Singular Integrals along Submanifolds of Finite Type

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1. Introduction

Let $n \in \mathbf{N}$, $n \geq 2$, and $y \in \mathbf{R}^n$. Let $K(y)$ be a Calderón–Zygmund kernel, that is,

$$K(y) = \frac{\Omega(y)}{|y|^n}, \tag{1.1}$$

where Ω is homogeneous of degree 0 and satisfies

$$\int_{\mathbf{S}^{n-1}} \Omega(y)\, d\sigma(y) = 0. \tag{1.2}$$

Let $B(0,1)$ denote the unit ball centered at the origin in $\mathbf{R}^n$, let $d \in \mathbf{N}$, and let $\Phi\colon B(0,1) \to \mathbf{R}^d$ be a C^∞ mapping. Define the singular integral operator T_Φ on $\mathbf{R}^d$ by

$$(T_\Phi f)(x) = \text{p.v.} \int_{B(0,1)} f(x - \Phi(y))K(y)\, dy. \tag{1.3}$$

The following L^p boundedness theorem can be found in Stein [7].

THEOREM A. *Let T_Φ be given as above. Suppose that*

(i) Φ *is of finite type at* 0, *and*

(ii) $\Omega \in C^1(\mathbf{S}^{n-1})$.

Then, for $1 < p < \infty$, there exists a constant $C_p > 0$ such that

$$\|T_\Phi f\|_{L^p(\mathbf{R}^d)} \leq C_p \|f\|_{L^p(\mathbf{R}^d)} \tag{1.4}$$

for every $f \in L^p(\mathbf{R}^d)$.

It is well known that T_Φ may fail to be bounded on L^p for any p if condition (i) is removed (the precise definition of a finite type mapping will be reviewed in the next section). The purpose of this paper is to establish the L^p boundedness of T_Φ when condition (ii) is replaced by the following weaker condition:

(ii′) $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q > 1$.

This yields the following theorem.

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THEOREM B. *Let T_Φ be given as before. Suppose that*

(i) Φ *is of finite type at* 0, *and*
(ii′) $\Omega \in L^q(\mathbf{S}^{n-1})$ *for some* $q > 1$.

Then T_Φ is a bounded operator from $L^p(\mathbf{R}^d)$ to itself for $1 < p < \infty$.

We shall also establish the L^p boundedness for the corresponding maximal truncated singular integrals.

THEOREM C. *Let*

$$(T_\Phi^* f)(x) = \sup_{\varepsilon>0}\left|\int_{\varepsilon\le|y|<1} f(x-\Phi(y))K(y)\,dy\right|. \tag{1.5}$$

Suppose Φ and Ω satisfy conditions (i) *and* (ii′), *respectively. Then the operator T_Φ^* is bounded from $L^p(\mathbf{R}^d)$ to itself for $1 < p < \infty$.*

We shall first establish an estimate for some oscillatory integrals.

2. Oscillatory Integrals

We shall begin with a definition.

DEFINITION 2.1. Let U be an open set in $\mathbf{R}^n$ and $\phi\colon U \to \mathbf{R}^d$ a smooth mapping. For $x_0 \in U$ we say that ϕ is of *finite type* at x_0 if, for each unit vector $\eta \in \mathbf{R}^d$, there is a multi-index α with $|\alpha| \ge 1$ so that

$$\partial_x^\alpha[\phi(x)\cdot\eta]|_{x=x_0} \ne 0. \tag{2.1}$$

The following lemma is a special case of Lemma 3.2 in [5].

LEMMA 2.2. *Let $\psi \in C^\infty(\mathbf{R})$, $\varphi \in C_0^\infty(\mathbf{R})$, $a < b$, and $k \in \mathbf{N}$. Assume that $|\psi^{(k)}(x)| \le r \le M$ for $x \in [a,b]$ and $|\psi^{(k+1)}(x)| \le M$ for $x \in [a-r, b+r]$. Then there exists a positive constant C which depends only on k, M, and φ such that*

$$\left|\int_a^b e^{i\lambda\psi(x)}\varphi(x)\,dx\right| \le C|\lambda|^{-\varepsilon/k}\int_{a-r}^{b+r}|\psi^{(k)}(x)|^{-\varepsilon(1+1/k)}\,dx \tag{2.2}$$

holds for $\lambda \in \mathbf{R}$ and $\varepsilon \in [0,1]$.

LEMMA 2.3. *Let $\Phi\colon B(0,1) \to \mathbf{R}^d$ be a smooth mapping and let Ω be a homogeneous function of degree* 0. *Suppose that Φ is of finite type at zero and $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q > 1$. Then there are $\delta, C > 0$, $N \in \mathbf{N}$, and $j_0 \in \mathbf{Z}_-$ such that*

$$\left|\int_{2^{j-1}\le|y|<2^j} e^{-i\xi\cdot\Phi(y)}\frac{\Omega(y)}{|y|^n}\,dy\right| \le C(2^{Nj}|\xi|)^{-\delta} \tag{2.3}$$

for all $j \le j_0$ and $\xi \in \mathbf{R}^d$.

Proof. For any $\eta_0 \in \mathbf{S}^{d-1}$, there exists a nonzero multi-index $\alpha_0 = \alpha(\eta_0)$ such that

$$\partial_y^{\alpha_0}[\eta_0 \cdot \Phi(y)]|_{y=0} \neq 0. \tag{2.4}$$

Let $k = |\alpha_0|$ and define $G_k\colon B(0,1) \times \mathbf{S}^{d-1} \to \mathbf{R}$ by

$$G_k(y,\eta) = \sum_{|\alpha|=k} [\eta \cdot \partial_y^\alpha \Phi(y)] y^\alpha. \tag{2.5}$$

Then, by (2.4) and (2.5) we have

$$\frac{\partial^{\alpha_0} G_k}{\partial y^{\alpha_0}}(0,\eta_0) \neq 0 \quad \text{and} \quad \frac{\partial^{\beta} G_k}{\partial y^{\beta}}(0,\eta_0) = 0$$

for all β with $|\beta| \le k-1$.

Let V_k be the space of homogeneous polynomials of degree k in n variables and let $d(k) = \dim(V_k)$. Then there are $d(k)$ vectors $e_1, \dots, e_{d(k)} \in \mathbf{S}^{n-1}$ such that

$$\mathcal{B} = \{(e_1 \cdot y)^k, (e_2 \cdot y)^k, \dots, (e_{d(k)} \cdot y)^k\}$$

forms a basis of V_k. Thus there exists an $e \in \{e_1, \dots, e_{d(k)}\}$ such that

$$\begin{cases} (e \cdot \nabla_y)^l G_k(y,\eta)|_{(0,\eta_0)} = 0 & \text{for } 0 \le l \le k-1; \\ (e \cdot \nabla_y)^k G_k(y,\eta)|_{(0,\eta_0)} \neq 0. \end{cases} \tag{2.6}$$

By using a rotation if necessary, we may assume that $e = (1, 0, \dots, 0)$. Let $y' = (y_2, \dots, y_n)$. Then, by (2.6) and the Malgrange preparation theorem [4], there exist $h > 0$, an open neighborhood $W_0 \subset \mathbf{S}^{d-1}$ of η_0, smooth functions $a_0(y',\eta), \dots, a_{k-1}(y',\eta)$ on $[-h,h]^{n-1} \times W_0$, and a nonzero smooth function $c(y,\eta)$ on $[-h,h]^n \times W_0$ such that

$$G_k(y,\eta) = c(y,\eta)(y_1^k + a_{k-1}(y',\eta) y_1^{k-1} + \cdots + a_0(y',\eta)) \tag{2.7}$$

for $(y,\eta) \in [-h,h]^n \times W_0$. Thus, for any $\varepsilon < 1/k$ and any open neighborhood W of η_0 satisfying $\bar{W} \subset W_0$, we have

$$\sup_{\eta \in W} \int_{|y| \le h/2} |G_k(y,\eta)|^{-\varepsilon}\, dy = C(h,\varepsilon,W) < \infty. \tag{2.8}$$

By the compactness of $\mathbf{S}^{d-1}$, there exist $h_0 \in (0,1/4)$, δ_0, $A > 0$, and $k_0 \in \mathbf{N}$ such that, for any $\eta \in \mathbf{S}^{d-1}$,

$$\int_{|y| \le h_0} |G_k(y,\eta)|^{-\delta_0}\, dy \le A \tag{2.9}$$

holds for some $k \in \{1, 2, \dots, k_0\}$.

Let

$$B = \max_{|y| \le 1/2} \sum_{|\beta| \le k_0} |\partial_y^\beta \Phi(y)| \quad \text{and}$$

$$j_0 = \max\{\, j \in \mathbf{Z} \mid 2^j \le \min[(4B)^{-1}, h_0/4]\,\}.$$

For $\xi \in \mathbf{R}^d\backslash\{0\}$, choose $k \in \{1, \ldots, k_0\}$ so that (2.9) holds for $\eta = \xi/|\xi|$. By letting $\varepsilon = \delta_0/(2q')$ and applying Lemma 2.2, we obtain for all $j \le j_0$

$$\begin{aligned}
&\left| \int_{2^{j-1} \le |y| < 2^j} e^{-i\xi \cdot \Phi(y)} \frac{\Omega(y)}{|y|^n} \, dy \right| \\
&\qquad \le C|\xi|^{-\varepsilon/k} \int_{\mathbf{S}^{n-1}} |\Omega(y)| \left[\int_{1/4}^{5/4} |G_k(2^j t y, \eta)|^{-\varepsilon(1+1/k)} \, dt \right] d\sigma(y) \\
&\qquad \le C 2^{-j(1+(n-1)/q')} |\xi|^{-\varepsilon/k} \int_{|y| \le h_0} |\Omega(y)| |y|^{-(n-1)/q} |G_k(y, \eta)|^{-2\varepsilon} \, dy \\
&\qquad \le C \|\Omega\|_{L^q(\mathbf{S}^{n-1})} (2^{Nj} |\xi|)^{-\varepsilon/k},
\end{aligned}$$

where $N = [\varepsilon^{-1} k(1 + (n-1)/q')] + 1$. By letting $\delta = \varepsilon/k_0$ we see that (2.3) holds when $2^{Nj}|\xi| \ge 1$. Because (2.3) always holds when $2^{Nj}|\xi| < 1$, Lemma 2.3 is proved. □

LEMMA 2.4. *Let $m \in \mathbf{N}$ and let $R(\cdot)$ be a real-valued polynomial on $\mathbf{R}^n$ with* $\deg(R) \le m - 1$. *Suppose*

$$P(y) = \sum_{|\alpha|=m} a_\alpha y^\alpha + R(y), \tag{2.10}$$

Ω is homogeneous of degree 0, and $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q > 1$. Then there exists a $C = C(m, n) > 0$ such that

$$\left| \int_{2^{j-1} \le |y| < 2^j} e^{iP(y)} \frac{\Omega(y)}{|y|^n} \, dy \right| \le C \|\Omega\|_q \left[2^{mj} \sum_{|\alpha|=m} |a_\alpha| \right]^{-1/2q'm} \tag{2.11}$$

holds for any $j \in \mathbf{Z}$ and $\{a_\alpha\} \subset \mathbf{R}$.

Proof. Let

$$I(y) = \int_{1/2}^{1} \exp\left\{ i \left[(2^j t)^m \sum_{|\alpha|=m} a_\alpha y^\alpha + R(2^j t y) \right] \right\} \frac{dt}{t}.$$

Then $|I(y)| \le 1$. By van der Corput's lemma [8] we also have

$$|I(y)| \le C 2^{-j} \left| \sum_{|\alpha|=m} a_\alpha y^\alpha \right|^{-1/m},$$

which implies

$$|I(y)| \le C 2^{-j/2q'} \left| \sum_{|\alpha|=m} a_\alpha y^\alpha \right|^{-1/2q'm}.$$

Thus

$$
\begin{aligned}
&\left|\int_{2^{j-1}\le|y|<2^j} e^{iP(y)}\frac{\Omega(y)}{|y|^n}\,dy\right| \\
&\qquad\le \int_{\mathbf{S}^{n-1}}|\Omega(y)I(y)|\,d\sigma(y) \\
&\qquad\le C2^{-j/2q'}\|\Omega\|_q\left[\int_{\mathbf{S}^{n-1}}\left|\sum_{|\alpha|=m}a_\alpha y^\alpha\right|^{-1/2m}d\sigma(y)\right]^{1/q'} \\
&\qquad\le C\|\Omega\|_q\left[2^{mj}\sum_{|\alpha|=m}|a_\alpha|\right]^{-1/2mq'},
\end{aligned}
$$

where the last inequality follows from a result of Ricci and Stein [6, p. 183, Cor. 2]. □

3. Maximal functions and Singular Integrals

We shall need the following result from [2] (see also [1] and [3]).

LEMMA 3.1. *Let $l, m \in \mathbf{N}$ and let $\{\sigma_{s,k} : 0 \le s \le l$ and $k \in \mathbf{Z}\}$ be a family of measures on $\mathbf{R}^m$ with $\sigma_{0,k} = 0$ for every $k \in \mathbf{Z}$. Let $\{\alpha_{sj} : 1 \le s \le l$ and $1 \le j \le 2\} \subset \mathbf{R}^+$, $\{\eta_s : 1 \le s \le l\} \subset \mathbf{R}^+\setminus\{1\}$, $\{N_s : 1 \le s \le l\} \subset \mathbf{N}$, and $L_s : \mathbf{R}^m \to \mathbf{R}^{N_s}$ be linear transformations for $1 \le s \le l$. Suppose:*

(i) $\|\sigma_{s,k}\| \le 1$ *for* $k \in \mathbf{Z}$ *and* $1 \le s \le l$;

(ii) $|\hat{\sigma}_{s,k}(\xi)| \le C(\eta_s^k|L_s\xi|)^{-\alpha_{s2}}$ *for* $\xi \in \mathbf{R}^m$, $k \in \mathbf{Z}$, *and* $1 \le s \le l$;

(iii) $|\hat{\sigma}_{s,k}(\xi) - \hat{\sigma}_{s-1,k}(\xi)| \le C(\eta_s^k|L_s\xi|)^{\alpha_{s1}}$ *for* $\xi \in \mathbf{R}^m$, $k \in \mathbf{Z}$, *and* $1 \le s \le l$; *and*

(iv) *for some* $q > 1$ *there exists* $A_q > 0$ *such that*

$$\left\|\sup_{k\in\mathbf{Z}}\big||\sigma_{s,k}| * f\big|\right\|_{L^q(\mathbf{R}^m)} \le A_q\|f\|_{L^q(\mathbf{R}^m)}$$

for all $f \in L^q(\mathbf{R}^m)$ *and* $1 \le s \le l$.

Then, for every $p \in \left(\frac{2q}{q+1}, \frac{2q}{q-1}\right)$, there exists a positive constant C_p such that

$$\left\|\sum_{k\in\mathbf{Z}}\sigma_{l,k} * f\right\|_{L^p(\mathbf{R}^m)} \le C_p\|f\|_{L^p(\mathbf{R}^m)} \tag{3.1}$$

and

$$\left\|\left(\sum_{k\in\mathbf{Z}}|\sigma_{l,k} * f|^2\right)^{1/2}\right\|_{L^p(\mathbf{R}^m)} \le C_p\|f\|_{L^p(\mathbf{R}^m)} \tag{3.2}$$

hold for all $f \in L^p(\mathbf{R}^m)$. The constant C_p is independent of the linear transformations $\{L_s\}_{s=1}^l$.

For given Φ and Ω we define the maximal operator $\mathcal{M}_{\Omega,\Phi}$ by

$$(\mathcal{M}_{\Omega,\Phi}f)(x) = \sup_{k\in\mathbf{Z}_-}\left|\int_{2^{k-1}\le|y|<2^k} f(x-\Phi(y))\frac{\Omega(y)}{|y|^n}\,dy\right|. \tag{3.3}$$

The next lemma follows immediately from [9, p. 477, Prop. 1] (see also [10]).

LEMMA 3.2. *Let $\mathcal{P} = (P_1, \dots, P_d)$, where P_j is a real-valued polynomial on $\mathbf{R}^n$ and $\deg(\mathcal{P}) = \max_{1\le j\le d} \deg(P_j)$. Suppose that $\Omega \in L^1(\mathbf{S}^{n-1})$. Then the operator $\mathcal{M}_{\Omega,\mathcal{P}}$ is bounded on $L^p(\mathbf{R}^d)$ for $1 < p \le \infty$. The bound for $\|\mathcal{M}_{\Omega,\mathcal{P}}\|_{p,p}$ may depend on n, d, $\|\Omega\|_1$, and $\deg(\mathcal{P})$, but it is independent of the coefficients of the polynomials $P_j(\cdot)$.*

In what follows we shall establish the L^p boundedness for the maximal operator $\mathcal{M}_{\Omega,\Phi}$ when $\Omega \in L^q$ $(q > 1)$ and Φ is a smooth mapping of finite type. This can be viewed as an extension of [9, p. 476, Thm. 1] (which corresponds to the case $\Omega \in L^\infty$).

THEOREM 3.3. *Suppose that $\Phi\colon B(0,1) \to \mathbf{R}^d$ is smooth and of finite type at 0 and that Ω is homogeneous of degree 0 with $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q > 1$. Then the operator $\mathcal{M}_{\Omega,\Phi}$ is bounded on $L^p(\mathbf{R}^d)$ for all p satisfying $1 < p \le \infty$.*

Proof. Without loss of generality we may assume that $\Omega \ge 0$. For $k \in \mathbf{Z}_-$, we define the measures $\sigma_{\Phi,k}$ on $\mathbf{R}^d$ by

$$\int_{\mathbf{R}^d} F\, d\sigma_{\Phi,k} = \int_{2^{k-1}\le |y|<2^k} f(\Phi(y)) \frac{\Omega(y)}{|y|^n}\, dy. \tag{3.4}$$

By Lemma 2.3, there exist $\delta, C > 0$, $N \in \mathbf{N}$, and $k_0 \in \mathbf{Z}_-$ such that

$$|\hat{\sigma}_{\Phi,k}(\xi)| \le C(2^{Nk}|\xi|)^{-\delta} \tag{3.5}$$

for all $\xi \in \mathbf{R}^d$ and $k \le k_0$. For $\Phi = (\Phi_1, \dots, \Phi_d)$ we let $\mathcal{P} = (P_1, \dots, P_d)$, where

$$P_j(y) = \sum_{|\beta|\le N-1} \frac{1}{\beta!} \frac{\partial^\beta \Phi_j}{\partial y^\beta}(0) y^\beta \tag{3.6}$$

for $1 \le j \le d$. Then we have

$$|\hat{\sigma}_{\Phi,k}(\xi) - \hat{\sigma}_{\mathcal{P},k}(\xi)| \le C(2^{Nk}|\xi|), \tag{3.7}$$

where $\sigma_{\mathcal{P},k}$ is given by (3.4) with Φ replaced by $\mathcal{P}$.

We now choose a $\psi \in \mathcal{S}(\mathbf{R}^d)$ such that $\hat{\psi}(\xi) \equiv 1$ for $|\xi| \le 1/2$ and $\hat{\psi}(\xi) \equiv 0$ for $|\xi| \ge 1$. Let $\psi_t(x) = t^{-d}\psi(x/t)$ for $t > 0$ and define the measures $\{\nu_k\}$ by

$$\nu_k = \sigma_{\Phi,k} - \sigma_{\mathcal{P},k} * \psi_{2^{Nk}}. \tag{3.8}$$

Then, by (3.5) and (3.7), we obtain

$$|\hat{\nu}_k(\xi)| \le C \min\{(2^{Nk}|\xi|)^{-\delta}, 2^{Nk}|\xi|\} \tag{3.9}$$

for $\xi \in \mathbf{R}^d$ and $k \le k_0$. If we let Sf denote the square function

$$(Sf)(x) = \left(\sum_{k\le k_0} |\nu_k * f(x)|^2\right)^{1/2}, \tag{3.10}$$

then we have

$$\sup_{k\le k_0}|(\sigma_{\Phi,k}*f)(x)|\le (Sf)(x)+C(\mathcal{M}_{\Omega,\mathcal{P}}\mathcal{M}_{\mathrm{HL}}f)(x) \tag{3.11}$$

and

$$\sup_{k\le k_0}\big|(|\nu_k|*f)(x)\big|\le (Sf)(x)+2C(\mathcal{M}_{\Omega,\mathcal{P}}\mathcal{M}_{\mathrm{HL}}f)(x) \tag{3.12}$$

where $\mathcal{M}_{\mathrm{HL}}$ denotes the Hardy–Littlewood maximal operator on $\mathbf{R}^d$. By (3.9), (3.10), and Plancherel's theorem,

$$\|Sf\|_2\le C\|f\|_2; \tag{3.13}$$

when combined with Lemma 3.2 and (3.12), this implies that

$$\Big\|\sup_{k\le k_0}\big||\nu_k|*f\big|\Big\|_2\le C\|f\|_2. \tag{3.14}$$

By (3.9), (3.14), and Lemma 3.1, we get

$$\|Sf\|_p\le C_p\|f\|_p \tag{3.15}$$

for all p satisfying $4/3<p<4$. By repeating the arguments in (3.13) $\to$ (3.14) $\to$ (3.15) with $p=2$ replaced by $p=4/3+\varepsilon$ ($\varepsilon\to 0^+$), we obtain that

$$\|Sf\|_p\le C_p\|f\|_p \tag{3.16}$$

for $8/7<p<8$. By such arguments we eventually obtain that S is bounded on L^p for $1<p<\infty$, which implies that

$$\Big\|\sup_{k\le k_0}|\sigma_{\Phi,k}*f|\Big\|_p\le C_p\|f\|_p \tag{3.17}$$

for $1<p<\infty$. This shows that $\mathcal{M}_{\Omega,\Phi}$ is bounded on L^p for $1<p<\infty$. Since $\|\mathcal{M}_{\Omega,\Phi}f\|_\infty\le C\|f\|_\infty$ holds trivially, the proof of Theorem 3.3 is complete. □

We shall now give a proof of our main result.

Proof of Theorem B. Let δ, N, and $\mathcal{P}$ be given as in the proof of Theorem 3.3. For $1\le j\le d$ we let $a_{j\beta}=(1/\beta!)\partial^\beta\Phi_j/\partial y^\beta(0)$. For $0\le s\le N$ we define $\mathcal{Q}^s=(Q_j^s,\dots,Q_d^s)$ by

$$Q_j^s(y)=\sum_{|\beta|\le s}a_{j\beta}y^\beta,\quad j=1,\dots,d \tag{3.18}$$

when $0\le s\le N-1$ and $\mathcal{Q}^N=\Phi$. Let $\sigma_{s,k}=\sigma_{\mathcal{Q}^s,k}$. Then, by (3.18) and Lemma 2.4, we have

$$|\hat\sigma_{s,k}(\xi)-\hat\sigma_{s-1,k}(\xi)|\le C\bigg(2^{sk}\sum_{|\beta|=s}\bigg|\sum_{j=1}^d a_{j\beta}\xi_j\bigg|\bigg) \tag{3.19}$$

and

$$|\hat\sigma_{s,k}(\xi)|\le C\bigg[2^{sk}\sum_{|\beta|=s}\bigg|\sum_{j=1}^d a_{j\beta}\xi_j\bigg|\bigg]^{-1/2q's} \tag{3.20}$$

for $k \le k_0$ and $1 \le s \le N-1$. By (1.2), (3.5), (3.7), (3.19)–(3.20), Lemmas 3.1–3.2, and Theorem 3.3, we obtain that

$$\Big\| \sum_{k \le k_0} \sigma_{\Phi,k} * f \Big\|_p \le C_p \|f\|_p$$

for $1 < p < \infty$. Therefore T_Φ is a bounded operator on $L^p(\mathbf{R}^d)$ for $1 < p < \infty$. □

Finally, we point out that Theorem C can be proved by combining the estimates obtained here and the techniques in [1] and [3]. We omit the details.

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