Properly Immersed Singly Periodic Minimal Cylinders in $\mathbb{R}^3$

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1. Introduction

In 1867, Riemann [16] discovered a 1-parameter family of minimal surfaces foliated by circles and lines in parallel planes. Since then, many other mathematicians have characterized these examples from different points of view. Some of these characterizations can be seen in [1; 2; 4; 6; 14; 19].

Very recently, Meeks, Pérez, and Ros [9] have characterized the plane, the catenoid, the helicoid, and the Riemann examples as the only properly embedded genus-0 minimal surfaces with an infinite number of symmetries, and it is conjectured (see [8] and [18]) that this result remains valid without the hypothesis of an infinite number of symmetries.

In particular, the Riemann examples are the only properly embedded minimal tori with a finite number of planar ends in $\mathbb{R}^3/\mathcal{T}$, where $\mathcal{T}$ is the group generated by a nontrivial translation, which improves the aforementioned results.

Previously, Pérez and Ros [15] had proved that there are no properly embedded minimal surfaces of genus 1 and a finite number of planar ends in $\mathbb{R}^3/\mathcal{S}_\theta$, where $\mathcal{S}_\theta$ is a group generated by a screw motion of angle $\theta \neq 0$.

Observe that the Meeks–Pérez–Ros theorem can be stated by saying that any properly embedded minimal torus in $\mathbb{R}^3/\mathcal{T}$ with $2n$ ends is a covering of a torus in $\mathbb{R}^3/(\mathcal{T}/n)$ with two ends.

In this paper we study the same kind of questions in the more general immersed case: *Is a properly immersed minimal torus with $2n$ ends in $\mathbb{R}^3/\mathcal{T}$ a covering of a torus in $\mathbb{R}^3/(\mathcal{T}/n)$ with two ends?* López, Ritoré, and Wei [6] have found all complete minimal immersed tori in $\mathbb{R}^3/\mathcal{T}$ with two parallel planar embedded ends. This moduli space consists of a countable number of regular curves and, with the exception of Riemann examples, each one of these curves contains at least one point that provides a surface with vertical flux, and hence they are not embedded (see [7] and [15]).

We give an affirmative answer to the question just posed when dealing with properly immersed minimal tori with four planar ends. Toward that end, we prove the following theorem.

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THEOREM A. *Any properly immersed minimal torus in $\mathbb{R}^3/\mathcal{T}$ with four parallel embedded planar ends is the lift to $\mathbb{R}^3/\mathcal{T}$ of a properly immersed minimal torus in $\mathbb{R}^3/(\mathcal{T}/2)$ with two ends.*

However, for properly immersed tori with six ends the answer is negative, and it is natural to think that the same occurs for an arbitrary number of ends as well. In this sense, Theorem A is the best possible. In fact, we have proved the following.

THEOREM B. *The space of properly immersed minimal tori with six horizontal embedded planar ends and vertical flux in $\mathbb{R}^3/\mathcal{T}$ consists of a countable number of regular curves.*

Any surface in these curves is invariant under either an order-3 screw motion or an order-3 rotation around a vertical axis. Moreover, the conjugate surface exists as a singly periodic minimal surface and belongs to this space, too.

The curves in Theorem B are parameterized by certain homology classes on a fixed compact genus-1 surface $\bar{M}_0$, and the underlying complex structure of any surface in these curves is always the same: the hexagonal torus. Moreover, each curve contains a point that gives a highly symmetric surface, and the whole curve is obtained by deforming it in the way described by López and Ros in [7].

This family contains the first known immersed tori with a finite number of planar ends in $\mathbb{R}^3/\mathcal{S}_\theta$, θ nontrivial. In this case, $\theta = 2\pi/3$ and the number of ends is two. These examples prove that Pérez–Ros theorem [15] about minimal tori invariant under a screw motion does not remain valid in the more general immersed case. We have included some pictures (see Figures 1, 2, 3, and 4) that correspond to the most symmetric examples in this family.

The fundamental tool used in this work is the Weierstrass representation for minimal surfaces.

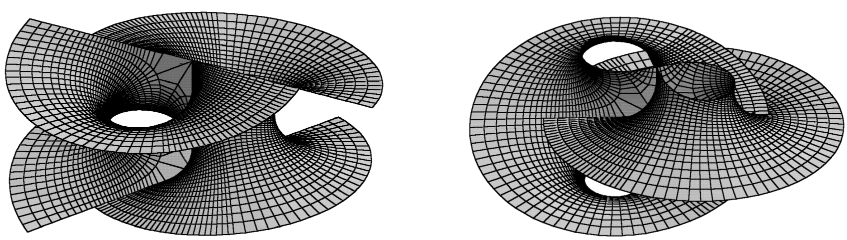

Figure 1 Two different fundamental domains with respect to the screw motion $\mathcal{S}_{2\pi/3}$ corresponding to the immersion X_{γ_1} given in Section 4. The image shows that the two planar ends intersect each other in a straight line, and they are asymptotic to the same plane at infinity. Furthermore, the surface contains another double straight line orthogonal to the ends, and the surface does not intersect itself further.

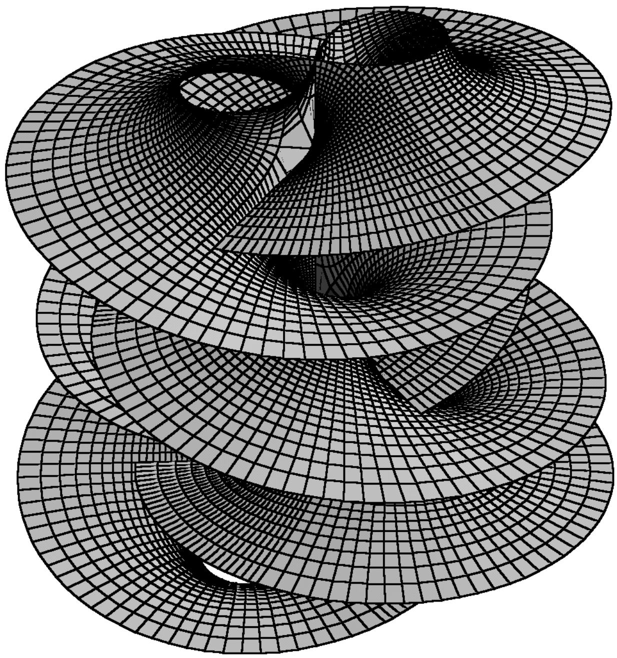

Figure 2 A translational fundamental domain of the example in Figure 1 as a surface in $\mathbb{R}^3/\mathcal{T}$, where $\mathcal{T}$ is generated by $\mathcal{S}^3_{2\pi/3}$. Note that pairs of ends come at the same height and are parallel.

This paper is laid out as follows. In Section 2 we recall some basic facts about singly periodic minimal surfaces, emphasizing the classic Weierstrass representation of minimal surfaces and the results of Osserman [12], Jorge–Meeks [5], and Meeks–Rosenberg [10]. In Section 3 we study the moduli space of properly immersed singly periodic minimal torus with four parallel embedded planar ends, and we also prove Theorem A (see Theorem 1). In Section 4, we prove that Theorem A is not true if we let the torus have six ends. To obtain this result, we construct the family of surfaces of Theorem B.

2. Preliminaries

In this work we use the Weierstrass representation for singly periodic minimal surfaces (see [10]).

Let $\tilde{X}\colon \tilde{M} \to \mathbb{R}^3$ be a proper minimal immersion of a surface $\tilde{M}$ in 3-dimensional Euclidean space, invariant under a cyclic group $\mathcal{T}$ of translations. We suppose that $\tilde{X}$ is not the covering of any surface. Using isothermal parameters, $\tilde{M}$ has a natural conformal structure, and we label $(\tilde{g}, \tilde{\eta})$ as its Weierstrass data. The

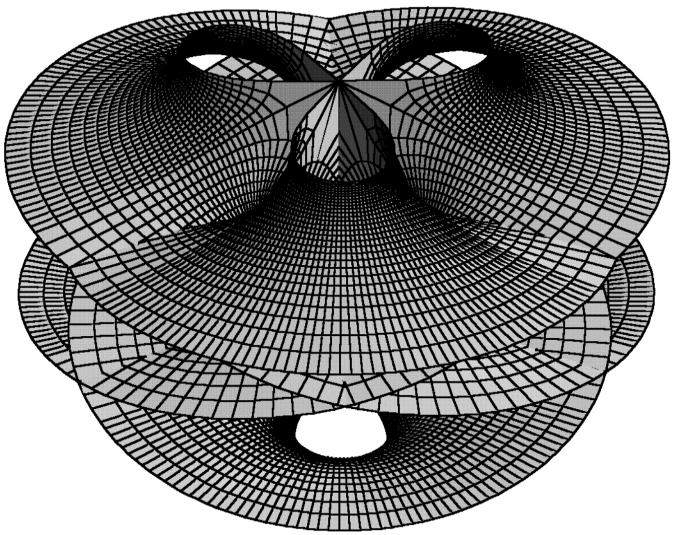

Figure 3 A fundamental domain in $\mathbb{R}^3/\mathcal{T}$ of the conjugate surface to the example given in Figure 2. The figure shows that the surface is invariant under an order-3 rotation, and there are two triads of ends asymptotic to the same plane at infinity.

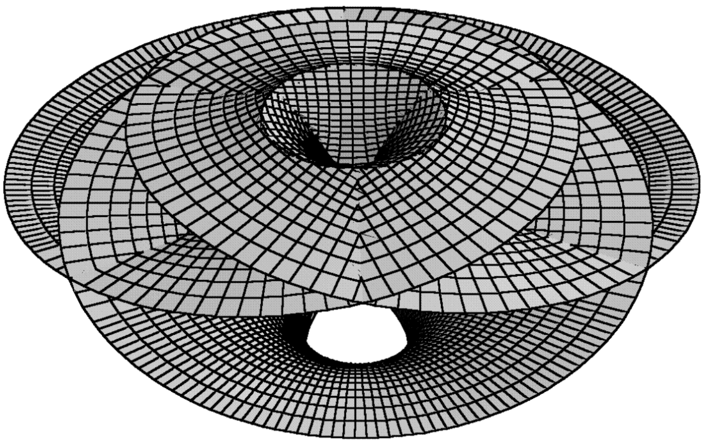

Figure 4 A half of the surface in Figure 3. The image shows that there exists a circle contained in a plane of symmetry of the surface.

Gauss map of $\tilde{X}$ is a meromorphic function on $\tilde{M}$, and $\tilde{\eta}$ is a holomorphic 1-form on $\tilde{M}$.

We can work with the quotient $M = \tilde{M}/\mathcal{T}$ because both the Gauss map and the holomorphic 1-form are translation-invariant, and so pass to the quotient. Then we have a minimal immersion $X\colon M \to \mathbb{R}^3/\mathcal{T}$ whose Weierstrass representation we label as (g, η).

Moreover, $X = \mathrm{Re}(\int(\phi_1, \phi_2, \phi_3))$, where

$$\phi_1 = \frac{1}{2}\eta(1-g^2), \quad \phi_2 = \frac{i}{2}\eta(1+g^2), \quad \phi_3 = \eta g \tag{1}$$

are holomorphic 1-forms on M satisfying

$$\sum_{j=1}^{3} |\phi_j|^2 \neq 0. \tag{2}$$

In particular, the group of periods

$$\left\{ \mathrm{Re}\left(\int_\gamma \Phi \right) : [\gamma] \in \mathcal{H}_1(M, \mathbb{Z}) \right\}$$

coincides with $\mathcal{T}$, where $\Phi = (\phi_1, \phi_2, \phi_3)$.

Conversely, if g and η are a meromorphic map and a holomorphic 1-form on the Riemann surface M and if $\Phi = (\phi_1, \phi_2, \phi_3)$ as in (1), then

$$\tilde{X} = \mathrm{Re}\left(\int \Phi \right)$$

defines in a natural way a branched conformal minimal immersion $\tilde{X}\colon \tilde{M} \to \mathbb{R}^3$, where $\tilde{M}$ is the Riemann surface associated to the multivaluated function on M: $\mathrm{Re}(\int \Phi)$. The immersion $\tilde{X}$ is unbranched if and only if (ϕ_1, ϕ_2, ϕ_3) satisfies (2). One can observe that $\tilde{X}$ induces a minimal immersion $X\colon M \to \mathbb{R}^3/\mathcal{T}$, where $\mathcal{T}$ is a subgroup generated by a nontrivial translation, if and only if the group of periods of $\tilde{X}$ coincides with $\mathcal{T}$. Moreover, X is not the *trivial* covering of any surface $X'\colon M' \to \mathbb{R}^3/\mathcal{T}$ (i.e., generically $X^{-1}(p)$ contains only one point $\forall p \in X(M)$) if and only if $\tilde{X}\colon \tilde{M} \to \mathbb{R}^3$ is not the covering of any surface.

Suppose that $X\colon M \to \mathbb{R}^3/\mathcal{T}$ is a properly immersed minimal torus with n embedded parallel planar ends, where $\mathcal{T}$ is a cyclic group generated by a nontrivial translation. Any properly embedded planar end has finite total curvature (for details see [10; 11]), and it has a well-defined limit normal vector.

Then, the surface M is conformally equivalent to a torus $\bar{M}$ punctured in n points, $M \equiv \bar{M} - \{P_1, \ldots, P_n\}$. Moreover, the Weierstrass data (g, η) extend meromorphically to $\bar{M}$, and the total curvature of M is given by

$$C(M) = \int_M K = 2\pi(\chi(M) - n) = -4n\pi.$$

Since $C(M) = -4\pi \deg(g)$, it follows that $\deg(g) = n$ (for details see [10] and [12]).

Taking into account that the ends of the surface are parallel, we can suppose, after a rotation in $\mathbb{R}^3$, that the normal vector at the ends are vertical. Because the ends of M are embedded and of planar type, the Gauss map is branched at the ends and the 1-form ϕ_3 is holomorphic (see [5], [10], and [12] for details). Since $\bar{M}$ is a torus, ϕ_3 has no zeroes (see [3]). Hence $g^{-1}(\{0, \infty\}) = \{P_1, \dots, P_n\}$ and the branching number of g at P_i is equal to 1 ($i = 1, \dots, n$); that is, all the ends are of Riemann type (see [12]). In particular, n is even and, up to relabeling, the classic divisors of g and ϕ_3 are

$$[g] = \frac{P_1^2 \cdots P_k^2}{Q_1^2 \cdots Q_k^2}, \qquad [\phi_3] = 1, \tag{3}$$

where $n = 2k$ and $Q_i = P_{k+i}$ ($i = 1, \dots, k$). See [3] for a complete exposition of this matter.

The ends of $X\colon M \to \mathbb{R}^3/\mathcal{T}$ are planar and of finite total curvature, so for any closed loop γ contained in a conformal disk of $\bar{M}$ and winding once around any end we have

$$\mathrm{Re}\left(\int_\gamma (\phi_1, \phi_2, \phi_3)\right) = 0.$$

Thus, elementary algebraic arguments lead to

$$\mathrm{Residue}(\eta, P_j) = \mathrm{Residue}(\eta g^2, Q_i) = 0$$

for all $i, j \in \{1, \dots, k\}$. In particular, the group of periods $\mathcal{T}$ of M can be written as follows:

$$\left\{ \mathrm{Re}\left(\int_\gamma \Phi\right) : [\gamma] \in \mathcal{H}_1(\bar{M}, \mathbb{Z}) \right\}.$$

Furthermore, there is a homology basis $\{\alpha, \beta\}$ of $\mathcal{H}_1(\bar{M}, \mathbb{Z})$ such that $\mathrm{Re}(\int_\alpha \Phi) = 0$ and $\mathrm{Re}(\int_\beta \Phi)$ generates $\mathcal{T}$.

As usual, we say that X has vertical flux in $\mathbb{R}^3$ if and only if $\tilde{X}$ does. This means that, for any closed curve $\tilde{\gamma}$ in $\tilde{M}$, the vector $\mathrm{Im}(\int_{\tilde{\gamma}} \tilde{\Phi})$ is vertical, where

$$\tilde{\Phi} = \left(\frac{1}{2}\tilde{\eta}(1 - \tilde{g}^2), \frac{i}{2}\tilde{\eta}(1 + \tilde{g}^2), \tilde{\eta}\tilde{g}\right).$$

In other words, $\int_\alpha \Phi = \lambda(0, 0, 1)$, $\lambda \in \mathbb{C}$.

We say that the surface $X\colon M \to \mathbb{R}^3/\mathcal{T}$ has *vertical flux* in $\mathbb{R}^3/\mathcal{T}$ if and only if

$$\mathrm{Im}\left(\int_\alpha \Phi\right) = \lambda_1(0, 0, 1), \quad \mathrm{Im}\left(\int_\beta \Phi\right) = \lambda_2(0, 0, 1), \quad \lambda_1, \lambda_2 \in \mathbb{R}.$$

These concepts have a natural physical interpretation. We refer, for instance, to [17] and [13] for a good exposition.

3. Singly Periodic Minimal Tori with Four Planar Ends

In this section we study the space of properly immersed minimal tori in $\mathbb{R}^3/\mathcal{T}$ with four parallel embedded planar ends, where $\mathcal{T}$ is a nontrivial group generated by a translation. Our main theorem asserts that any such surface can be made by attaching two copies of an immersed two-ended minimal torus in $\mathbb{R}^3/(\mathcal{T}/2)$. This last family of surfaces has been completely described by López, Ritoré, and Wei in [6].

In order to obtain these results, we need to describe both the complex structure and Weierstrass data arising out of a four-ended singly periodic torus (see the proof of Theorem 1).

Following the notation fixed in Section 2, we suppose that $X\colon M \to \mathbb{R}^3/\mathcal{T}$ is a properly immersed minimal torus with four parallel embedded planar ends, where $\mathcal{T}$ is the group generated by a nontrivial translation in $\mathbb{R}^3$. Furthermore, we assume that X is not a trivial covering of a two-ended minimal surface $X'\colon M' \to \mathbb{R}^3/\mathcal{T}$.

The main theorem of this section is as follows.

THEOREM 1. *The minimal immersion* $\tilde{X}\colon \tilde{M} \to \mathbb{R}^3$ *coincides with* $\tilde{Y}\colon \tilde{N} \to \mathbb{R}^3$, *where* $Y\colon N \to \mathbb{R}^3/(\mathcal{T}/2)$ *is a properly immersed minimal torus with two parallel embedded planar ends.*

As mentioned in the preceeding section, the 1-form ηg is holomorphic in $\bar{M}$ and, up to rigid motions, the divisor of the Gauss map is $[g] = (P_1^2 \cdot P_2^2)/(Q_1^2 \cdot Q_2^2)$, where $\{P_1, P_2, Q_1, Q_2\} \in \bar{M}$ are the ends of X.

To prove the theorem, we distinguish two different cases:

(a) there exists a meromorphic function z on $\bar{M}$ verifying $z^2 = g$;
(b) there does *not* exist any meromorphic function z on $\bar{M}$ satisfying $z^2 = g$.

Theorem 1 is a consequence of the following two lemmas.

LEMMA 1. *If* $g = z^2$, *where* z *is a degree*-2 *meromorphic function on* $\bar{M}$, *then there exists a holomorphic involution* $I\colon \bar{M} \to \bar{M}$ *without fixed points leaving* M *invariant and satisfying* $I^\star(\phi_j) = \phi_j$ $(j = 1, 2, 3)$.

Proof. The divisor of z is given by $[z] = (P_1 \cdot P_2)/(Q_1 \cdot Q_2)$. Furthermore, using the classic Riemann–Hurwitz relation (see [3, p. 102]), the function z has four ramification points—in this case, all with branching number 1. Let $\{R_1, R_2, R_3, R_4\}$ denote the set of ramification points of z, and label $z(R_i) = r_i \in \mathbb{C} - \{0\}$ $(i = 1, 2, 3, 4,\ r_i \neq r_j,\ i \neq j)$. Up to a homothetical change of variables, we can suppose that $r_1 \cdot r_2 \cdot r_3 \cdot r_4 = 1$. Classical theory of compact Riemann surfaces (see [3]) implies the existence of a meromorphic function w on $\bar{M}$ such that

$$\begin{aligned} w(P)^2 &= (z(P) - r_1) \cdot (z(P) - r_2) \cdot (z(P) - r_3) \cdot (z(P) - r_4) \\ &= z(P)^4 + a_3 z(P)^3 + a_2 z(P)^2 + a_1 z(P) + 1 \quad \forall P \in \bar{M}, \end{aligned}$$

and we can use this equation for representing our surface as follows:

$$\bar{M} \equiv \{\, (z, w) \in (\mathbb{C} \cup \{\infty\})^2 \mid w^2 = z^4 + a_3 z^3 + a_2 z^2 + a_1 z + 1 \}.$$

Up to this biholomorphism,

$$P_1 = (0, 1), \quad P_2 = (0, -1), \quad \{Q_1, Q_2\} = z^{-1}(\infty),$$

and the Weierstrass data of X are (recall that we have done a homothetical change of variables in the function z):

$$g = Az^2, \quad \eta g = B\frac{dz}{w}, \quad A, B \in \mathbb{C} - \{0\}.$$

Taking into account that Residue$(\eta, P_1) =$ Residue$(\eta g^2, Q_1) = 0$ (see Section 2), an easy computation gives $a_1 = a_3 = 0$.

Defining $I(z, w) = (-z, -w)$, the lemma holds. □

LEMMA 2. *If g does not have a well-defined square root on $\bar{M}$, then there exists a holomorphic involution $I : \bar{M} \to \bar{M}$ without fixed points leaving M invariant and satisfying $I^\star(\phi_j) = \phi_j$ $(j = 1, 2, 3)$.*

Proof. The Riemann–Roch theorem applied to the divisor $P_1/(Q_1 \cdot Q_2)$ (see [3, pp. 73–77]) and our hypothesis imply the existence of a degree-2 meromorphic function h on $\bar{M}$ satisfying

$$[h] = \frac{P_1 \cdot Z}{Q_1 \cdot Q_2},$$

where $Z \in \bar{M} - \{P_2, Q_1, Q_2\}$. Defining $y = g/h^2$, we have

$$[y] = \frac{P_2^2}{Z^2}.$$

Using the classic Riemann–Hurwitz relation (see [3]), the function y has four ramification points, all of them with branching number 1. The set of ramification points of y is given by $\{P_2, Z, R_1, R_2\}$, where $R_1, R_2 \in \bar{M} - \{P_2, Z\}$; as before, there exists a meromorphic function w on $\bar{M}$ such that

$$w(P)^2 = y(P) \cdot (y(P) - y(R_1)) \cdot (y(P) - y(R_2)) \quad \forall P \in \bar{M}.$$

We can use this equation for representing our surface as follows:

$$\bar{M} \equiv \{\, (y, w) \in (\mathbb{C} \cup \{\infty\})^2 \mid w^2 = y(y - y(R_1))(y - y(R_2)) \,\}.$$

Up to this biholomorphism,

$$P_2 = (0, 0), \quad Z = (\infty, \infty), \quad R_1 = (y(R_1), 0), \quad R_2 = (y(R_2), 0).$$

We shall prove that P_1 is a branch point of y, that is, $P_1 \in \{Z, R_1, R_2\}$.

Suppose that P_1 is not a branch point of y. Up to a homothetical change of variables, we assume that $y(P_1) = 1$. To determine the Weierstrass data of X, we compute the function h. A direct application of the Riemann–Roch theorem gives that the complex vector space of meromorphic functions on $\bar{M}$ having at most single poles at the points P_1 and Z, and no other singularities, has dimension 2. Furthermore, this space is generated by $\{1, (w + w(P_1))/(y - 1)\}$.

Since $1/h$ belongs to this space, there exist two constants $A, \lambda \in \mathbb{C}$ $(A \neq 0)$ such that

$$\frac{1}{h} = \frac{1}{\sqrt{A}}\left(\frac{w(P_1) + w}{y - 1} + \lambda\right).$$

Hence

$$g = A\frac{(y-1)^2 y}{(w(P_1)+w+\lambda(y-1))^2}, \qquad \eta g = B\frac{dy}{w},$$

where $A, B \in \mathbb{C} - \{0\}$. Taking into account that $\mathrm{Residue}(\eta, P_1) = 0$ (see Section 2), it is not hard to deduce that $\lambda = w(P_1)$. Therefore,

$$\frac{1}{h} = \frac{1}{\sqrt{A}}\frac{w(P_1)y+w}{y-1}$$

and thus P_2 is a pole of h, that is, $P_2 \in \{Q_1, Q_2\}$. This is obviously a contradiction.

Hence we can assume that $P_1 \in \{Z, R_1, R_2\}$. We distinguish two cases as follows.

(i) $P_1 \in \{R_1, R_2\}$.
(ii) $P_1 = Z$.

Consider case (i). Up to relabeling, we can suppose $P_1 = R_1$. By using, as before, the Riemann–Roch theorem, we obtain

$$\frac{1}{h} = \frac{1}{\sqrt{A}}\left(\frac{w}{y-y(P_1)} + \lambda\right),$$

where $A, \lambda \in \mathbb{C}$ $(A \neq 0)$. Therefore,

$$g = A\frac{(y-y(P_1))^2 y}{(w+\lambda(y-y(P_1))^2}, \qquad \eta g = B\frac{dy}{w},$$

where $A, B \in \mathbb{C} - \{0\}$. Since $\mathrm{Residue}(\eta, P_1) = 0$, an easy computation yields $\lambda = 0$. Thus,

$$\frac{1}{h} = \frac{1}{\sqrt{A}}\left(\frac{w}{y-y(P_1)}\right)$$

and so P_2 is a pole of h, which is absurd. This case is impossible.

Now consider case (ii). Because P_1 is a double zero of h and a double pole of y, we deduce that $1/h = (1/\sqrt{A})(y-\lambda)$, $\lambda \in \mathbb{C} - \{0, y(R_1), y(R_2)\}$, $A \in \mathbb{C} - \{0\}$. Up to a homothetical change of variables we can suppose $\lambda = 1$. In particular, $\{Q_1, Q_2\} = y^{-1}(1)$. Then, it is clear that

$$g = A\frac{y}{(y-1)^2}, \qquad \eta g = B\frac{dy}{w},$$

where $A, B \in \mathbb{C} - \{0\}$. Writing

$$(y - y(R_1)) \cdot (y - y(R_2)) = (y-1)^2 + a_1(y-1) + a_0,$$

we have, up to the sign, $\mathrm{Residue}(\eta g^2, Q_1) = AB(a_0 - a_1)/2\sqrt{a_0^3}$. Since this number vanishes (see Section 2), we deduce that $a_0 = a_1$.

Defining $I(y, w) = (1/y, -w/y^2)$, the lemma holds. $\square$

Proof of Theorem 1. From Lemmas 1 and 2, there exists a holomorphic involution without fixed points $I : \bar{M} \to \bar{M}$ leaving M invariant and satisfying $I^\star(\Phi) = \Phi$. If we label N as the Riemann surface $M/\langle I\rangle$, then the natural projection $\pi : M \to N = M/\langle I\rangle$ is a two-sheeted unbranched covering. On the other hand, for any $P \in M$,

$$X(I(P)) - X(P) = \mathrm{Re}\left(\int_P^{I(P)} \Phi\right) + \mathcal{T} = \mathrm{Re}\left(\int_{I(P)}^{P} I^\star(\Phi)\right) + \mathcal{T}$$
$$= -\mathrm{Re}\left(\int_P^{I(P)} \Phi\right) + \mathcal{T} = X(P) - X(I(P))$$

and so

$$2(X(I(P)) - X(P)) = \vec{0} + \mathcal{T}.$$

Hence, either $\mathrm{Re}(\int_P^{I(P)} \Phi) \in \mathcal{T}$ or $\mathrm{Re}(\int_P^{I(P)} \Phi) \in \mathcal{T}/2$. In the first case, $X\colon M \to \mathbb{R}^3/\mathcal{T}$ is a trivial covering of the surface $Y\colon N \to \mathbb{R}^3/\mathcal{T}$ defined by $X = Y \circ \pi$, which is absurd. In the second case, we define $Y\colon N \to \mathbb{R}^3/(\mathcal{T}/2)$ as

$$Y \circ \pi = \pi_0 \circ X,$$

where $\pi_0\colon \mathbb{R}^3/\mathcal{T} \to \mathbb{R}^3/(\mathcal{T}/2)$ is the natural projection.

It is clear now that Y is a properly immersed minimal tori with two parallel embedded planar ends and $\tilde{X} = \tilde{Y}$, which concludes the proof. □

Note that Theorem 1, together with [6], gives (in the embedded case) uniqueness for the Riemann examples only for the case of four ends. See [9] for a more general setting.

4. A Family of Singly Periodic Minimal Tori with Six Parallel Embedded Planar Ends

In the preceeding section we proved that the moduli space of singly periodic properly immersed tori in $\mathbb{R}^3$ with four parallel embedded planar ends can be identified with the corresponding space of tori with two ends. However, this result does not remain valid for tori with six ends, and it is natural to think that the same occurs in the general case of $2k$ ends, $k \geq 3$.

In this section we present a family of properly immersed minimal tori in $\mathbb{R}^3/\mathcal{T}$ with six horizontal embedded planar ends, where $\mathcal{T}$ is the group generated by a vertical translation. These examples have some interesting properties. Among them we emphasize that they have vertical flux in $\mathbb{R}^3/\mathcal{T}$, group $\mathcal{T}$ is generated by a vertical vector, the conjugate surface is well-defined as a singly periodic minimal surface, and they are invariant under a nontrivial group of symmetries. Of course, these surfaces are not embedded (see [7; 9; 15]). In fact, some of the planar ends contained in the surfaces are placed at the same height in $\mathbb{R}^3$. Finally, we prove that these surfaces can be characterized as the unique properly immersed minimal tori in $\mathbb{R}^3/\mathcal{T}$ with six horizontal embedded planar ends with vertical group of periods and vertical flux in $\mathbb{R}^3/\mathcal{T}$.

4.1. The New Family of Examples

Let $\bar{M}_0$ be the genus-1 compact Riemann surface:

$$\bar{M}_0 = \{\,(z, v) \in (\mathbb{C} \cup \{\infty\})^2 \mid v^3 = z^3 + 1\}.$$

Label $\{P_1, P_2, P_3\} = z^{-1}(\infty)$ and $Q_i = (0, \theta^{i-1})$ $(i = 1, 2, 3)$, where $\theta = e^{2\pi i/3}$.

Consider on $M_0 = \bar{M}_0 - \{P_1, P_2, P_3, Q_1, Q_2, Q_3\}$ the following Weierstrass data:

$$g = \frac{1}{z^2}, \quad \eta_B \cdot g = B\frac{dz}{v^2}, \quad B \in \mathbb{C},\ |B| = 1,\ \arg(B) \in [0, \pi[; \tag{4}$$

define $\Phi_B = (\frac{1}{2}\eta_B(1-g^2), \frac{i}{2}\eta_B(1+g^2), \eta_B g)$ as in (1).

Let α be a closed curve in $\bar{M}_0$. Suppose there exists another closed curve β in $\bar{M}_0$ such that $\{\alpha, \beta\}$ is a homology basis of $\mathcal{H}_1(\bar{M}_0, \mathbb{Z})$.

The main achievement of this section is the following theorem.

THEOREM 2. *There exists a unique* $B(\alpha) \in \mathbb{C}$, $|B(\alpha)| = 1$, $\arg(B(\alpha)) \in [0, \pi[$, *such that the meromorphic data* (4) *yield a minimal immersion*

$$X_\alpha \colon M_0 \to \mathbb{R}^3/\mathcal{T}_\alpha, \qquad X_\alpha(P) = \mathrm{Re}\left(\int^P \Phi_{B(\alpha)}\right) + \mathcal{T}_\alpha$$

satisfying

$$\mathrm{Re}\left(\int_\alpha \Phi_{B(\alpha)}\right) = 0, \qquad \mathcal{T}_\alpha = \left\langle \mathrm{Re}\left(\int_\beta \Phi_{B(\alpha)}\right)\right\rangle,$$

where β *is any closed curve in* $\bar{M}_0$ *such that* $\{\alpha, \beta\}$ *is a homology basis of* $\mathcal{H}_1(\bar{M}_0, \mathbb{Z})$.

In order to prove this theorem and obtain some geometrical consequences, we need to introduce some notation and make some topological comments.

First, define the following conformal mappings:

$$J, R, S, T \colon \bar{M}_0 \to \bar{M}_0,$$
$$J(z, v) = (\theta z, \theta^2 v), \qquad R(z, v) = \left(\frac{1}{z}, \frac{v}{z}\right),$$
$$S(z, v) = (\bar{z}, \bar{v}), \qquad T(z, v) = (z, \theta v);$$

as before, $\theta = e^{2\pi i/3}$.

It is clear that $J^3 = T^3 = R^2 = S^2 = 1_{\bar{M}_0}$. Furthermore,

$$\begin{aligned} R \circ S &= S \circ R, & R \circ T &= T \circ R,\\ R \circ J &= J^{-1} \circ R, & S \circ J &= J^{-1} \circ S,\\ T \circ J &= J \circ T, & T \circ S &= S \circ T^{-1}. \end{aligned} \tag{5}$$

An easy computation yields

$$\begin{aligned} T(Q_1) &= Q_2, & T(Q_2) &= Q_3,\\ S(Q_1) &= Q_1, & S(Q_2) &= Q_3,\\ J(Q_1) &= Q_3, & J(Q_2) &= Q_1. \end{aligned} \tag{6}$$

Note that, up to relabeling, $P_i = R(Q_i)$ $(i = 1, 2, 3)$ and hence, from (5), we have:

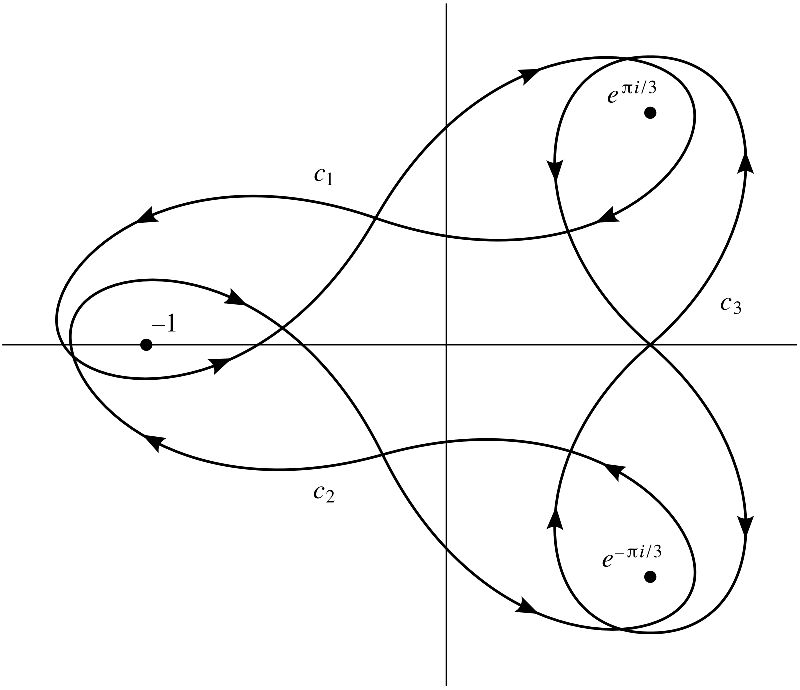

Figure 5 The curves c_1, c_2, and c_3.

$$\begin{aligned} T(P_1) &= P_2, & T(P_2) &= P_3, \\ S(P_1) &= P_1, & S(P_2) &= P_3, \\ J(P_1) &= P_2, & J(P_2) &= P_3. \end{aligned} \tag{7}$$

At this point we can describe a homology basis of $\bar{M}_0$. Let $c_i(t)$ $(i = 1, 2, 3)$ be the oriented closed curves in the z-plane illustrated in Figure 5. We assume that $c_i(0) \in \mathbb{R}$ and $c_i(0) < -1$ for $i = 1, 2$ and that $c_3(0) = re^{\pi i/3}$ for $r > 1$. Let $\gamma_i(t)$ $(i = 1, 2, 3)$ be the unique lift of $c_i(t)$ to $\bar{M}_0$ satisfying $\arg(v(\gamma_i(0))) = \pi/3$ $(i = 1, 2, 3)$.

In the following we identify γ and its homology class $[\gamma]$ for any closed curve γ lying in $\bar{M}_0$. Elementary topological arguments imply that the set

$$\{ (T^j)_\star(\gamma_i) \mid i = 1, 2, 3,\ j = 0, 1 \}$$

generates the group $\mathcal{H}_1(\bar{M}_0, \mathbb{Z})$.

The following lemma will allow us to describe a homology basis.

LEMMA 3. *The following formulae hold*:

$$\int_{\gamma_i} \frac{dz}{v^2} = -3\rho\theta^{i-1} \quad (i = 1, 2, 3),$$

where

$$\rho = \int_0^1 \frac{dt}{(1-t^3)^{2/3}} \in \mathbb{R}_+.$$

Proof. Given $a, b \in \mathbb{C}$, we label $[a, b]$ as the oriented segment with initial point a and final point b. An elementary analytic continuation argument gives:

$$\begin{aligned}\int_{\gamma_1} \frac{dz}{v^2} &= (e^{-4\pi i/3} - 1)\int_{[-1,0]} \frac{dz}{|v|^2} + (e^{-4\pi i/3} - 1)\int_{[0,e^{\pi i/3}]} \frac{dz}{|v|^2}\\ &= (e^{-4\pi i/3} - 1)\int_0^1 \frac{dt}{(1-t^3)^{2/3}}\\ &\quad + (e^{-4\pi i/3} - 1)e^{\pi i/3}\int_0^1 \frac{dt}{(1-t^3)^{2/3}} = -3\rho,\end{aligned}$$

with ρ as defined in the lemma.

Applying the same arguments to the closed curves γ_2 and γ_3, we obtain the other equalities. □

Let α_1, α_2 be two cycles in $\mathcal{H}_1(\bar{M}_0, \mathbb{Z})$. It is well known (see e.g. [3]) that the equality $\alpha_1 = \alpha_2$ holds if and only if $\int_{\alpha_1} \omega = \int_{\alpha_2} \omega$ for any holomorphic 1-form ω on $\bar{M}_0$. In our case, $\bar{M}_0$ is a torus and so the complex vector space of holomorphic 1-forms has dimension 1. Furthermore, the 1-form $\{dz/v^2\}$ is a basis of this space.This last remark, together with Lemma 3 and the identity $T^\star(dz/v^2) = \theta(dz/v^2)$, imply that

$$T_\star(\gamma_1) = \gamma_2, \quad T_\star(\gamma_2) = \gamma_3, \quad \gamma_1 + \gamma_2 + \gamma_3 = 0. \tag{8}$$

Hence, any of the sets $\{\gamma_i, \gamma_j\}$ $(i, j \in \{1, 2, 3\},\ i < j)$ is a homology basis of $\bar{M}_0$.

In what follows, we fix the basis of $\mathcal{H}_1(\bar{M}_0, \mathbb{Z})$ as $\{\gamma_1, \gamma_2\}$ and write

$$\alpha = m\gamma_1 + n\gamma_2, \quad m, n \in \mathbb{Z}, \quad \gcd(m, n) = 1.$$

We are now ready to prove Theorem 2.

Proof of Theorem 2. To solve the period problem, it suffices to find a constant $B(\alpha) \in \mathbb{C}$, $|B(\alpha)| = 1$, $\arg(B(\alpha)) \in [0, \pi[$, satisfying:

$$\mathrm{Re}\left(\int_\alpha \Phi_{B(\alpha)}\right) = 0.$$

If this equation holds, then the group of periods

$$\mathcal{T}_\alpha = \left\{ \mathrm{Re}\left(\int_\gamma \Phi_{B(\alpha)}\right) \,\middle|\, \gamma \in \mathcal{H}_1(\bar{M}_0, \mathbb{Z}) \right\}$$

would be cyclic and generated by the vector $\mathrm{Re}(\int_\beta \Phi_{B(\alpha)})$, where β is any closed curve of $\bar{M}_0$ such that $\{\alpha, \beta\}$ is a basis of $\mathcal{H}_1(\bar{M}_0, \mathbb{Z})$. These facts would conclude the proof.

In order to obtain this, observe first that, for any B, the 1-forms η_B and $\eta_B g^2$ are exact. In fact,

$$\eta_B = B\frac{z^2\,dz}{v^2} = d(Bv), \qquad \eta_B g^2 = B\frac{dz}{z^2v^2} = d\left(-B\frac{v}{z}\right). \tag{9}$$

It follows from Lemma 3 that

$$\begin{aligned}\mathrm{Re}\left(\int_\alpha \Phi_B\right) &= \left(0,0,\mathrm{Re}\left(\int_\alpha \eta_B g\right)\right) = \left(0,0,\mathrm{Re}\left(\int_{m\gamma_1+n\gamma_2} \eta_B g\right)\right)\\ &= m\left(0,0,\mathrm{Re}\left(B\int_{\gamma_1}\frac{dz}{v^2}\right)\right) + n\left(0,0,\mathrm{Re}\left(B\int_{\gamma_2}\frac{dz}{v^2}\right)\right)\\ &= (0,0,-3\rho\,\mathrm{Re}((m+n\theta)B)).\end{aligned}$$

Choosing the unique $B(\alpha)$ such that

$$B(\alpha)^2 = -\left(\frac{|m+n\theta|}{m+n\theta}\right)^2, \qquad \arg(B(\alpha)) \in [0,\pi[, \tag{10}$$

the theorem holds. □

In what follows we label $\eta_\alpha = \eta_{B(\alpha)}$ and $\Phi_\alpha = \Phi_{B(\alpha)}$.

REMARK 1. From (9), the group of periods of $X_\alpha\colon M_0 \to \mathbb{R}^3/\mathcal{T}_\alpha$ is vertical, and this surface has vertical flux in $\mathbb{R}^3/\mathcal{T}$. Thus, we can define a real 1-parameter deformation of X_α as follows:

$$\Sigma_\alpha = \{\, X_\alpha^\lambda\colon M_0 \to \mathbb{R}^3/\mathcal{T}_\alpha \mid \lambda \in]0,+\infty[\,\}$$

where X_α^λ is the immersion associated to the Weierstrass data $(M_0,\ g^\lambda = \lambda g,\ \eta_\alpha^\lambda = (1/\lambda)\eta_\alpha)$, $\lambda \in]0,+\infty[$. This deformation was first introduced by López and Ros [7], and it played a fundamental role in [6] as well.

As usual, we denote $\Phi_\alpha^\lambda = (\frac{1}{2}\eta_\alpha^\lambda(1-(g^\lambda)^2),\ \frac{i}{2}\eta_\alpha^\lambda(1+(g^\lambda)^2),\ \eta_\alpha^\lambda g^\lambda)$.

The next two remarks follow as a consequence of (10).

REMARK 2. The conjugate surface $(X_\alpha^\lambda)^\star$ associated to the meromorphic data $(M_0, g^\lambda, i\eta_\alpha^\lambda)$ is well-defined on M_0 as a singly periodic minimal surface and coincides, up to rigid motions, with the immersion $X_{\alpha^\star}^\lambda\colon M_0 \to \mathbb{R}^3/\mathcal{T}_{\alpha^\star}$, where $\alpha^\star = m^\star\gamma_1 + n^\star\gamma_2$, $m^\star = (m-2n)/d$, $n^\star = (2m-n)/d$, and $d = \gcd(m-2n, 2m-n)$.

REMARK 3. Up to rigid motions, the immersions $X_\alpha^\lambda \circ T$ and $X_\alpha^\lambda \circ T^2$ coincide with $X_{(T^2)_\star(\alpha)}^\lambda$ and $X_{T_\star(\alpha)}^\lambda$, respectively. Moreover, a precise expression for the curves $(T^2)_\star(\alpha)$ and $T_\star(\alpha)$ can be derived from (8). Hence these two immersions can be identified with X_α^λ.

Note also that $B((T^2)_\star(\alpha))^2 = \theta^2 B(\alpha)^2$ and $B(T_\star(\alpha))^2 = \theta B(\alpha)^2$ (see (10)). Then, if we use this normalization, we could suppose without loss of generality that $\arg(B(\alpha)) \in [0,\pi/3[$.

We shall now study the symmetry of the surfaces in Σ_α.

First, denote $M_i = (-\theta^{i-1}, 0)$ $(i = 1, 2, 3)$, where $\theta = e^{2\pi i/3}$. Observe that $\{M_1, M_2, M_3\} \subset M_0$ is the set of ramification points of the meromorphic function z on $\bar{M}_0$. Furthermore, an easy computation gives:

$$\begin{aligned} J(M_1) &= M_2, & J(M_2) &= M_3, \\ S(M_1) &= M_1, & S(M_2) &= M_3, \\ R(M_1) &= M_1, & R(M_2) &= M_3. \end{aligned} \tag{11}$$

Up to translations, we assume that $X_\alpha^\lambda(M_1) = 0 + \mathcal{T}_\alpha$, that is,

$$X_\alpha^\lambda(P) = \mathrm{Re}\left(\int_{M_1}^{P} \Phi_\alpha^\lambda\right) + \mathcal{T}_\alpha \quad \forall P \in M_0.$$

Using (9) and $v(M_1) = 0$, we deduce that

$$\begin{aligned} &X_\alpha^\lambda(P) \\ &= \mathrm{Re}\left(B(\alpha)\left(\frac{1}{2}\left(\frac{1}{\lambda}v(P) + \lambda\frac{v(P)}{z(P)}\right), \frac{i}{2}\left(\frac{1}{\lambda}v(P) - \lambda\frac{v(P)}{z(P)}\right), \int_{M_1}^{P}\frac{dz}{v^2}\right)\right) + \mathcal{T}_\alpha. \end{aligned}$$

Proposition 1. *If* $m + n \equiv 0 \pmod 3$, *then the automorphism* J *induces on* $X_\alpha^\lambda(M_0)$ *a rotation around the* x_3*-axis by an angle of* $2\pi/3$.

If $m+n \not\equiv 0 \pmod 3$, *then* J *induces a screw motion of order* 3 *that is the product of a translation along the* x_3*-axis and a rotation about this axis by an angle of* $2\pi/3$.

Proof. Note that

$$J^\star(g^\lambda) = \theta g^\lambda, \qquad J^\star(\eta_\alpha^\lambda g^\lambda) = \eta_\alpha^\lambda g^\lambda,$$

where $\theta = e^{2\pi i/3}$. Then

$$J^\star({}^t\Phi_\alpha^\lambda) = \mathcal{J} \cdot {}^t\Phi_\alpha^\lambda, \tag{12}$$

where $\mathcal{J}$ is the matrix

$$\mathcal{J} = \begin{pmatrix} \cos\left(\frac{2\pi}{3}\right) & -\sin\left(\frac{2\pi}{3}\right) & 0 \\ \sin\left(\frac{2\pi}{3}\right) & \cos\left(\frac{2\pi}{3}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Taking (11) and (12) into account, we obtain

$${}^tX_\alpha^\lambda(J(P)) = {}^t\vec{v}_0 + \mathcal{J} \cdot {}^tX_\alpha^\lambda(P) \quad \forall P \in M_0,$$

where $\vec{v}_0 = \mathrm{Re}(\int_{M_1}^{M_2} \Phi)$. Since $J^3 = 1_{M_0}$, it follows that $3\vec{v}_0 \in \mathcal{T}_\alpha$.

On the other hand, if we write $\beta = p\gamma_1 + q\gamma_2$ and $qm - pn = 1$ $(p, q \in \mathbb{Z})$ then, from Lemma 3, we have

$$\mathcal{T}_\alpha = \langle (0, 0, -3\rho\,\mathrm{Re}(B(\alpha)(p + q\theta))) \rangle$$

and so, from (10),

$$\mathcal{T}_\alpha = \left\langle\left(0, 0, \frac{3\sqrt{3}\rho}{2|m + n\theta|}\right)\right\rangle.$$

To compute $\vec{v}_0$, let c be the curve in the z-plane given by $c(t) = e^{ti}$ ($t \in [\pi, 5\pi/3]$) and take γ as the unique lift of the curve c to M_0 satisfying

$$\arg(v(\gamma(4\pi/3))) = 0.$$

Thus, by Stokes's theorem, we obtain

$$\begin{aligned}\vec{v}_0 &= \left(0, 0, \operatorname{Re}\left(B(\alpha)\int_\gamma \frac{dz}{v^2}\right)\right)\\ &= \operatorname{Re}\left(B(\alpha)\left(0, 0, \int_{[-1,0]} \frac{dz}{|v|^2} + \int_{[0, e^{-\pi i/3}]} \frac{dz}{|v|^2}\right)\right).\end{aligned}$$

Using (10) and the definition of ρ in Lemma 3 yields

$$\vec{v}_0 = \pm\left(0, 0, \frac{(m+n)\rho\sqrt{3}}{2|m+n\theta|}\right).$$

By looking at our last expression for $\mathcal{T}_\alpha$, we see that the proposition holds. □

REMARK 4. A straightforward computation yields the following.

(i) If J induces a *rotation* on $X_\alpha^\lambda(M_0)$, then the ends P_1, P_2, P_3 are placed at one height and the ends Q_1, Q_2, Q_3 are placed at another height.

(ii) If J induces a *screw motion* on $X_\alpha^\lambda(M_0)$, then there exist three couples of ends such that each couple is placed at different heights, but the ends in each couple are placed at the same height.

COROLLARY 1. *The automorphism J induces a rotation on $X_\alpha^\lambda\colon M_0 \to \mathbb{R}^3/\mathcal{T}_\alpha$ if and only if J induces a screw motion on the conjugate surface $(X_\alpha^\lambda)^\star\colon M_0 \to \mathbb{R}^3/\mathcal{T}_{\alpha^\star}$.*

Proof. From Remark 2, the conjugate surface coincides with $X_{\alpha^\star}^\lambda$, where $\alpha^\star = m^\star\gamma_1 + n^\star\gamma_2$, $m^\star = (m-2n)/d$, $n^\star = (2m-n)/d$, and $d = \gcd(m-2n, 2m-n)$.

First, observe that either $d = 1$ or $d = 3$. In order to obtain this, note that $m^\star + n^\star = 3(m-n)/d$ and $m^\star - n^\star = -(m+n)/d$ and take into account that $\gcd(m,n) = \gcd(m+n, m-n) = 1$. On the other hand, $dm^\star + 3n = m+n$ and $dn^\star - 3m = -(m+n)$.

If $m+n$ is a multiple of 3, then $d = 3$ and so $m^\star + n^\star = m - n$. Since $\gcd(m+n, m-n) = 1$, $m^\star + n^\star$ is not a multiple of 3. Conversely, if $m^\star + n^\star = 3(m-n)/d$ is not a multiple of 3, then $d = 3$. Taking into account once again that $dm^\star + 3n = m+n$, it follows that $m+n$ is a multiple of 3. Using Proposition 1, the corollary holds. □

To finish this section, we make some more comments about the symmetry of the surfaces in Σ_α.

(1) The automorphism R extends to a symmetry of X_α^λ if and only if $\lambda = 1$. In this case we note that $R(M_1) = M_1$ and $R^\star(\Phi_\alpha) = \mathcal{R}\cdot\Phi_\alpha$, where

$$\mathcal{R} = \begin{pmatrix} 1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}.$$

Hence R induces a reflection about the x_1-axis, and this axis is not contained in $X_\alpha(M_0)$. Therefore, the reflections about the straight lines $J(x_1 - \text{axis})$ and $J^2(x_1 - \text{axis})$ leave $X_\alpha(M_0)$ invariant, too.

(2) The automorphism S extends to a symmetry of X_α^λ if and only if $B(\alpha)^2 \in \mathbb{R}$, that is, either $\alpha = \gamma_1$ or $\alpha = \gamma_1 + 2\gamma_2$. In this case, $S(M_1) = M_1$ and $S^\star(\Phi_\alpha^\lambda) = \mathcal{S} \cdot \Phi_\alpha^\lambda$, where either

$$\mathcal{S} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

if $B(\alpha) = i$ and $\alpha = \gamma_1$, or

$$\mathcal{S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

if $B(\alpha) = 1$ and $\alpha = \gamma_1 + 2\gamma_2$. Thus S induces on $X_\alpha^\lambda(M_0)$ either a reflection about the x_2-axis that is contained in the surface (if $B(\alpha) = i$) or a symmetry with respect to the (x_1, x_3)-plane (if $B(\alpha) = 1$). Moreover, either $X_\alpha(M_0)$ contains the straight lines $J(x_2 - \text{axis})$ and $J^2(x_2 - \text{axis})$ (if $B(\alpha) = i$) or $X_\alpha(M_0)$ is invariant under the symmetry with respect to the planes $J((x_1, x_3) - \text{plane})$ and $J^2((x_1, x_3) - \text{plane})$ (if $B(\alpha) = 1$).

In the case $B(\alpha) = i$, the three straight lines are self-intersection curves. To see this, note that the fixed point set of S in $\bar{M}_0$ consists on a regular closed curve containing the points P_1 and Q_1.

(3) The case $\lambda = 1$ and $B(\alpha)^2 \in \mathbb{R}$ is particularly interesting. The automorphism $S \circ R$ induces (in addition) on $X_\alpha(M_0)$ either a reflection about the x_3-axis (if $B(\alpha) = i$) or a symmetry with respect to the (x_1, x_2)-plane (if $B(\alpha) = 1$). In both cases, the surface has twelve symmetries.

If $B(\alpha) = i$ then the x_3-axis is a double line also contained in $X_{\gamma_1}(M_0)$. For, observe that $S \circ R$ fixes the points M_1 and $(1, \sqrt[3]{2})$ and that $X_{\gamma_1}(M_1) = X_{\gamma_1}((1, \sqrt[3]{2}))$ (see Figures 1 and 2). If $B = 1$ then the image under the immersion contains a circle that lies in a horizontal plane of symmetry of the surface (see Figure 4).

We give a brief outline of how to prove this. The projection to the z-plane of the nodal set $\mathcal{N}$ associated to the harmonic function $\mathrm{Im}(zv + z^2\bar{v})$ in M_0 (see Figure 6) is a real algebraic variety of dimension 1 that has only one compact irreducible component in $\mathbb{C} - \{0\}$. The lift $c(s)$ of this component to $\mathcal{N}$ is a curve that satisfies the following conditions.

(1) $c(s)$ is a planar curve (i.e., $x_3(c(s)) = \text{constant}$), and it is contained in the nodal set of the Shiffman field (i.e., its planar curvature is constant).
(2) The set $\{c(s), s \in \mathbb{R}\}$ is invariant under the automorphism $S \circ R$.
(3) The immersion $X_{(\gamma_1 + 2\gamma_2)}(c(s))$ covers four times a circle in $\mathbb{R}^3$, and this circle is hence a self-intersection curve (see Figure 7).

Note that the fixed point set of $S \circ R$ gives also a planar curve included in a horizontal plane of symmetry, and that the distance between this plane and the one containing the circle is a half of the length of the period vector (see Figures 3 and 4).

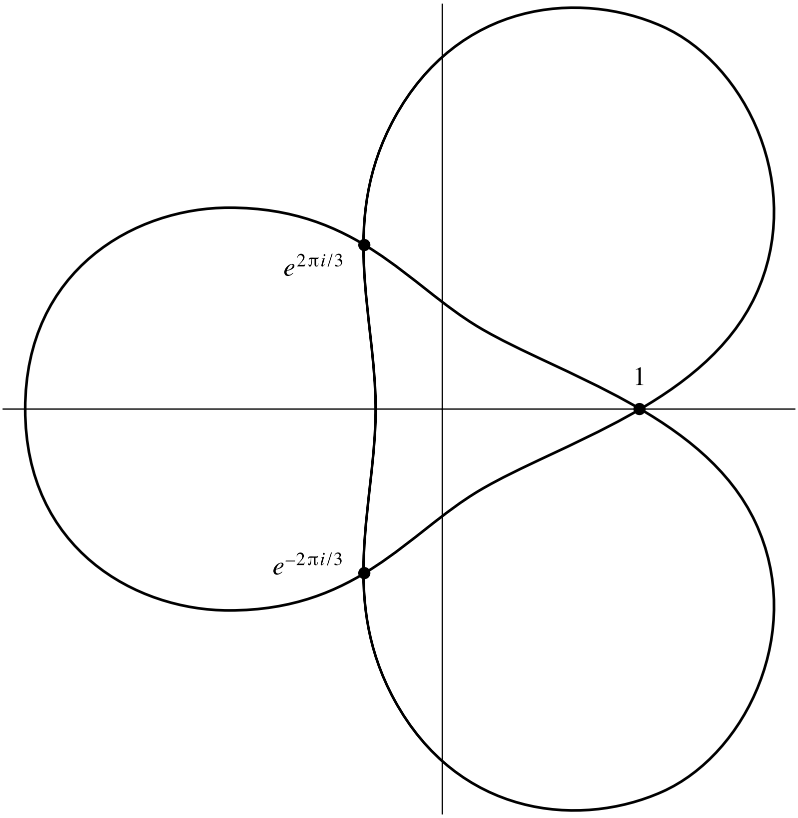

Figure 6 The projection on the z-plane of the curve $c(s)$ in M_0.

4.2. Uniqueness of the New Family of Examples

We next present a uniqueness theorem for the family of surfaces $\{\Sigma_\alpha : \alpha \in \mathcal{B}, \mathcal{B}$ is a basis of $\mathcal{H}_1(\bar{M}_0, \mathbb{Z})\}$.

Let $X : M \to \mathbb{R}^3/\mathcal{T}$ be a properly immersed minimal torus with n planar parallel embedded ends, where $\mathcal{T}$ is a cyclic group generated by a translation. As mentioned in Section 2, $n = 2k$ for $k \geq 1$. Without loss of generality, we assume that the normal vectors at the ends of M are vertical.

THEOREM 3. *Suppose that X has vertical flux in $\mathbb{R}^3/\mathcal{T}$, and that group $\mathcal{T}$ is generated by a vertical translation. Then $k \geq 3$. Moreover, the equality holds if and only if the immersion X coincides—up to biholomorphisms, rigid motions, and scaling—with X_α^λ, where $\lambda \in]0, +\infty[$ and $\alpha \in \mathcal{B}$, $\mathcal{B}$ is a basis of $\mathcal{H}_1(\bar{M}_0, \mathbb{Z})$.*

Proof. Label (M, g, η) as the Weierstrass data of the immersion X, and define $\Phi = (\phi_1, \phi_2, \phi_3)$ as in (1). We know (see Section 2) that

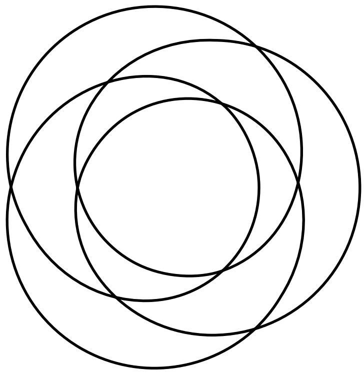

Figure 7 A curve in the surface $X_{\gamma_1+2\gamma_2}(M_0)$. This curve is the intersection of the surface with a horizontal plane that is very close to a plane of symmetry of the surface. This plane of symmetry intersects with $X_{\gamma_1+2\gamma_2}(M_0)$ in circle $X_{\gamma_1+2\gamma_2}(c(s))$.

$$M \equiv \bar{M} - \{P_1, \ldots, P_k, Q_1, \ldots, Q_k\},$$

where $\bar{M}$ is a compact genus-1 Riemann surface. Furthermore, (g, η) extends meromorphically to $\bar{M}$, and we can suppose without loss of generality that the classic divisors associated to g and η are given by

$$[g] = \frac{P_1^2 \cdots P_k^2}{Q_1^2 \cdots Q_k^2}, \qquad [\eta] = \frac{Q_1^2 \cdots Q_k^2}{P_1^2 \cdots P_k^2}.$$

Since the generator of $\mathcal{T}$ is vertical, the real part of the periods of the 1-forms ϕ_1, ϕ_2 vanish. On the other hand, X has vertical flux in $\mathbb{R}^3/\mathcal{T}$ and so these periods are real numbers. We deduce that any period of ϕ_1, ϕ_2 vanishes and these 1-forms are exact, that is to say, η and $g^2\eta$ are exact.

If X has four ends (i.e. $k = 4$) then it is not hard to prove, using elementary arguments, that η cannot be exact. Anyway, in this case and from Theorem 1 we can identify X with a two-sheeted covering of a torus with only two ends. However, in this case $(k = 1)$ the 1-forms η and $g^2\eta$ are not exact (there are no meromorphic functions of degree 1 on a torus). Therefore, we conclude that $k \geq 3$.

In what follows, we suppose $k = 3$. We want to find the Weierstrass representation for the immersion X when there are six ends. The following deductions have this purpose.

Write $\eta = dh$, where

$$[h - h(Q_i)] = \frac{Q_i^3}{P_1 \cdot P_2 \cdot P_3}, \quad i = 1, 2, 3.$$

The function $f = 1/(g \prod_{i=1}^{3}(h - h(Q_i)))$ satisfies

$$[f] = \frac{P_1 \cdot P_2 \cdot P_3}{Q_1 \cdot Q_2 \cdot Q_3}$$

and so $g = Kf^2$ for $K \in \mathbb{C}$.

On the other hand, the Riemann–Roch theorem applied to the divisor $Q_3/(Q_1 \cdot Q_2)$ (see [3, pp. 73–77]) implies the existence of a meromorphic function y verifying

$$[1/y] = \frac{Q_3 \cdot Z}{Q_1 \cdot Q_2},$$

where $Z \in \bar{M} - \{Q_1, Q_2\}$. An easy computation gives $[(h - h(Q_3)) f y] = Q_3/Z$ and so $Z = Q_3$; that is,

$$f = \frac{C}{y(h - h(Q_3))}, \quad C \in \mathbb{C} - \{0\}. \tag{13}$$

If we label $\{Q_3, S_1, S_2, S_3\}$ as the set of ramification points of y, then the classic theory of compact Riemann surfaces (see [3, p. 102]) gives the existence of a meromorphic function w satisfying

$$w^2 = \prod_{i=1}^{3}(y - y(S_i)) = y^3 + a_2 y^2 + a_1 y + a_0;$$

we can use this equation to represent, up to biholomorphisms, the torus $\bar{M}$.

Up to a homothetical change of variables, we can suppose $a_0 = 1$ (observe that $y^{-1}(0) = \{Q_1, Q_2\}$ contains two different points and that 0 is not the image of any ramification point of y).

The Riemann–Roch theorem implies that the vector space of meromorphic functions having at Q_3 at most one pole of order 3 and no other singularities has dimension 3. Furthermore, it is generated by $\{1, y, w\}$. Since $1/(h - h(Q_3))$ belongs to this space, it is not hard to conclude that

$$h - h(Q_3) = \frac{D}{\mu_1 y + \mu_0 + w} \tag{14}$$

for suitable constants $D, \mu_0, \mu_1 \in \mathbb{C}$. Substituting $h - h(Q_3)$ for this function in (13) yields $f = (C/D)(\mu_1 y + \mu_0 + w)/y$, and so

$$g = A' \frac{(\mu_1 y + \mu_0 + w)^2}{y^2}, \qquad \eta = \frac{B'}{A'} \frac{y^2}{(\mu_1 y + \mu_0 + w)^2} \frac{dy}{w}, \tag{15}$$

where $A' = KC^2/D^2$ and $B' \in \mathbb{C} - \{0\}$. Recalling that $dh = \eta$ and using (14) and (15), we obtain $a_1 = a_2 = \mu_1 = 0$. From (15), we deduce that

$$g^2 \eta = A'B' \frac{(\mu_0 + w)^2}{y^2 w} dy.$$

Because this 1-form is also exact, the same holds for the 1-form

$$\tau = B'A' \frac{y^3 + 1 + \mu_0^2}{y^2 w} dy = g^2 \eta + 2B'A' \mu_0 d(1/y).$$

Thus,

$$\tau - 2A'B' d(w/y) = A'B' \frac{\mu_0^2 + 3}{y^2 w} dy$$

is exact, which implies $\mu_0^2 = -3$. Up to the change $(y, w) \to (y, -w)$, we can suppose $\mu_0 = \sqrt{3}i$.

Define

$$z = \sqrt{3}\sqrt[3]{2}i\,\frac{y}{w + \sqrt{3}i} \quad \text{and} \quad v = \frac{w - \sqrt{3}i}{w + \sqrt{3}i}.$$

Then a straightforward computation gives

$$v^3 = z^3 + 1, \quad g = \frac{A}{z^2}, \quad g\eta = B\frac{dz}{v^2}$$

for suitable constants $A, B \in \mathbb{C} - \{0\}$. Therefore, $\bar{M}$ is biholomorphic to the surface $\bar{M}_0$ defined in Section 4.1, and up to this biholomorphism, $M = M_0$. Moreover, up to rigid motions and scaling, we can assume that $A = \lambda \in \mathbb{R}_+$ and $|B| = 1$, $\arg(B) \in [0, \pi[$.

If we label $\{\alpha, \beta\}$ as a basis of $\mathcal{H}_1(\bar{M}_0, \mathbb{Z})$ such that $\mathrm{Re}(\int_\alpha \Phi) = 0$, then Theorem 2 yields $B = B(\alpha)$ and, as a consequence of Remark 1, $X = X_\alpha^\lambda$. This fact completes the proof. □

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