# Analytic Varieties with Boundaries in Totally Real Tori

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## 1. Introduction

Let $\partial D$ be the unit circle in $\mathbf{C}$ and let $\pi_2 : \mathbf{C}^2 \to \mathbf{C}$ be the projection $\pi_2(z, w) = w$. Let $T_1$ and $T_2$ be disjoint maximal real smooth tori in $\partial D \times \mathbf{C}$ such that, for each $\xi \in \partial D$ and $j = 1, 2$, the fiber

$$T_{j,\xi} := \pi_2(T_j \cap (\{\xi\} \times \mathbf{C}))$$

of $T_j$ over $\xi$ is a smooth Jordan curve in $\mathbf{C}$. Also, let $V$ be a two-sheeted analytic variety over $D$ with boundary in $T_1 \cup T_2$; that is, there exist $p$ and $q$ holomorphic functions on $D$, continuous up to the boundary $\partial D$, such that

$$V = \{ (z, w) \in \bar{D} \times \mathbf{C};\ w^2 - p(z)w + q(z) = 0 \}$$

and such that, for every boundary point $\xi \in \partial D$, each curve $T_{1,\xi}$ and $T_{2,\xi}$ contains exactly one root of the equation $w^2 - p(\xi)w + q(\xi) = 0$.

In this paper we consider the question of when it is possible to perturb variety $V$ along $T_1 \cup T_2$. More precisely, we are interested in geometric conditions on $T_1 \cup T_2$ and $V$ such that it is possible to parameterize all nearby two-sheeted varieties over $D$ with boundaries in $T_1 \cup T_2$. The method we apply is the method of partial indices, which has been successfully used in problems of perturbing analytic discs along maximal real boundaries by several authors [4; 6; 7; 8; 9]. The geometric conditions we obtain are expressed in terms of the winding numbers of the normals to the fibers $T_{j,\xi}$ $(j = 1, 2)$ along the roots of the equation $w^2 - p(\xi)w + q(\xi) = 0$ $(\xi \in \partial D)$. A typical result is the following.

Theorem. *Let $V$ be an irreducible two-sheeted analytic variety over $D$ with boundary in the disjoint union $T_1 \cup T_2$ of two maximal real tori fibered over $\partial D$. Let $\alpha_1$ and $\alpha_2$ be the complex functions on $\partial D$ representing the boundary roots of the variety $V$ such that, for every $\xi \in \partial D$, we have $\alpha_j(\xi) \in T_{j,\xi}$ $(j = 1, 2)$; let $\Delta = \alpha_1 - \alpha_2$. Also, let $\nu_1(\xi)$ and $\nu_2(\xi)$ be normals to the fibers $T_{1,\xi}$ and $T_{2,\xi}$ at the points $\alpha_1(\xi)$ and $\alpha_2(\xi)$, respectively. If $W(\nu_1) + W(\nu_2) \geq -1$, then the family of two-sheeted analytic varieties over $D$ with boundaries in $T_1 \cup T_2$ that are close to $V$ is a $C^1$ submanifold of the space of two-sheeted analytic varieties over $D$ of dimension* $2(W(\nu_1) + W(\nu_2) + W(\Delta)) + 2$.

Received January 8, 1997. Revision received August 26, 1997.
This work was supported in part by a grant from the Ministry of Science of the Republic of Slovenia.
Michigan Math. J. 45 (1998).

See Section 3 for more details. Here, $W(\gamma)$ denotes the winding number of a nonzero continuous complex function $\gamma$ on $\partial D$.

The method we apply also allows small perturbations $\tilde{T}_1$ and $\tilde{T}_2$ of the tori $T_1$ and $T_2$ (respectively) and hence our results also prove the existence of two-sheeted analytic varieties over $D$ with boundaries in the perturbed union $\tilde{T}_1 \cup \tilde{T}_2$. Recall that, for a single maximal real torus $T$ over $\partial D$, it is a well-known result of Forstnerič [7] that the existence of a holomorphic function $f$ from the disc algebra $A(D)$ such that for each $\xi \in \partial D$ the value $f(\xi)$ lies in the interior of the bounded component of $\mathbf{C} \setminus T_\xi$ implies the existence of a holomorphic function $a \in A(D)$ such that $a(\xi) \in T_\xi$ for each $\xi \in \partial D$. Actually Forstnerič proves much more. In fact, the whole polynomial hull of $T$ over $D$ is given as the union of the graphs of the analytic discs $\tilde{a} \in A(D)$ such that $\tilde{a}(\xi) \in T_\xi$ for each $\xi \in \partial D$; see [7] for more details. See also [1; 2; 3; 13; 14; 15] for results related to the question on the polynomial hull of a compact fibration over $\partial D$.

ACKNOWLEDGMENT. The author wishes to thank Prof. Franc Forstnerič and Prof. Josip Globevnik for stimulating discussions.

## 2. Partial Indices

We will just recall some of the facts related to the partial indices of a closed $C^\alpha$ $(0 < \alpha < 1)$ path $\phi$ in a $C^2$ maximal real manifold $T$ in $\mathbf{C}^n$. One may also consider a $C^2$ maximal real fibration $\{T_\xi\}_{\xi \in \partial D}$ over $\partial D$. In the latter case each fiber $T_\xi$ is a maximal real submanifold of $\mathbf{C}^n$ and $\phi(\xi) \in T_\xi$ for every $\xi \in \partial D$. More details on partial indices can be found in [8; 9] and [16; 17]; see also [11].

For each $\xi \in \partial D$ let $A(\xi)$ denote a matrix whose columns span the tangent space of $T$ (or of the fiber $T_\xi$) at the point $\phi(\xi)$. Then there exist $n$ integers $k_1, \dots, k_n$ called the partial indices of the closed path $\phi$ in $T$, uniquely determined up to their order, and a holomorphic matrix function $\Phi \in (A^\alpha(D))^{n \times n}$ such that $\Phi\colon \bar{D} \to GL(n, \mathbf{C})$ and such that, on $\partial D$,

$$B(\xi) := A(\xi)\overline{A(\xi)^{-1}} = \Phi(\xi) \begin{pmatrix} \xi^{k_1} & 0 & \dots & 0 \\ 0 & \xi^{k_2} & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \xi^{k_n} \end{pmatrix} \overline{\Phi(\xi)^{-1}};$$

that is, the holomorphic $j$th column $v_j(\xi)$ of the matrix function $\Phi(\xi)$ solves the equation

$$B(\xi)\overline{v_j(\xi)} = \xi^{k_j} v_j(\xi)$$

on $\partial D$. If the tangent bundle of $T$ is trivial (orientable) along $\phi(\partial D)$, one may choose $A(\xi)$ to be $C^\alpha$ on $\partial D$. However, any choice of a matrix $A(\xi)$ whose columns span the tangent space of $T$ (or of the fiber $T_\xi$) at the point $\phi(\xi)$ will result in the same matrix function $B(\xi)$.

The sum $k_1 + \dots + k_n = k = W(\det(B))$ is called the *total index* of the closed path $\phi$ in $T$. The total index is even if $T$ is trivial along $\phi(\partial D)$ and odd otherwise. In the case all partial indices are greater than or equal to $-1$, we call the curve

*$\phi$ regular*. It is known from results of Globevnik [8; 9] and Oh [11] that, in this case, for every maximal real manifold $\tilde{T}$ close to $T$ the family of small $(A^\alpha(D))^n$ analytic perturbations of $\phi$ along $\tilde{T}$ is a $C^1$ submanifold of the space $(A^\alpha(D))^n$ of dimension $n+k$. Also, these manifolds depend smoothly on $\tilde{T}$. See [8; 9; 11] for more details.

## 3. Two-Sheeted Varieties over $D$ with Boundaries in $T_1 \cup T_2$

Let $T_1$ and $T_2$ be disjoint maximal real tori over $\partial D$ such that, for $j = 1, 2$ and each $\xi \in \partial D$, the fiber

$$T_{j,\xi} = \pi_2(T_j \cap (\{\xi\} \times \mathbf{C}))$$

is a smooth Jordan curve in $\mathbf{C}$. More precisely, let $S^1 = \mathbf{R}/\mathbf{Z}$. Then there exist $r_1, r_2 \in C^2(\partial D \times S^1)$ such that the mapping

$$(\xi, t) \in \partial D \times S^1 \mapsto (\xi, r_j(\xi, t)) \in T_j$$

is a parameterization of $T_j$ $(j = 1, 2)$. Also, for each $j = 1, 2$ and each $\xi \in \partial D$, the mapping

$$t \in S^1 \mapsto r_j(\xi, t)$$

is a $C^2$ parameterization of the fiber $T_{j,\xi}$. In particular $\frac{\partial}{\partial t} r_j(\xi, t) \neq 0$ for any $j = 1, 2$, $\xi \in \partial D$, and $t \in S^1$.

Given $T_1, T_2$ and their parameterizations $r_1, r_2$, we define $\Sigma(T_1, T_2)$, a 3-torus in $\partial D \times \mathbf{C}^2 \subseteq \mathbf{C}^3$, as the set of all points $(\xi, w_1, w_2) \in \partial D \times \mathbf{C}^2$ such that

$$w_1 = r_1(\xi, t) + r_2(\xi, s) \quad \text{and} \quad w_2 = \tfrac{1}{2}(r_1(\xi, t) - r_2(\xi, s))^2$$

for some $s, t \in S^1$.

Proposition 1. *Let $\pi_3 \colon \mathbf{C}^3 \to \mathbf{C}^2$ be the projection $\pi_3(z, w_1, w_2) = (w_1, w_2)$. If $T_1 \cap T_2 = \emptyset$, then $\Sigma(T_1, T_2)$ is a maximal real 3-torus in $\mathbf{C}^3$ and every fiber $\Sigma(T_1, T_2)_\xi = \pi_3(\Sigma(T_1, T_2) \cap (\{\xi\} \times \mathbf{C}^2))$, $\xi \in \partial D$, is a maximal real 2-torus in $\mathbf{C}^2$.*

*Proof*. Every pair of complex numbers $w_1$ and $w_2$ uniquely determines an unordered pair of complex numbers $\alpha_1, \alpha_2$ such that

$$w_1 = \alpha_1 + \alpha_2 \quad \text{and} \quad w_2 = \tfrac{1}{2}(\alpha_1 - \alpha_2)^2.$$

Because $T_1$ and $T_2$ are disjoint, $\Sigma(T_1, T_2)$ is a 3-torus parameterized by the mapping

$$(\xi, t, s) \in \partial D \times S^1 \times S^1 \mapsto \left(\xi, r_1(\xi, t) + r_2(\xi, s), \tfrac{1}{2}(r_1(\xi, t) - r_2(\xi, s))^2\right).$$

Let $D(\xi, t, s) := r_1(\xi, t) - r_2(\xi, s)$. For each $\xi \in \partial D$, the tangent space to the fiber $\Sigma(T_1, T_2)_\xi$ at the point $(r_1(\xi, t) + r_2(\xi, s), \frac{1}{2} D(\xi, t, s)^2)$ is the $\mathbf{R}$-linear span of the columns of the matrix

$$A = \begin{pmatrix} \frac{\partial}{\partial t} r_1(\xi, t) & \frac{\partial}{\partial s} r_2(\xi, s) \\ D(\xi, t, s)\, \frac{\partial}{\partial t} r_1(\xi, t) & -D(\xi, t, s)\, \frac{\partial}{\partial s} r_2(\xi, s) \end{pmatrix}. \tag{1}$$

Since $T_1$ and $T_2$ are disjoint, the determinant

$$\det(A) = -2D(\xi, t, s)\, \frac{\partial}{\partial t} r_1(\xi, t)\, \frac{\partial}{\partial s} r_2(\xi, s)$$

is nonzero and $\Sigma(T_1, T_2)_\xi$ is a maximal real 2-torus in $\mathbf{C}^2$. Also, $\Sigma(T_1, T_2)$ is a maximal real 3-torus in $\mathbf{C}^3$. □

Let $\alpha_1$ and $\alpha_2$ be two $C^\alpha$ complex functions on $\partial D$ such that, for every $\xi \in \partial D$,

$$\alpha_1(\xi) \in T_{1,\xi} \quad \text{and} \quad \alpha_2(\xi) \in T_{2,\xi}.$$

To a pair of such closed curves in $\mathbf{C}$ we associate the closed curve

$$\xi \mapsto \left(\alpha_1(\xi) + \alpha_2(\xi), \tfrac{1}{2}(\alpha_1(\xi) - \alpha_2(\xi))^2\right) \tag{2}$$

in $\mathbf{C}^2$ such that, for each $\xi \in \partial D$, the point $\left(\alpha_1(\xi) + \alpha_2(\xi), \tfrac{1}{2}(\alpha_1(\xi) - \alpha_2(\xi))^2\right)$ lies in the maximal real torus $\Sigma(T_1, T_2)_\xi$. To each curve in a maximal real fibration in $\mathbf{C}^2$ over $\partial D$ one can associate two partial indices, hence one can define the partial indices of a pair of $C^\alpha$ closed curves $\alpha_1$ in $T_1$ and $\alpha_2$ in $T_2$ as the partial indices of the curve (2) in the maximal real fibration $\{\Sigma(T_1, T_2)_\xi\}_{\xi \in \partial D}$.

Let $\nu_j(\xi) := -i \frac{\partial}{\partial t} r_j(\xi, t)$, evaluated at the point $r_j(\xi, t) = \alpha_j(\xi)$, denote a normal to the curve $T_{j,\xi}$ at the point $\alpha_j(\xi)$ ($j = 1, 2$), and let $\Delta(\xi) := \alpha_1(\xi) - \alpha_2(\xi)$. Then the matrix (1) along the curve (2) can be written in the form

$$A(\xi) = i \begin{pmatrix} 1 & 1 \\ \Delta(\xi) & -\Delta(\xi) \end{pmatrix} \begin{pmatrix} \nu_1(\xi) & 0 \\ 0 & \nu_2(\xi) \end{pmatrix},$$

and the corresponding matrix $B(\xi) := A(\xi)\overline{A(\xi)^{-1}}$ is

$$B(\xi) = -\frac{1}{2\overline{\Delta(\xi)}} \begin{pmatrix} 1 & 1 \\ \Delta(\xi) & -\Delta(\xi) \end{pmatrix} \begin{pmatrix} \tau_1(\xi) & 0 \\ 0 & \tau_2(\xi) \end{pmatrix} \begin{pmatrix} \overline{\Delta(\xi)} & 1 \\ \overline{\Delta(\xi)} & -1 \end{pmatrix},$$

where

$$\tau_j(\xi) = \frac{\nu_j(\xi)^2}{|\nu_j(\xi)|^2}$$

for $j = 1, 2$.

If $l \in \mathbf{Z}$ is a partial index of the curves $\alpha_1$ and $\alpha_2$ then there exist $C^\alpha$ functions $a$ and $b$ on $\bar{D}$, holomorphic on $D$ and with no common zeros on $\bar{D}$, such that on $\partial D$ we have

$$B(\xi) \begin{pmatrix} \overline{a(\xi)} \\ \overline{b(\xi)} \end{pmatrix} = \xi^l \begin{pmatrix} a(\xi) \\ b(\xi) \end{pmatrix}.$$

Therefore, on $\partial D$ the following two equations hold:

$$\tau_1 \Delta \overline{(\Delta a + b)} = -\xi^l \bar{\Delta}(\Delta a + b), \tag{3}$$

$$\tau_2 \Delta \overline{(\Delta a - b)} = -\xi^l \bar{\Delta}(\Delta a - b). \tag{4}$$

We know that the total index is given as the winding number of the determinant

$$\det(B) = \Delta\bar{\Delta}^{-1}\tau_1\tau_2.$$

Hence the total index is

$$k = 2W(\Delta) + W(\tau_1) + W(\tau_2) = 2(W(\Delta) + W(\nu_1) + W(\nu_2)).$$

We assume from now on that $\alpha_1$ and $\alpha_2$ represent the boundary values of some two-sheeted analytic variety $V$ over $D$ with boundary in $T_1 \cup T_2$. That is, we assume there exist $C^\alpha$ functions $p, q$ on $\bar{D}$, holomorphic on $D$, such that $V$ is given by

$$w^2 - p(z)w + q(z) = 0$$

and such that

$$p(\xi) = \alpha_1(\xi) + \alpha_2(\xi), \qquad q(\xi) = \alpha_1(\xi)\alpha_2(\xi)$$

for every $\xi \in \partial D$ ($p$ and $q$ are actually in $C^{2-0}(\bar{D})$ according to [5]). This condition also implies that, for every symmetric polynomial $P$ of two variables, the function $\xi \mapsto P(\alpha_1(\xi), \alpha_2(\xi))$ on $\partial D$ has a holomorphic extension into $D$. In particular this implies that

$$W(\Delta) = \tfrac{1}{2}W(\Delta^2) \geq 0$$

since, as is well known, the winding number of a disc algebra function, which is nonzero on $\partial D$, is a nonnegative integer.

Multiplying equations (3) and (4) yields

$$\tau_1\tau_2\Delta^2\overline{(\Delta^2 a^2 - b^2)} = \xi^{2l}\overline{\Delta^2}(\Delta^2 a^2 - b^2) \tag{5}$$

on $\partial D$. We denote

$$W(\nu_1) = n_1, \quad W(\nu_2) = n_2, \quad W(\Delta) = n_{12}.$$

Thus there exist real functions $u_1, u_2, u_{12}$, and $v_{12}$ on $\partial D$ such that

$$\tau_1(\xi) = \xi^{2n_1}e^{2iu_1(\xi)}, \qquad \tau_2(\xi) = \xi^{2n_2}e^{2iu_2(\xi)}, \tag{6}$$

and

$$\Delta(\xi) = \xi^{n_{12}}e^{v_{12}(\xi)+iu_{12}(\xi)}. \tag{7}$$

Substituting (6) and (7) into (5), we have

$$e^{2iu(\xi)}\overline{(\Delta^2 a^2 - b^2)(\xi)} = \xi^{2(l-2n_{12}-n_1-n_2)}(\Delta^2 a^2 - b^2)(\xi),$$

where $u = u_1 + u_2 + 2u_{12}$. Let $Hu$ denote the unique harmonic conjugate of $u$ for which $(Hu)(0) = 0$, and let $K := e^{-i(u+iHu)}$. Then $K$ has a holomorphic extension into $D$ with no zeros on $\bar{D}$ and is such that, on $\partial D$,

$$e^{2iu} = K^{-1}\bar{K}.$$

Therefore,

$$\overline{K(\Delta^2 a^2 - b^2)} = \xi^{2(l-n_{12})-k}K(\Delta^2 a^2 - b^2) \tag{8}$$

on $\partial D$. Because $\alpha_1$ and $\alpha_2$ are the boundary roots of a two-sheeted variety $V$ over $D$, the function $K(\Delta^2 a^2 - b^2)$ has a holomorphic extension into $D$.

Let us assume for a moment that

$$2(l-n_{12})-k>0. \tag{9}$$

Then the left-hand side of (8) is an antiholomorphic function on $D$ and the right-hand side of (8) is a holomorphic function on $D$ with a zero at 0. This is only possible if

$$(\Delta^2 a^2 - b^2) = 0$$

on $\bar{D}$. Observe that $\Delta^2$ is a well-defined function on $\bar{D}$. If $a$ had a zero on $\bar{D}$ then $b$ would have a zero at the same point, which is impossible. Thus $a$ has no zeros on $\bar{D}$ and $\Delta^2 = b^2/a^2$ on $\bar{D}$. Hence the variety $V$ is reducible. We may assume $\Delta = b/a$. Then equation (3) implies that

$$\tau_1 \frac{b}{a}\overline{2b} = -\xi^l \overline{\left(\frac{b}{a}\right)} 2b$$

on $\partial D$ and hence

$$l = W(\tau_1) = 2W(\nu_1) = 2n_1.$$

The other partial index is

$$k-l = 2(W(\nu_2)+W(\Delta)) = 2n_2 + 2n_{12}.$$

Using the value of $l$ in the inequality (9), we see that this case can only happen if $n_1 - n_2 > 2n_{12}$.

In case the inequality (9) does not hold for any of the partial indices of $V$ (e.g., when $V$ is an irreducible variety), we must have

$$2(l-n_{12})-k \le 0 \quad \text{or} \quad l \le k/2 + n_{12}$$

for both partial indices. Hence we also have

$$k-l \le k/2 + n_{12}.$$

THEOREM 1. *Let $V$ be a two-sheeted analytic variety over $D$ with boundary in the disjoint union $T_1 \cup T_2$ of two maximal real tori fibered over $\partial D$. Let $\alpha_1$ and $\alpha_2$ be the complex functions on $\partial D$ representing the boundary roots of the variety $V$ such that, for every $\xi \in \partial D$, we have $\alpha_j(\xi) \in T_{j,\xi}$ $(j=1,2)$. Also, let $\nu_1(\xi)$ and $\nu_2(\xi)$ be normals to the fibers $T_{1,\xi}$ and $T_{2,\xi}$ at the points $\alpha_1(\xi)$ and $\alpha_2(\xi)$, respectively.*

(1) *If $V$ is reducible and $|W(\nu_1)-W(\nu_2)| \ge W(\Delta)$, then the partial indices of $V$ are $2\max\{W(\nu_1), W(\nu_2)\}$ and $2W(\Delta)+2\min\{W(\nu_1), W(\nu_2)\}$.*

(2) *In the cases*

(a) *$V$ is reducible and $|W(\nu_1)-W(\nu_2)| < W(\Delta)$ or*
(b) *$V$ is irreducible,*

*the partial indices are bounded by*

$$W(\nu_1)+W(\nu_2) \le k_1, k_2 \le W(\nu_1)+W(\nu_2)+2W(\Delta).$$

*In either case, the total index is $k = 2(W(\nu_1)+W(\nu_2)+W(\Delta))$.*

*Proof*. The only part of the theorem we still have to check is the first case. We may assume $W(\nu_1) - W(\nu_2) \geq W(\Delta)$. Using the notation from (6) and (7), let

$$K_1 := e^{i((u_1+u_{12})+iH(u_1+u_{12}))}, \quad K_2 := e^{i((u_2+u_{12})+iH(u_2+u_{12}))}, \quad L := e^{i(u_1+iHu_1)}.$$

For $l = 2W(\nu_1)$, a pair of holomorphic functions $a$ and $b$ that solve (3) and (4) is

$$a := iL, \qquad b := \Delta a = i\Delta L.$$

Recall that $V$ is reducible and hence $\Delta$ has a holomorphic extension into $D$. Let $N = 2(W(\nu_1) - W(\nu_2))$. Since $W(\nu_1) - W(\nu_2) \geq W(\Delta)$, there exists a polynomial $P(\xi) = \sum_{j=0}^{j=N} c_j \xi^j$ such that $c_{N-j} = \overline{c_j}$ for each $j = 0, \ldots, N$ and such that $\Delta$ divides $PK_1 + K_2$. Then a pair of holomorphic functions $a$ and $b$ that solve (3) and (4) for $l = 2W(\Delta) + 2W(\nu_2)$ is

$$a := \frac{i}{\Delta}(PK_1 + K_2), \qquad b := i(PK_1 - K_2).$$

Observe that

$$\det\begin{pmatrix} iL & \frac{i}{\Delta}(PK_1 + K_2) \\ i\Delta L & i(PK_1 - K_2) \end{pmatrix} = 2LK_2$$

is nonzero on $\bar{D}$. □

Results from [8; 9] imply that when both partial indices are greater than or equal to $-1$ there exist a neighborhood $N$ of $(p, q)$ in $(A^\alpha(D))^2$ and a neighborhood $U$ of $(r_1, r_2)$ in $(C^2(\partial D \times S^1))^2$ such that, for each pair $(\tilde{r}_1, \tilde{r}_2) \in U$, the set of discs $(\tilde{p}, \tilde{q}) \in N$ such that

$$\left(\tilde{p}(\xi), \tfrac{1}{2}(\tilde{p}(\xi)^2 - 4\tilde{q}(\xi))\right) \in \Sigma(\tilde{T}_1, \tilde{T}_2)_\xi$$

for every $\xi \in \partial D$ ($\tilde{T}_1$ and $\tilde{T}_2$ are the 2-tori in $\mathbf{C}^2$ defined by the parameterizations $\tilde{r}_1$ and $\tilde{r}_2$ respectively) is a $C^1$ submanifold of $N$ of dimension $2(W(\nu_1) + W(\nu_2) + W(\Delta)) + 2$.

Identifying the space of two-sheeted analytic varieties over $D$ with the space of analytic discs $(A^\alpha(D))^2$, we have the following corollary.

COROLLARY 1. *Let $V$ be a two-sheeted variety over $D$ with boundary in $T_1 \cup T_2$.*

(1) *If $V$ is reducible and $|W(\nu_1) - W(\nu_2)| \geq W(\Delta)$, let $2\max\{W(\nu_1), W(\nu_2)\} \geq 0$ and $2W(\Delta) + 2\min\{W(\nu_1), W(\nu_2)\} \geq 0$.*
(2) *If either $V$ is reducible and $|W(\nu_1) - W(\nu_2)| < W(\Delta)$ or if $V$ is irreducible, let $W(\nu_1) + W(\nu_2) \geq -1$.*

*Then, for every pair $\tilde{T}_1$ and $\tilde{T}_2$ of maximal real tori over $\partial D$ close to $T_1$ and $T_2$, respectively, the family of two-sheeted analytic varieties over $D$ with boundaries in $\tilde{T}_1 \cup \tilde{T}_2$ that are close to $V$ is a $C^1$ submanifold of the space of two-sheeted analytic varieties over $D$ of dimension $2(W(\nu_1) + W(\nu_2) + W(\Delta)) + 2$. These manifolds depend smoothly on $\tilde{T}_1$ and $\tilde{T}_2$.*

There are two major cases of the positions of the tori $T_1$ and $T_2$ that one may consider. One is the case where $T_2$ lies in the unbounded component of $(\partial D \times \mathbf{C}) \setminus T_1$

and the other is the case where $T_2$ lies in the bounded component of $(\partial D \times \mathbf{C}) \setminus T_1$. Of course the roles of $T_1$ and $T_2$ can be exchanged, but this does not produce any new different cases. We will first consider the second case under the assumption that there exists a function $c \in A^\alpha(D)$ such that, for each $\xi \in \partial D$,

$$c(\xi) \in \operatorname{Int} \widehat{T_{2,\xi}} \subset\subset \operatorname{Int} \widehat{T_{1,\xi}}. \tag{10}$$

Here, $\widehat{T_{j,\xi}}$ denotes the closure of the bounded simply connected domain in $\mathbf{C}$ that is bounded by $T_{j,\xi}$; that is, $\widehat{T_{j,\xi}}$ is the polynomial hull of $T_{j,\xi}$ in $\mathbf{C}$ $(j = 1, 2)$. Condition (10) is biholomorphically equivalent to the case

$$0 \in \operatorname{Int} \widehat{T_{2,\xi}} \subset\subset \operatorname{Int} \widehat{T_{1,\xi}}$$

(i.e., $c = 0$). In this case we have the following equalities:

$$W(\Delta) = W(\alpha_1) = W(\nu_1), \qquad W(\alpha_2) = W(\nu_2).$$

Because $\alpha_1$ and $\alpha_2$ are the boundary roots of a two-sheeted analytic variety over $D$, their sum $\alpha_1 + \alpha_2$ and their product $\alpha_1\alpha_2$ have holomorphic extensions into $D$. Thus

$$0 \le W(\alpha_1 + \alpha_2) = W(\alpha_1) = W(\nu_1) \tag{11}$$

and

$$0 \le W(\alpha_1\alpha_2) = W(\alpha_1) + W(\alpha_2) = W(\nu_1) + W(\nu_2). \tag{12}$$

COROLLARY 2. *If the torus $T_2$ lies in the bounded component of $(\partial D \times \mathbf{C}) \setminus T_1$ and there exists a function $c \in A^\alpha(D)$ such that* (10) *holds, then every two-sheeted analytic variety $V$ over $D$ with boundary in $T_1 \cup T_2$ is regular. Also, $W(\nu_1) \ge 0$ and $W(\nu_1) + W(\nu_2) \ge 0$.*

Here (and hereafter), the regularity is meant in the sense of Section 2. That is, both associated partial indices are greater than or equal to $-1$, and not in the usual sense of regularity of a variety.

Henceforth we assume that

$$\widehat{T_{1,\xi}} \cap \widehat{T_{2,\xi}} = \emptyset \tag{13}$$

for every $\xi \in \partial D$. We also assume that there exists a function $c \in A^\alpha(D)$ with no zeros on $\partial D$ such that $W(c)$ is an even integer and such that

$$\gamma_1(\xi) \in \operatorname{Int} \widehat{T_{1,\xi}} \quad \text{and} \quad \gamma_2(\xi) \in \operatorname{Int} \widehat{T_{2,\xi}} \tag{14}$$

for every $\xi \in \partial D$. Here, $\gamma_1$ and $\gamma_2$ are the square roots of $c$ over $\partial D$, that is, the $C^\alpha$ functions on $\partial D$ such that $\gamma_1(\xi)^2 = \gamma_2(\xi)^2 = c(\xi)$ and $\gamma_1(\xi) = -\gamma_2(\xi)$ for every $\xi \in \partial D$.

REMARKS. (1) It is enough to assume that $c$ is from the disc algebra.

(2) The assumption on the existence of such a function $c$ is biholomorphically equivalent to the assumption that there exists a two-sheeted analytic variety $V_o$ over $D$ defined by functions from the disc algebra and with boundary roots $\gamma_1$ and $\gamma_2$ such that, for each $\xi \in \partial D$ we have $\gamma_j(\xi) \in \operatorname{Int} \widehat{T_{j,\xi}}$ $(j = 1, 2)$.

(3) Using a biholomorphism we may even assume that $c$ is a finite Blaschke product.

We write

$$\alpha_1 = \gamma_1 + \tilde{\alpha}_1 \quad \text{and} \quad \alpha_2 = \gamma_2 + \tilde{\alpha}_2 \tag{15}$$

for some nonzero $C^\alpha$ functions $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ for whose winding numbers we have

$$W(\tilde{\alpha}_1) = W(\nu_1) = n_1 \quad \text{and} \quad W(\tilde{\alpha}_2) = W(\nu_2) = n_2,$$

respectively. Furthermore, for $j = 1, 2$ we have

$$\alpha_j^2 = \gamma_j^2 + \tilde{\alpha}_j(2\gamma_j + \tilde{\alpha}_j).$$

Because (13) holds, the functions

$$\xi \mapsto 2\gamma_j(\xi) + \tilde{\alpha}_j(\xi) \quad (j = 1, 2)$$

are nonzero on $\partial D$. Condition (13) actually implies much more; given that the winding number is homotopy invariant, we conclude for $j = 1, 2$ that

$$W(2\gamma_j + \tilde{\alpha}_j) = W(2\gamma_j) = \tfrac{1}{2}W(c). \tag{16}$$

Denote $A_j := \tilde{\alpha}_j(2\gamma_j + \tilde{\alpha}_j)$, $j = 1, 2$. Then $A_1$ and $A_2$ are nonzero $C^\alpha$ functions on $\partial D$ such that, for $j = 1, 2$,

$$\alpha_j^2 = \gamma_j^2 + A_j = c + A_j \tag{17}$$

and

$$W(A_j) = W(\nu_j) + \tfrac{1}{2}W(c). \tag{18}$$

Since $\alpha_1$ and $\alpha_2$ represent the boundary roots of a two-sheeted analytic variety over $D$, (17) implies that the functions

$$\xi \mapsto A_1(\xi) + A_2(\xi) \quad \text{and} \quad \xi \mapsto A_1(\xi)A_2(\xi)$$

have holomorphic extensions into $D$. Thus

$$W(A_1A_2) = W(A_1) + W(A_2) = W(\nu_1) + W(\nu_2) + W(c) \geq 0.$$

Also, the homotopy invariance of the winding number implies that

$$W(\Delta) = W(\gamma_1 - \gamma_2) = \tfrac{1}{2}W(c). \tag{19}$$

PROPOSITION 2. *If conditions* (13) *and* (14) *hold, then* $W(\nu_1) + W(\nu_2) + W(c) \geq 0$ *and* $W(\Delta) = W(c)/2$. *Also, the partial indices* $k_1$ *and* $k_2$ *of an irreducible two-sheeted analytic variety* $V$ *over* $D$ *with boundary in* $T_1 \cup T_2$ *satisfy the following inequalities*:

$$-W(c) \leq W(\nu_1) + W(\nu_2) \leq k_1, k_2 \leq W(\nu_1) + W(\nu_2) + W(c).$$

More can be said when the holomorphic function $c$ has a holomorphic square root. Adding and multiplying equations (15), we see that the functions

$$\tilde{\alpha}_1 + \tilde{\alpha}_2 \quad \text{and} \quad \gamma_1(\tilde{\alpha}_2 - \tilde{\alpha}_1) + \tilde{\alpha}_1\tilde{\alpha}_2$$

on $\partial D$ have holomorphic extensions into $D$. Multiplying the first function by $\gamma_1$ (which has a holomorphic extension into $D$) and adding and subtracting it from the second function, we get that the functions

$$\tilde{\alpha}_2(2\gamma_1+\tilde{\alpha}_1) \quad \text{and} \quad \tilde{\alpha}_1(-2\gamma_1+\tilde{\alpha}_2)$$

holomorphically extend into $D$. We observe that these two functions have no zeros on $\partial D$ and that condition (13) implies (16). Hence

$$W(\tilde{\alpha}_2(2\gamma_1+\tilde{\alpha}_1)) = W(\tilde{\alpha}_2)+\tfrac{1}{2}W(c) \geq 0$$

and

$$W(\tilde{\alpha}_1(-2\gamma_1+\tilde{\alpha}_2)) = W(\tilde{\alpha}_1)+\tfrac{1}{2}W(c) \geq 0.$$

PROPOSITION 3. *If, in addition to condition* (14), *the function $c$ has a holomorphic square root, then*

$$W(\nu_1) \geq -\tfrac{1}{2}W(c) \quad \textit{and} \quad W(\nu_2) \geq -\tfrac{1}{2}W(c).$$

PROPOSITION 4. *If $W(c)=0$ then every two-sheeted analytic variety over $D$ with boundary in $T_1 \cup T_2$ is reducible.*

*Proof.* Let $V$ be a two-sheeted analytic variety over $D$ with boundary in $T_1 \cup T_2$. We know from (19) that

$$W(\Delta) = \tfrac{1}{2}W(c) = 0.$$

Therefore, the winding number of the discriminant $\Delta^2$ of the variety $V$ is 0 and so it has a holomorphic square root. Hence $V$ is reducible. □

## 4. Examples

EXAMPLE 1. Let $a_1 \neq a_2$ be two positive real numbers and let $T_1 = \partial D \times a_1(\partial D)$ and $T_2 = \partial D \times a_2(\partial D)$. Let $n \in \mathbf{Z}$ be a nonnegative integer and let $V$ be the variety given by

$$V = \{\,(z,w) \in \bar{D} \times \mathbf{C};\ w^2-(a_1+a_2)z^n w + a_1 a_2 z^{2n} = 0\,\},$$

where $V$ is a variety with boundary in $T_1 \cup T_2$. The winding numbers of the corresponding normals are $W(\nu_1) = W(\nu_2) = n$ and $W(\Delta) = n$. Also, a short calculation shows that the partial indices are $2n$ and $4n$. Thus the total index is $6n$ and, for each pair of maximal real tori close to $T_1 \cup T_2$, there exists a $(6n+2)$-parameter family of two-sheeted analytic varieties over $D$ close to $V$. Each reducible two-sheeted analytic variety over $D$ close to $V$ with boundary in $T_1 \cup T_2$ is given by an equation of the form

$$(w-a_1 e^{i\varphi}B_1(z)\cdots B_n(z))(w-a_2 e^{i\psi}C_1(z)\cdots C_n(z)) = 0,$$

where $\varphi, \psi \in \mathbf{R}$ and $B_1, \ldots, B_n$ and $C_1 \ldots, C_n$ are automorphisms of the unit disc $D$ close to the identity with the leading factor equal to 1. Hence the family of reducible two-sheeted analytic varieties over $D$ with boundary in $T_1 \cup T_2$ is a submanifold of the codimension $2n = (6n+2)-(1+1+2n+2n)$ of the manifold of all two-sheeted analytic varieties over $D$ with boundaries in $T_1 \cup T_2$ that are close to $V$. Thus, most of the two-sheeted varieties over $D$ with boundaries in $T_1 \cup T_2$ and close to $V$ are irreducible.

EXAMPLE 2. Let $c \in A^2(D)$ be such that it has no zeros on $\partial D$ and its winding number $W(c)$ is an even integer. Let $a > 0$ be a positive constant such that $a < \min_{\partial D}|c(z)|$, and let

$$T_1 \cup T_2 = \{ (\xi, w) \in \partial D \times \mathbf{C};\ |w^2 - c(\xi)| = a \}.$$

Let $B$ be a finite Blaschke product, and let variety $V$ with boundary in $T_1 \cup T_2$ be given by the equation

$$w^2 = c(z) + aB(z).$$

The winding numbers of the normals to the fibers of $T_1 \cup T_2$ along the boundary of $V$ are then $W(\nu_1) = W(\nu_2) = W(B) - \frac{1}{2}W(c)$ and $W(\Delta) = \frac{1}{2}W(c)$. The partial indices are $2W(B) - W(c)$ and $2W(B)$, and the total index is $4W(B) - W(c)$.

EXAMPLE 3. Let $T_1 = \{ (\xi, w) \in \partial D \times \mathbf{C};\ |w| = \frac{1}{2} \}$ and $T_2 = \{ (\xi, w) \in \partial D \times \mathbf{C};\ |w| = 1 \}$. Let $p$ be a disc algebra function such that

$$p(\partial D) \subseteq \{ (x, y) \in \mathbf{R}^2 = \mathbf{C};\ \tfrac{4}{9}x^2 + 4y^2 = 1 \},$$

and let $V$ be given by the equation

$$w^2 - p(z)w + \tfrac{1}{2} = 0. \tag{20}$$

The solutions of the equation (20) over $\partial D$ are

$$\alpha_1 = \tfrac{4}{3}(p - \tfrac{1}{2}\bar{p}) \quad \text{and} \quad \alpha_2 = \tfrac{1}{2}\overline{\alpha_1}.$$

Also,

$$\begin{aligned}\alpha_1\overline{\alpha_1} &= \tfrac{16}{9}(p - \tfrac{1}{2}\bar{p})(\bar{p} - \tfrac{1}{2}p)\\ &= \tfrac{4}{9}(\operatorname{Re} p)^2 + 4(\operatorname{Im} p)^2 = 1.\end{aligned}$$

Hence $V$ is a two-sheeted analytic variety over $D$ with boundary in $T_1 \cup T_2$. The winding numbers of the corresponding normals to the fibers are

$$W(\nu_1) = W(p) \quad \text{and} \quad W(\nu_2) = -W(p).$$

Thus, one of the winding numbers of the normals to the fibers can be an arbitrary negative integer—that is, there is no lower bound as in Proposition 3. Recall that Corollary 2 implies that every two-sheeted analytic variety over $D$ with boundary in $T_1 \cup T_2$ is regular and that it is always the case that $W(\nu_1) + W(\nu_2) \geq 0$.

EXAMPLE 4. Let $T_1 \cup T_2 = \{ (\xi, w) \in \partial D \times \mathbf{C};\ |w^2 - \xi^2| = \frac{1}{2} \}$. Let $V = \{ (z, w) \in D \times \mathbf{C};\ w^2 = z^2 + \frac{1}{2} \}$ be a variety with boundary in $T_1 \cup T_2$. As shown in Example 2, the winding numbers of the corresponding normals are both $-1$ and the partial indices are 0 and $-2$ (variety $V$ is not regular!). On the other hand, if we just slightly perturb one of the tori closer to its center (i.e., for $\frac{1}{2} > \varepsilon > 0$

let $\tilde{T}_1$ be the component of $T_1 \cup T_2$ closer to the curve $\xi \mapsto (\xi, \xi)$ and let $\tilde{T}_2$ be the component of $\{ (\xi, w) \in \partial D \times \mathbf{C};\ |w^2 - \xi^2| = \frac{1}{2} - \varepsilon \}$ closer to the curve $\xi \mapsto (\xi, -\xi))$, then there is no two-sheeted analytic variety over $D$ with boundary in $\tilde{T}_1 \cup \tilde{T}_2$ close to $V$. Indeed, let $\tilde{V}$ be a two-sheeted analytic variety over $D$ with boundary in $\tilde{T}_1 \cup \tilde{T}_2$. Then, on $\partial D$, we have

$$\tilde{\alpha}_1^2(\xi) = \xi^2 + \tilde{A}_1(\xi) \quad \text{and} \quad \tilde{\alpha}_2^2(\xi) = \xi^2 + \tilde{A}_2(\xi),$$

where $\tilde{A}_1$ and $\tilde{A}_2$ are $C^\alpha$ functions on $\partial D$ such that $|\tilde{A}_1(\xi)| = \frac{1}{2}$ and $|\tilde{A}_2(\xi)| = \frac{1}{2} - \varepsilon$ for every $\xi \in \partial D$. Hence

$$\begin{aligned} W(\tilde{A}_1) &= W(\tilde{A}_1 - \tilde{A}_2) = W(\tilde{\alpha}_1^2 - \tilde{\alpha}_2^2) \\ &= W(\tilde{\alpha}_1 - \tilde{\alpha}_2) + W(\tilde{\alpha}_1 + \tilde{\alpha}_2) \geq 1 + 0 = 1. \end{aligned}$$

On the other hand, we know from (18) that $W(\tilde{A}_1) = W(\tilde{\nu}_1) + 1$ and thus

$$W(\tilde{\nu}_1) \geq 0; \tag{21}$$

that is, at least one of the winding numbers of the normals to the fibers of $\tilde{T}_1 \cup \tilde{T}_2$ along the boundary roots of $\tilde{V}$ is greater than or equal to 0. Hence whatever $\frac{1}{2} > \varepsilon > 0$ we choose, none of the varieties $\tilde{V}$ can be uniformly close to $V$. Observe also that the inequality (21), together with Proposition 3, shows that every two-sheeted analytic variety $\tilde{V}$ over $D$ with boundary in $\tilde{T}_1 \cup \tilde{T}_2$ is regular.

EXAMPLE 5. Let $T$ be a maximal real torus in $\partial D \times \mathbf{C}$ such that, for each $\xi \in \partial D$, the fiber $T_\xi = \pi_2(T \cap (\{\xi\} \times \mathbf{C}))$ of $T$ over $\xi$ is a disjoint union of two Jordan curves $J_\xi^1$ and $J_\xi^2$ in $\mathbf{C}$. Let $V$ be a two-sheeted variety over $D$ with boundary in $T$—that is, there exist functions $p$ and $q$ from $A^\alpha(D)$ such that

$$V = \{ (z, w) \in \bar{D} \times \mathbf{C};\ w^2 - p(z)w + q(z) = 0 \}$$

and such that, for every $\xi \in \partial D$, each curve $J_\xi^1$ and $J_\xi^2$ contains exactly one root of the equation $w^2 - p(\xi)w + q(\xi) = 0$. Similarly as before, one defines a 3-dimensional maximal real manifold $\Sigma(T) \subseteq \partial D \times \mathbf{C}^2$ whose each fiber $\Sigma(T)_\xi = \pi_3(\Sigma(T) \cap (\{\xi\} \times \mathbf{C}^2))$ is a maximal real 2-torus in $\mathbf{C}^2$ as well as an analytic disc

$$z \mapsto \big(p(z), \tfrac{1}{2}(p(z)^2 - 4q(z))\big) \tag{22}$$

with boundary in the maximal real fibration $\{\Sigma(T)_\xi\}_{\xi \in \partial D}$. One can again define the partial indices of a two-sheeted variety $V$ over $D$ with boundary in $T$ as the partial indices of the disc (22) with boundary in $\{\Sigma(T)_\xi\}_{\xi \in \partial D}$.

Let $F\colon \mathbf{C}^2 \to \mathbf{C}^2$ be defined as $F(z, w) := (z^2, w)$. Then the preimage $F^{-1}(T) = T_1^o \cup T_2^o$ is the union of two disjoint maximal real tori over $\partial D$. Also, $V^o := F^{-1}(V)$ is a two-sheeted variety over $D$ with boundary in $T_1^o \cup T_2^o$. Let $k_1 \geq k_2$ be the partial indices of the variety $V$ with boundary in $T$ and let $k_1^o \geq k_2^o$ be the partial indices of $V^o$ with boundary in $T_1^o \cup T_2^o$. Then the form of the map $F$ implies that

$$k_1^o = 2k_1 \quad \text{and} \quad k_2^o = 2k_2,$$

and one can easily prove statements similar to those before. For example, if there exists a function $c \in A^\alpha(D)$ with no zeros on $\partial D$ such that $W(c)$ is an odd integer and such that, for every $\xi \in \partial D$,

$$\sqrt{c(\xi)} \in \operatorname{Int} \widehat{J^1_\xi} \quad \text{and} \quad -\sqrt{c(\xi)} \in \operatorname{Int} \widehat{J^2_\xi},$$

then

$$k_1 \geq -W(c) \quad \text{and} \quad k_2 \geq -W(c).$$

These inequalities imply that when $W(c) = 1$—for example, if $c(\xi) = \xi$, which is (modulo a biholomorphism) a canonical case for $W(c) = 1$—then every two-sheeted variety over $D$ with boundary in $T$ is regular. Together with the area bounds, which are not too hard to obtain, we may apply Gromov's compactness theorem [10; 12, Thm. 4.2.1, p. 247] to obtain the existence of a two-sheeted analytic variety $V$ over $D$ with boundary in $T$. This, however, is nothing new! The result of Forstnerič [7] implies that there exists an analytic function $a \in A^\alpha(D)$ such that $a(\xi) \in T^o_{1,\xi}$ for every $\xi \in \partial D$. Let $\Gamma(a)$ be the graph of $a$. Then $V = F(\Gamma(a))$ is a two-sheeted variety over $D$ with boundary in $T$.

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