2 is holomorphically convex because it can be defined by $\max(\lambda, \tau) \le 0$, where λ is provided by Lemma 2 and τ is a plurisubharmonic function defining W_2. The set K_1 is obviously holomorphically convex (locally K_1 is either a totally real surface or strictly pseudoconvex). On $K_1 \cap K_2$ the hypotheses of Section 2.1 are satisfied (same reason for $K_1 \cup K_2$ near $K_1 \cap K_2$). Hence 2.1 shows us the existence of λ.

REMARK. The following simple example illustrates the usefulness of the hypothesis of complex tangency in Lemma 2.

In $\mathbb{C}^2$, take the totally real surface

$$\Sigma = \left\{ \left(\frac{e^{i\theta}}{\sqrt{2}}, \frac{t}{\sqrt{2}} \right), 1 \le t \le 2 \right\}$$

attached to the unit sphere along the circle $(e^{i\theta}/\sqrt{2}, 1/\sqrt{2})$. Every function that is holomorphic in a neighborhood of the union of the unit ball and Σ extends to a holomorphic function on a neighborhood of $\{(\zeta, t), |\zeta| \le 1/\sqrt{2} \le t \le 2\}$. More local examples can be constructed as well.

References

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