Holomorphic Motions of Hyperbolic Sets

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0. Introduction

Let M be a complex Hermitian manifold and $\{f_a\}_{a\in\mathbf{D}}$ a holomorphic family of endomorphisms of M, where $\mathbf{D}$ is the unit disk. This means that the map $\mathbf{D}\times M \to M$, defined by $(a,x)\to f_a(x)$, is holomorphic. Suppose that $f = f_0$ has a compact surjectively invariant subset K, that is, $f(K) = K$. For example, K could be a fixed point or a periodic orbit, but also a more complicated set such as the Julia set of a rational function. We may then ask if K is persistent under the perturbation f_a of the map f. For instance, if K is a fixed point of f, then we ask if f_a has a fixed point K_a near K for a small enough. A sufficient (albeit not necessary) condition for this is that the fixed point K be hyperbolic, meaning that the derivative of f at K has no eigenvalue of modulus 1.

There is a natural notion of hyperbolicity for general sets K. Let us first consider the case when the maps f_a are diffeomorphisms. The precise definition (which can be found e.g. in [R]) will not be stated here, but it says that the tangent bundle over K splits continuously into two invariant subbundles on which the derivative of f is expanding and contracting, respectively.

One basic result in real dynamics is that hyperbolic sets are persistent under perturbations in the map f (see [R]). In our case this means that if a is small enough, then f_a has a hyperbolic set K_a close to K, and there exists a homeomorphism h_a close to the identity conjugating $f|_K$ to $f_a|_{K_a}$.

If K is a hyperbolic fixed point, then it follows from the implicit function theorem that the fixed point K_a of f_a depends holomorphically on a. The natural generalization of this to more general sets K is the notion of a holomorphic motion, the definition of which is given in Section 1.

THEOREM A. *Let $\{f_a\}_{a\in\mathbf{D}}$ be a holomorphic family of diffeomorphisms of a Hermitian manifold M parameterized by the unit disk $\mathbf{D}$. Suppose that $f = f_0$ has a hyperbolic subset K. Then K moves holomorphically with the parameter a at $a = 0$. More precisely, there exist $r > 0$ and a holomorphic motion $h\colon \mathbf{D}_r \times K \to M$ such that, for each $a \in \mathbf{D}_r$:*

(1) *$K_a := h(a,K)$ is a hyperbolic subset for f_a;*
(2) *the map $h_a := h(a,\cdot)\colon K \to K_a$ is a homeomorphism and $f_a \circ h_a = h_a \circ f$.*

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Let us now return to the situation of a holomorphic family $\{f_a\}$ of endomorphisms of a Hermitian manifold M. There is a notion of a hyperbolic set K in this setting too [R]. Again, we will not give the precise definition, but let us note that it involves the set $\hat{K} = \{(x_k)_{k\leq 0};\ x_k \in K,\ f(x_k) = x_{k+1}\}$ of backwards orbits in K.

The real theory [R] tells us that, for a small enough, f_a has a hyperbolic set K_a close to K and there exists a continuous surjective map $h_a\colon \hat{K} \to K_a$. Now K and K_a need not be homeomorphic, so K does, in general, not move holomorphically with a. Nevertheless, the dependence of K_a on a reflects the complex structure; one way of saying this is that $a \to K_a$ is a strongly analytic multifunction, the definition of which is given in Section 1.

THEOREM B. *Let $\{f_a\}_{a\in\mathbf{D}}$ be a holomorphic family of endomorphisms of a Hermitian manifold M parameterized by the unit disk $\mathbf{D}$. Suppose that $f = f_0$ has a hyperbolic subset K. Then $\hat{K}$ moves holomorphically with the parameter a at $a = 0$ and $a \to K_a$ is a strongly analytic multifunction. More precisely, there exist $r > 0$ and a continuous map $h\colon \mathbf{D}_r \times \hat{K} \to M$ such that:*

(1) *for each $a \in \mathbf{D}_r$, $K_a := h_a(\hat{K})$ is a hyperbolic set for f_a, where $h_a = h(a, \cdot)$;*
(2) *for each $a \in \mathbf{D}_r$, the map h_a satisfies the relation $f_a \circ h_a = h_a \circ \hat{f}$ and lifts to a homeomorphism $\widehat{h_a}\colon \hat{K} \to \widehat{K_a}$, which is just the identity for $a = 0$;*
(3) *the map $h(\cdot, \hat{x})\colon \mathbf{D}_r \to M$ is holomorphic for each $\hat{x} \in \hat{K}$;*
(4) *the set $\bigcup_a(\{a\} \times K_a)$ in $\mathbf{D}_r \times M$ is foliated by holomorphic graphs over $\mathbf{D}_r$.*

Sometimes a hyperbolic set K does move holomorphically with the parameter even for endomorphisms. An important situation when this happens is when K is a repellor, meaning that the derivative of f is expanding on the whole tangent bundle over K (see Definition 1.3).

THEOREM C. *If $\{f_a\}$ is a holomorphic family of endomorphisms and K is a repellor for $f = f_0$, then K moves holomorphically with a at $a = 0$ in the sense of Theorem A.*

Theorem C applies to show that the Julia set of a rational function moves holomorphically with the parameter on the open set of parameter space consisting of hyperbolic maps. In [MSS], the authors prove that in fact one has a holomorphic motion for a (possibly larger) open dense set of parameter space.

1. Definitions

In this section we recall the definitions of holomorphic motions and analytic multifunctions. For notational simplicity we will let these be parameterized by the unit disk.

DEFINITION 1.1. Let $\mathbf{D}$ be the unit disk, M a complex manifold, and X a subset of M. Then a *holomorphic motion* of X parameterized by $\mathbf{D}$ is a continuous map $\phi\colon \mathbf{D} \times X \to M$ such that:

(1) $\phi(0, \cdot) = \mathrm{id}$;
(2) $\phi(\cdot, x)\colon \mathbf{D} \to M$ is holomorphic for every $x \in X$;
(3) $\phi(a, \cdot)\colon X \to M$ is injective for every $a \in \mathbf{D}$.

Holomorphic motions have mostly been studied for subsets of the Riemann sphere. In [MSS], Mañe, Sad, and Sullivan proved the celebrated λ-lemma, which states that each map $\phi(a, \cdot)$ is quasiconformal and that the continuity assumption on ϕ is redundant. Słodkowski [S], strengthening previous results, later proved that a holomorphic motion of any subset X of $\hat{\mathbf{C}}$ can be extended to a holomorphic motion of the whole Riemann sphere.

In higher-dimensional complex manifolds, such extension and continuity properties do not hold in general. Indeed, it is easy to construct a holomorphic motion of a subset X of $\mathbf{C}^2$ such that all the maps $\phi(a, \cdot)$ are discontinuous for $a \neq 0$. Moreover, the role of quasiconformality is not clear, at least not for arbitrary sets X. Some results on quasiconformality and holomorphic motions in higher dimension can be found in [ABR].

Next we discuss analytic multifunctions. Let M be a complex manifold. Then a multifunction from $\mathbf{D}$ to M is a map K from $\mathbf{D}$ to the set $\mathcal{K}(M)$ of compact subsets of M. The map K is called continuous (upper semicontinuous) if it is continuous (usc) in the Hausdorff metric on $\mathcal{K}(M)$. Its graph is defined by $\Gamma(K) = \bigcup_{a\in\mathbf{D}}(\{a\} \times K(a))$, and it is easy to see that K is usc iff $\Gamma(K)$ is closed in $\mathbf{D} \times M$.

DEFINITION 1.2. A *strongly analytic* multifunction is an usc multifunction K such that $\Gamma(K)$ is the union of graphs of holomorphic maps from $\mathbf{D}$ to M.

From the definition it follows that a strongly analytic multifunction K is both continuous and an analytic multifunction in the sense of [A]. The latter statement means that if $D \subset\subset \mathbf{D}$ is open and ψ is plurisubharmonic in a neighborhood of $\Gamma(K|_D)$, then $\phi(\lambda) := \sup\{\psi(\lambda, x); x \in K(\lambda)\}$ is subharmonic on D. Also note that a holomorphic motion can be viewed as a strongly analytic multifunction K such that $\Gamma(K)$ is the union of *disjoint* graphs.

Analytic multifunctions appear naturally in complex dynamics. For example, Baribeau and Ransford [BR] proved that if f_a is a holomorphic family of rational functions then $a \to J_a^*$ is an analytic multifunction, where J_a^* is the usc regularization of the Julia set J_a of f_a; that is, the graph $\Gamma(a \to J_a^*)$ is the closure of the graph $\Gamma(a \to J_a)$.

Let us finally give the definition of a repellor as is needed in the statement of Theorem C.

DEFINITION 1.3. Let f be a holomorphic endomorphism of a Hermitian manifold M, and let K be a compact invariant set. Then K is said to be a *repellor* if there exist $c > 0$ and $\lambda > 1$ such that $|f_*^n v| \geq c\lambda^n |v|$ for all tangent vectors v over K and all $n \geq 1$.

2. Proofs

Proof of Theorem A. From the real theory [R] we know that we may find an $r > 0$ and, for all $a \in \mathbf{D}_r$, a continuous map $h_a \colon K \to M$ such that $K_a := h_a(K)$ is a hyperbolic subset for f_a; $h_a \colon K \to K_a$ is a homeomorphism and the relation $f_a \circ h_a = h_a \circ f$ holds. Moreover, h_0 is the inclusion $K \hookrightarrow M$, and the map $a \to h_a$ is C^∞ as a map from D to the real Banach manifold $C(K, M)$ of continuous functions of K into M. All of this is proved using the implicit function theorem on $C(K, M)$.

We want to prove that the map $a \to h_a(x)$ is holomorphic for all $x \in K$ and depends continuously on x. But the smoothness of $a \to h_a$ implies that $a \to h_a(x)$ is C^∞ and that all derivatives of $h_a(x)$ with respect to a depend continuously on x. Fix $b \in \mathbf{D}_r$ and let μ be the section of the tangent bundle of M over K_b defined by $\mu(h_b(x)) := \frac{\partial}{\partial \bar{a}} h_a(x)|_{a=b}$; this makes sense since h_b is a homeomorphism. Then μ is a continuous (and hence bounded) section of TM over the compact set K_b. We want to prove that $\mu \equiv 0$. From the relation $f_a \circ h_a = h_a \circ f$ we easily obtain $\mu \circ f_b = (f_b)_* \mu$, where $(f_b)_*$ is the derivative of f_b. But then the following lemma tells us that $\mu \equiv 0$, which completes the proof. □

LEMMA 2.1. *Let K be a hyperbolic set for an endomorphism f of a Riemannian manifold M, and let $(x_i)_{i \in \mathbf{Z}}$ be an orbit in K. Suppose that μ is a bounded section of the tangent bundle over (x_i) (i.e., $\mu(x_i) \in T_{x_i} M$) with the property $\mu(x_{i+1}) = f_*(x_i)\mu(x_i)$. Then $\mu(x_i) = 0$ for all i.*

Proof. We prove the lemma in the case when f is a diffeomorphism—the modifications in the endomorphism case are left to the reader. There is a continuous f_*-invariant splitting of the tangent bundle over K into unstable and stable bundles E^u and E^s, respectively, so we may write $\mu = \mu_u + \mu_s$, where μ_u and μ_s are bounded sections over (x_i) of E^u and E^s, respectively. We then have that $\mu_u(x_{i+1}) = f_*(x_i)\mu_u(x_i)$ and $\mu_s(x_{i+1}) = f_*(x_i)\mu_s(x_i)$. Suppose that $\mu_u(x_i) \neq 0$ for some i. Then the expansion along E^u gives that $|\mu_u(x_{i+n})| = |f_*^n(x_i)\mu_u(x_i)| \to \infty$ as $n \to \infty$. This contradicts the assumption that μ_u was bounded. Hence $\mu_u \equiv 0$. In the same way, we see that $\mu_s \equiv 0$ and so $\mu \equiv 0$. □

Proof of Theorem B. The proof is very similar to that of Theorem A. The existence of r and h satisfying (1)–(2) follows from the real theory [R]. This time h is constructed using the implicit function theorem on the real Banach manifold $C(\hat{K}, M)$ of continuous functions from $\hat{K}$ to M. To prove (3), we take b in $\mathbf{D}_r$ and consider the map μ from $\widehat{K_b}$ to TK_b defined by $\mu(\widehat{h_b}(\hat{x})) := \frac{\partial}{\partial \bar{a}} h_a(\hat{x})|_{a=b}$. Then μ is well-defined since $\widehat{h_b}$ is a homeomorphism. Moreover, μ is continuous (and hence bounded) and satisfies the relation $\mu \circ \widehat{f_b} = (f_b)_*\mu$. Therefore, if (x_i) is any orbit in K_b, then Lemma 2.1 shows that $\mu((x_i)) = 0$. This proves (3). Finally, (4) follows immediately from (3). $\square$

Proof of Theorem C. Let h_a be as in Theorem B. We claim that there exists a homeomorphism $g_a\colon K \to K_a$ such that $g_a \circ \pi = h_a$, where $\pi\colon \hat{K} \to K$ is the projection $\pi((x_k)) = x_0$. To see this, take any $x \in K$ and let $\hat{x} = (x_k)$ and $\hat{y} = (y_k)$ be two points in $\hat{K}$ with $\pi(\hat{x}) = \pi(\hat{y}) = x$ (i.e., $x_0 = y_0 = x$). We must show that $h_a(\hat{x}) = h_a(\hat{y})$. Suppose this is not the case and let $x(a) = h_a(\hat{x})$ and $y(a) = h_a(\hat{y})$. Then, for $n \geq 0$, we have

$$d(f_a^n(x(a)), f_a^n(y(a))) \leq d(f_a^n(x(a)), f^n(x)) + d(f_a^n(y(a)), f^n(x)) \leq c(a),$$

where $c(a) \to 0$ as $a \to 0$. Hence the forward orbits of $x(a)$ and $y(a)$ are very close if a is small. Because of the expansion, this is only possible if $x(a) = y(a)$. Therefore, the map $h_a\colon K \to K_a$ is well-defined. It remains to be shown that $a \to h_a(x)$ is holomorphic for all $x \in K$, but this follows immediately because the maps $a \to h_a(\hat{x})$ are holomorphic. $\square$

It is also possible to give a direct proof of Theorem C without using Theorem B. Let us sketch how to do this. The idea is to use Sullivan's telescope construction as described in [HO]. For simplicity we assume that the constant c in the definition of a repellor is equal to 1; this can be achieved by changing the metric on M slightly (a construction originally due to Mather). Let $U_0(x)$ be the ball of radius $\varepsilon > 0$ centered at $x \in K$. The expansion implies that $f^{-1}(U_0(f(x)))$ has a unique component contained in $U_0(x)$ for $x \in K$ if ε is small enough. Call this component $U_1(x)$. Inductively we find a nested sequence (telescope) of open sets $\{U_n(x)\}_{n \geq 0}$ for $x \in K$, and the expansion implies that the diameter of $U_n(x)$ is uniformly exponentially small. In particular, the intersection $\bigcap_{n \geq 0} U_n(x)$ (the focus of the telescope) is the single point x. If a is small enough, then we may construct a perturbed telescope $\{U_{n,a}(x)\}_{n \geq 0}$ for $x \in K$ so that $U_{n,a}(x)$ is a connected component of $f_a^{-n}(U_0(f^n(x)))$. We will still have that the diameter of $U_{n,a}(x)$ is uniformly exponentially small, so the focus of the telescope is a well-defined point $h_a(x)$. It is easy to see, using that the expansion on K is bounded above, that $h_a(x)$ depends continuously on x—in fact, h_a is Hölder continuous. Exchanging the roles of f and f_a, we see that (for a small enough) h_a is a homeomorphism, which is bi-Hölder. Define $K_a := h_a(K)$. It is clear from the construction that h_a conjugates f on K to f_a on K_a. Finally, for fixed x, $h_a(x)$ is given as a uniform limit of functions holomorphic in a; hence $a \to h_a(x)$ is holomorphic. This completes the second proof of Theorem C.

3. Examples

Our first example concerns polynomial diffeomorphisms of $\mathbf{C}^2$, for which we use [BS] as a reference. We consider only diffeomorphisms that are conjugate to finite compositions of (generalized) Hénon maps.

A polynomial diffeomorphism of $\mathbf{C}^2$ is said to be *hyperbolic* if it is hyperbolic on its nonwandering set; in this case the nonwandering set consists of a basic set J of unstable dimension 1 and a finite number of repelling or attracting periodic points.

It follows from Theorem A that if $\{f_a\}_{a\in\mathbf{D}}$ is a holomorphic family of polynomial diffeomorphisms of $\mathbf{C}^2$ and $f = f_0$ is hyperbolic, then J moves holomorphically with a at $a = 0$.

The second example is of a polynomial endomorphism f of $\mathbf{C}^2$, defined by $f(z, w) = (z^2, w^2)$. The nonwandering set Ω of f is the union $\Omega_0 \cup \Omega_1 \cup \Omega_2$, where $\Omega_0 = \{(0, 0)\}$, $\Omega_1 = \{|w| = 1, z = 0\} \cup \{|z| = 1, w = 0\}$, and $\Omega_2 = \{|z| = |w| = 1\}$. In this case f is hyperbolic on all of Ω, and it has unstable dimension i on Ω_i.

We now embed f in a holomorphic family $\{f_a\}$ of endomorphisms of $\mathbf{C}^2$ with $f_0 = f$. It then follows from Theorem C that the set Ω_2 moves holomorphically with a for a small enough; the same is true for Ω_0. On the other hand, the set Ω_1 does not move holomorphically in general. To see this, consider the component $K = \{|z| = 1, w = 0\}$ of Ω_1. We embed f_0 in the holomorphic family $\{f_a\}$ defined by $f_a(z, w) = (z^2, w^2 + az)$, $|a| < 1/4$. Then the Riemann surface $V_a = \{w^2 = r^2 z\}$ is invariant, where $r = 1/2 - \sqrt{1/4 - a}$ and the branch of the root is chosen so that $\sqrt{1/4} = 1/2$. If we use z as a variable on V_a, then the dynamics on V_a is given by $z \to z^2$. Hence $K_a = \{|z| = 1, w^2 = r^2 z\}$. For $a \neq 0$, this is a fiber bundle over the circle $\{|z| = 1, w = 0\}$ with a two-point set as a fiber, and it is clear that K_a is not a holomorphic motion of K.

In fact, the discontinuity of K_a in this example is misleadingly simple. If we take $f_a(z, w) = (z^2, w^2 + w/10 + az)$, then one can see that the set K_a, which is a perturbation of the set $K_0 = \{|z| = 1, w = 0\}$ for small $a \neq 0$, is a fiber bundle over the circle $|z| = 1$ with Cantor sets as fibers.

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References

[A] B. Aupetit, *Analytic multifunctions and their applications,* Complex potential theory (P. M. Gauthier, G. Sabidussi, eds.), pp. 1–74, Kluwer, Dordrecht, 1994.

[ABR] K. Astala, Z. Balogh, and H. M. Reimann, *Lempert mappings and holomorphic motions in* $\mathbf{C}^n$, preprint.

[BR] L. Baribeau and T. J. Ransford, *Meromorphic multifunctions in complex dynamics,* Ergodic Theory Dynamical Systems 12 (1992), 39–52.

[BS] E. Bedford and J. Smillie, *Polynomial diffeomorphisms of* $\mathbf{C}^2$: *Currents, equilibrium measure and hyperbolicity,* Invent. Math. 103 (1991), 69–99.

[HO] J. H. Hubbard and W. Oberste-Vorth, *Hénon mappings in the complex domain II,* Real and complex dynamical systems (B. Branner, P. Hjorth, eds.), pp. 89–132, Kluwer, Dordrecht, 1995.

[MSS] R. Mañé, P. Sad, and D. Sullivan, *On the dynamics of rational maps*, Ann. Sci. École Norm. Sup. (4) 16 (1983), 193–217.

[R] D. Ruelle, *Elements of differentiable dynamics and bifurcation theory*, Academic Press, Boston, 1989.

[S] Z. Słodkowski, *Holomorphic motions and polynomial hulls*, Proc. Amer. Math. Soc. 111 (1991), 347–357.

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