# Pluricomplex Green Functions and the Dirichlet Problem for the Complex Monge–Ampère Operator

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## 0. Preliminaries

Let us recall some important notions which will be used here. Let $D$ be an open subset in $\mathbb{C}^n$, and denote by PSH($D$) the cone of plurisubharmonic functions $u: D \to [-\infty, +\infty[$ on $D$ not identically equal to $-\infty$ on any component of $D$.

Let $u \in \mathrm{PSH}(D)$. For $a \in D$ and $0 < r < d_a := \mathrm{dist}(a; \mathbb{C}^n \setminus D)$, we set

$$M_u(a, r) := \int_{|\xi|=1} u(a + r\xi)\, d\sigma(\xi), \tag{0.1}$$

where $d\sigma(\xi)$ is the normalized area measure on the unit Euclidean sphere in $\mathbb{C}^n$. It is well known that the function $r \mapsto M_u(a, r)$ is increasing and convex in $\log r$. Then the following limit exists:

$$\nu(u; a) := \lim_{r \to 0^+} \frac{M_u(a, r)}{\log r}. \tag{0.2}$$

By [Ki1], (0.2) coincides with the following definition [L1]:

$$\nu(u; a) := \lim_{r \to 0^+} \frac{\sigma_u(B(a, r))}{\omega_{2n-2} r^{2n-2}}, \tag{0.3}$$

where $\omega_{2n-2}$ is the volume of the unit ball in $\mathbb{C}^{n-1}$ and $\sigma_u := \frac{1}{2\pi} \Delta u \beta_n = \frac{1}{2\pi} dd^c u \wedge \beta_{n-1}$; $\beta$ is the standard Kälherian form of $\mathbb{C}^n$ and $\beta_{n-1} := \beta^{n-1}/(n-1)!$.

The number defined by (0.3) is called the *Lelong number* of the current $dd^c u$ at the point $a$, or the *density* of $u$ at the point $a$. It is well known that the Lelong number is independent of holomorphic changes of coordinates [S; D3]. Thus it is possible to define this number for plurisubharmonic functions on complex manifolds. In fact, the definition (0.3) is meaningful in this case.

The function $\nu(u; \cdot): a \mapsto \nu(u; a)$ defined by (0.3) is upper semicontinuous on $D$, with values in $\mathbb{R}_+$. If $u(a) > -\infty$ then $\nu(u; a) = 0$. If $u = \log|f|$, where $f$ is a holomorphic function such that $f(a) = 0$ and not identically zero on a neighborhood of $a$, then $\nu(\log|f|; a)$ is an integer equal to the multiplicity of the zero of $f$ at the point $a$.

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Although we will not need to use it in the sequel, it is interesting to recall the following deep theorem of Siu [S; Ki1].

THEOREM. *Let $u \in \mathrm{PSH}(D)$. Then, for any $c > 0$, the set*

$$A(u,c) := \{z \in D;\ \nu(u;a) \geq c\}$$

*is an analytic subset of $D$. In particular, if $u^{-1}(-\infty) \Subset D$ then the sets $A(u,c)$ $(c > 0)$ are finite subsets of $D$.*

We only need to use this particular case, which will be proved independently in Section 3 (see Lemma 3.2).

## 1. Hyperconvex Manifolds and Admissible Plurisubharmonic Functions

Let us recall some definitions that will be needed later. The following definition is due to Stehlé [St].

DEFINITION 1.1. A complex analytic manifold $D$ is called a *hyperconvex* manifold if there exists a plurisubharmonic function $\rho\colon D \to [-1, 0[$ such that, for every $\varepsilon > 0$, $D_\varepsilon := \{z \in D;\ \rho(z) < -\varepsilon\} \Subset D$. Such a function will be called a *bounded exhaustion* of $D$.

After Kerzman and Rosay [KeR], any bounded pseudoconvex domain of $\mathbb{C}^n$ with smooth boundary is hyperconvex. More generally, Demailly [D2] has proved that any bounded pseudoconvex domain of $\mathbb{C}^n$ of Lipschitz boundary is hyperconvex.

DEFINITION 1.2. Let $D$ a hyperconvex manifold of pure dimension $n$. A plurisubharmonic function $\varphi\colon D \to [-\infty, +\infty[$ will be called an *admissible* plurisubharmonic function on $D$ if it satisfies the following conditions:

(i) the function $e^\varphi$ is continuous on $D$ and the pluripolar set of $\varphi$ defined by

$$S_\varphi := \{z \in D;\ \varphi(z) = -\infty\} \tag{1.1}$$

is compact;

(ii) the density set of $\varphi$ defined by

$$A_\varphi := \{a \in D;\ \nu(\varphi;a) > 0\} \tag{1.2}$$

is dense in $S_\varphi$ and meets each component of $D$.

The need for condition (ii) in this definition will be explained after the proof of Theorem 2.1. For now, let us show that it is always possible to construct such a function on any pseudoconvex domain of $\mathbb{C}^n$. More precisely, we have the following result.

LEMMA 1.3. *Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex domain and let $K$ be a closed complete pluripolar subset of $\Omega$. Then, given any sequence $(a_j)_{j\in J}$ $(J \subset \mathbb{N})$ of points of $K$ and any sequence $(\nu_j)_{j\in J}$ of positive numbers such that $\sum_{j\in J} \nu_j < +\infty$,* *there exists a continuous plurisubharmonic function $\varphi$ on $\Omega$ such that $S_\varphi = K$ and $\nu(\varphi; a_j) = \nu_j$ for any $j \in J$.*

*Proof.* It is well known [Z1] that there exists a continuous plurisubharmonic function $\psi: \Omega \to [-\infty, +\infty[$ such that $S_\psi = K$. Let $\chi: \mathbb{R} \to \mathbb{R}$ be any increasing convex function such that $\lim_{t \to -\infty} \chi(t) = -\infty$ and $\lim_{t \to -\infty} \chi(t)/t = 0$. We may take, for example, the following function: $\chi(t) = -\log(1-t)$ if $t < 0$ and $\chi(t) = t$ if $t \geq 0$. Then the function $\chi \circ \psi$ is plurisubharmonic and continuous on $\Omega$, $S_{\chi \circ \psi} = K$, and $\nu(\chi \circ \psi; z) \equiv 0$ on $\Omega$.

On the other hand, consider the function

$$u(z) := \sum_{j \in J} \nu_j \log|z - a_j|, \quad z \in \Omega.$$

This function is clearly plurisubharmonic on $\Omega$ and continuous on $\Omega \setminus K$. Let us prove that $\nu(u; a_j) = \nu_j$ for any $j \in J$. Indeed, it is clear that $\nu(u; a_j) \geq \nu_j$ for any $j \in J$. If $n = 1$ then the equality is an easy consequence of the formula $\nu(u; a_j) = (1/2\pi)\Delta u(\{a_j\}) = \nu_j$, where $\Delta$ is the Laplace operator in $\mathbb{C}$. To prove the same result in $\mathbb{C}^n$, observe first that if $\zeta \in \mathbb{C}^n \setminus \{0\}$ and $j \in J$ are fixed then the one complex variable function $\lambda \to u_\zeta(\lambda) := u(a_j + \lambda\zeta)$, which is the restriction of $u$ to the corresponding domain of the complex line $L_\zeta := \{a_j + \lambda\zeta; \lambda \in \mathbb{C}\}$, is subharmonic and satisfies $\nu(u_\zeta; 0) = \nu_j$, where $\nu(u_\zeta; 0)$ is the point mass of the Laplacian of the subharmonic function $u_\zeta$ at 0. Now our claim follows from a result of Siu that asserts that $\nu(u; a_j) = \nu(u_\zeta; a_j)$ for almost all $\zeta \in \mathbb{C}^n$ (see [S; H]).

Now, to get a function $\varphi$ satisfying the conditions of the lemma, it is enough to put $\varphi := \chi(\psi) + u$. □

## 2. Pluricomplex Green Function on Hyperconvex Manifolds

Introduced in [Z2] was a generalized pluricomplex Green function associated with any admissible plurisubharmonic function on a hyperconvex domain in $\mathbb{C}^n$, together with some applications to the problem of interpolation-approximation of holomorphic functions. Here we want to study in more detail the properties of the pluricomplex Green function and its connection with the degenerate Dirichlet problem for the complex Monge–Ampère operator.

From now on, $D$ will be a hyperconvex manifold of pure dimension $n$, and $\varphi: D \to [-\infty, +\infty[$ will be an admissible plurisubharmonic function on $D$ (cf. Definition 1.2).

We can define a generalized pluricomplex Green function given by

$$G_D(z; \varphi) := \sup\{u(z);\ u \in P_0(D, \varphi)\}, \quad z \in D, \tag{2.1}$$

where $P_0(D, \varphi)$ denotes the class of PSH functions $u$ on $D$ such that $u \leq 0$ on $D$ and $\nu(u; \cdot) \geq \nu(\varphi; \cdot)$ on $D$. The function defined by (2.1) will be called the *weighted pluricomplex Green function* of $D$ associated to the admissible plurisubharmonic function $\varphi$.

We now state the following result [Z2].

THEOREM 2.1. *The weighted pluricomplex Green function $G := G_D(\cdot;\varphi)$ satisfies the following properties:*

$$G \in \mathrm{PSH}(D) \cap L^\infty_{\mathrm{loc}}(D \setminus K), \tag{2.2}$$

*where $K := S_\varphi$;*

$$G(z) \to 0 \quad as \ z \to \partial D; \tag{2.3}$$

$$\nu(G,a) = \nu(\varphi,a) \quad \forall a \in D, \tag{2.4}$$

*in particular, $G(a) = -\infty$ if $\nu(\varphi,a) > 0$; and*

$$(dd^c G)^n = 0, \tag{2.5}$$

*in the sense of currents on $D \setminus K$.*

*Proof.* First let us prove that $P_0(D,\varphi) \neq \emptyset$. Indeed, let $\rho\colon D \to [-1,0[$ be a bounded exhaustion of $D$ and let $\omega$ be an open set such that $A_\varphi \Subset \omega$. Choose $c_1$ and $c_2$ such that $\sup_{\bar\omega}\rho < c_1 < c_2$. Then, if $\omega_c := \{z \in D;\ \rho(z) < c\}$ we have $\omega \Subset \omega_{c_1} \Subset \omega_{c_2}$. Choose $\alpha > 0$, and $\beta \in \mathbb{R}$ such that $\alpha\rho + \beta \le \varphi$ on $\partial\omega_{c_1}$ and $\alpha\rho + \beta \ge \varphi$ on $\partial\omega_{c_2}$. Then the function defined by

$$\tilde\varphi(z) := \begin{cases} \varphi(z) - \beta & \text{if } z \in \bar\omega_{c_1} \\ \sup\{\alpha\rho(z), \varphi(z) - \beta\} & \text{if } z \in \omega_{c_2} \setminus \bar\omega_{c_1} \\ \alpha\rho(z) & \text{if } z \in D \setminus \omega_{c_2} \end{cases} \tag{2.6}$$

is plurisubharmonic on $D$, $\tilde\varphi \le 0$ on $D$, and $\nu(\tilde\varphi;\cdot) = \nu(\varphi;\cdot)$ on $D$. Thus $\tilde\varphi \in P_0(D,\varphi)$ and we have

$$\tilde\varphi \le G_D(\cdot,\varphi) \ \text{ on } D. \tag{2.7}$$

This proves that $P_0(D,\varphi) \neq \emptyset$ and gives a subsolution. Let us write $G := G_D(\cdot;\varphi)$. By a classical result of Lelong, its upper semicontinuous regularization $G^*$ is PSH on $D$. From (2.7) and the fact that $\tilde\varphi + \beta = \varphi$ on a neighborhood of $S_\varphi$, we see that $\nu(G;\cdot) \le \nu(\varphi;\cdot)$ on $D$.

By a classical result of Choquet (see [L1]), there exists a sequence $\{u_j\}_{j\ge1}$ from the class $P_0(D,\varphi)$ such that

$$u := \sup_{j\ge1} u_j \le G_D \ \text{ and } \ u^* = G_D^* \ \text{ on } D \setminus K. \tag{2.8}$$

Observe that the class $P_0(D,\varphi)$ has the following lattice property:

$$u, v \in P_0(D,\varphi) \Rightarrow \sup\{u,v\} \in P_0(D,\varphi).$$

From this property it follows that we can suppose the sequence $\{u_j\}_{j\ge1}$ to be increasing.

Fix $a \in D$. For each $j \ge 1$, the function $r \to M_{u_j}(a,r)$ is a convex function of $\log r$; hence, for $0 < r < r_0 < d_a$, we have

$$\frac{M_{u_j}(a,r) - M_{u_j}(a,r_0)}{\log r - \log r_0} \ge \nu(u_j,a) \ge \nu(\varphi,a). \tag{2.9}$$

Because $u_j \uparrow u^* = G^*$ on $D \setminus X$ where $X := \{u < u^*\}$, and since (by [L1]) the intersection of $X$ with each sphere of $\mathbb{C}^n$ is of zero area, we deduce from (2.9) that, for $0 < r < r_0 < d_a$, the following formula holds when $j \to +\infty$:

$$\frac{M_{G^*}(a, r) - M_{G^*}(a, r_0)}{\log r - \log r_0} \geq \nu(\varphi, a).$$

From this inequality it follows that, when $r \to 0$, we have

$$\nu(G^*, \cdot) \geq \nu(\varphi, \cdot) \text{ on } D. \tag{2.10}$$

From (2.8) and (2.10) we deduce that $G^* \in P_0(D, \varphi)$, $G^* = G$, and $\nu(G; \cdot) = \nu(\varphi; \cdot)$ on $D$.

This immediately implies (2.4), whence (2.2) and (2.3) are a consequence of (2.7). The property (2.5) can be proved in a classical way by showing that the function $G$ is maximal on $D \setminus K$ (see [BT2] for this kind of argument). A more general property will be proved later. □

Let us explain why we need condition (ii) in Definition 1.2 of "admissibility".

REMARK. It is clear that (2.1) makes sense for any plurisubharmonic function $\varphi$ on $D$ such that $K_\varphi := \overline{S_\varphi}$ is compact. Then Theorem 2.1 is valid in this situation and so the pluricomplex Green function $G := G_D(\cdot; \varphi)$ associated to $\varphi$ by (2.1) satisfies the equality $A_G = A_\varphi$. Hence $G(z) = -\infty$ for any $z \in A_\varphi$.

On the other hand, let a continuous function $\varphi$ plurisubharmonic on $D$ be given such that $S_\varphi \Subset D$. Then, as we will prove in Lemma 3.2, the set $A_\varphi$ is countable. Therefore, applying Lemma 1.3, we obtain an admissible plurisubharmonic function $\varphi'$ on $D$ such that $A_{\varphi'} = A_\varphi$. Then $G_D(\cdot; \varphi') = G_D(\cdot; \varphi)$ on $D$. Because (by Theorem 2.1) the function $G_D(\cdot; \varphi')$ is locally bounded on $D \setminus S'_\varphi = D \setminus \overline{A_\varphi}$, the pluripolar set of the Green function associated to $\varphi$ is contained in $\overline{A_\varphi}$, which may be smaller than the pluripolar set $S_\varphi$ of the given function $\varphi$.

Since we are mainly interested in obtaining a pluricomplex Green function with a prescribed compact complete pluripolar set $K$, it is natural to consider an admissible plurisubharmonic function $\varphi$ with $\overline{A_\varphi} = S_\varphi = K$. We will make this question more precise in what follows, after the proof of Theorem 2.6.

We now give a generalized version of the classical Schwarz lemma, which is an easy consequence of the definition of the Green function.

PROPOSITION 2.2. *Let $u \in \mathrm{PSH}(D)$ such that $\nu(u; \cdot) \geq \nu(\varphi; \cdot)$ on $D$ and $u \leq M$ on $D$. Then*

$$u(z) \leq M + G_D(z; \varphi) \quad \forall z \in D.$$

EXAMPLE 2.3. Let $D$ be a hyperconvex domain in $\mathbb{C}^n$, $a \in D$, and $\varphi_a(z) := \log|z - a|$. Then the function $G_D(\cdot; \varphi_a)$ coincides with the pluricomplex Green function $G_D(\cdot; a)$ with a logarithmic pole at the point $a$, which has been studied by several authors [L1; L2; K1; D2]. For example, if $\|\cdot\|$ is a norm on $\mathbb{C}^n$ and $B_r := \{z \in \mathbb{C}^n;\ \|z\| < r\}$, $r > 0$, then

$$G_{B_r}(z; 0) = \log \frac{\|z\|}{r}, \quad z \in B_r.$$

More generally, let $A := \{(a_1, \nu_1), \dots, (a_p, \nu_p)\} \subset D \times \mathbb{R}_+^\star$ and set

$$\varphi_A(z) := \sum_{j=1}^{j=p} \nu_j \log|z - a_j|, \quad z \in D.$$

Then the weighted pluricomplex Green function $G_D(\cdot, \varphi_A) =: G_D(\cdot; A)$ associated to this admissible function is nothing more than the pluricomplex Green function with a finite number of weighted poles considered earlier in [L2; Po; Za]. For more examples and interesting geometric applications of the pluricomplex Green function, see [JP; K3].

By [D2; L2], the function $G_D(\cdot, A)$, is continuous and satisfies the following complex Monge–Ampère equation:

$$(dd^c G_D(\cdot, A))^n = (2\pi)^n \sum_{j=1}^{p} \nu_j^n \delta_{a_j}, \tag{2.11}$$

in the sense of currents on $D$.

Let us now give a generalization of the preceding example.

EXAMPLE 2.4. Let $(a_j)_{j\geq 1}$ be an infinite sequence of distinct points in $\mathbb{C}^n \setminus \{0\}$ converging to 0. Then $K := \{a_j;\ j \geq 1\} \cup \{0\}$ is a compact set. Let $D$ be a hyperconvex domain of $\mathbb{C}^n$ such that $K \subset D$. Consider a sequence $(\varepsilon_j)_{j\geq 0}$ of positive numbers such that $\sum_{j=0}^{+\infty} \varepsilon_j < +\infty$, and define the following function:

$$\varphi(z) := \varepsilon_0 \log|z| + \sum_{j=1}^{+\infty} \varepsilon_j \log|z - a_j|, \quad z \in D.$$

It is clear that $\varphi$ is continuous and plurisubharmonic on $D$. Moreover, as in the proof of Lemma 1.3, one can show that $\nu(\varphi; a_j) = \varepsilon_j$ for $j \geq 1$ and that $\nu(\varphi; 0) = \varepsilon_0$. This means that $A_\varphi = K = S_\varphi$. Thus $\varphi$ is an admissible plurisubharmonic function on $D$ and, by Theorem 2.1, the corresponding weighted pluricomplex Green function $G := G_D(\cdot; \varphi)$ satisfies the property $S_G = K$. Moreover, by Remark 2.7, $G$ is a continuous function on $D$.

EXAMPLE 2.5. Let $D$ be a bounded domain in $\mathbb{C}$, regular with respect to the classical Dirichlet problem, and let $K$ be a polar compact subset of $D$. By a classical result [T], there exist a sequence $\{a_j\}_{j\geq 1}$ of extremal points in $K$ and a sequence $\{\varepsilon_j\}_{j\geq 1}$ of positive real numbers such that the function defined by

$$\psi(z) := \sum_{j=1}^{+\infty} \varepsilon_j \log|z - a_j|,$$

is subharmonic on $\mathbb{C}$, harmonic on $\mathbb{C} \setminus K$, and satisfies $S_\psi := \psi^{-1}(-\infty) = K$ (Evans's potential of $K$). Hence the Green function $G := G_D(\cdot; \psi)$ of $D$ associated to the admissible subharmonic function $\psi$ coincides with

$$G'(z) := \sum_{j=1}^{+\infty} \varepsilon_j G_D(z, a_j), \quad z \in D,$$

and satisfies $S_G := \{z \in D; G(z) = -\infty\} = S_\psi = K$.

Indeed, it is clear that $\Delta G' = \sum_{j=1}^{+\infty} \varepsilon_j \delta_{a_j} = \Delta\psi$ and that $\nu(\psi, a_j) = \varepsilon_j$ for any $j \geq 1$. It is easy to see that $\Delta G = \sum_{a \in A_\psi} \nu(\psi, a)\delta_a$. Hence $\Delta G = \Delta G' = \Delta\psi$ in the sense of measures on $D$. This means that $G - G'$ and $G' - \psi$ are harmonic on $D$. Hence $S_G = S'_G = S_\psi = K$ and since $G$ and $G'$ tend to zero at the boundary of $D$, it follows by the maximum principle that $G = G'$. Hence $S_G = K$, that is, the polar set of the Green function coincides with the given polar compact set $K$.

The foregoing example shows that the situation is very simple in one complex variable, since a weighted Green function with prescribed poles on a given polar compact subset can always be constructed on a regular domain, thanks to the existence of an Evans' potential.

In the pluricomplex case, the situation is more complicated. Before discussing this case, let us first prove the following result about the continuity of the Green function. Denote by $P_0^c(D; \varphi)$ the class of continuous PSH functions $u$ such that $u \leq 0$ on $D$ and $\nu(u; \cdot) \geq \nu(\varphi; \cdot)$ on $D$. Then we have the following fundamental result.

THEOREM 2.6. *Let $D$ be a hyperconvex domain and $\varphi$ an admissible plurisubharmonic function on $D$. Then*

$$G_D(z; \varphi) = \sup\{u(z);\ u \in P_0^c(D; \varphi)\} \quad \forall z \in D \setminus S_\varphi. \tag{2.12}$$

*In particular, $G_D(\cdot; \varphi)$ is continuous on $D \setminus S_\varphi$.*

*Proof.* Let $G := G_D(\cdot; \varphi)$ and $G' := \sup P_0^c(D; \varphi)$. Then we see that $G' \leq G$ on $D$. To prove the converse inequality, consider the continuous exhausting plurisubharmonic function $\tilde{\varphi}$ defined from $\varphi$ by (2.6), and consider its open sublevel sets $D_\alpha := \{z \in D; \tilde{\varphi}(z) < -\alpha\} \Subset D$ for $\alpha > 0$. Let $u \in P_0(D; \varphi)$. By the fundamental approximation theorem of Demailly [D3], there exists a sequence $(u_m)_{m\in\mathbb{N}}$ of continuous plurisubharmonic functions on $D$ satisfying the following estimates:

$$u(z) - \frac{c_1}{m} \leq u_m(z) \leq \sup_{|\zeta - z| < r} u(\zeta) + \frac{1}{m} \log \frac{c_2}{r^n} \quad \forall z \in D^{(r)}, \tag{2.13}$$

where $D^{(r)} := \{z \in D; \operatorname{dist}(z, \partial D) > r\}$; and

$$\nu(u; z) - \frac{n}{m} \leq \nu(u_m; z) \leq \nu(u; z) \quad \forall z \in D. \tag{2.14}$$

Fix $\varepsilon > 0$ and $\alpha > 0$ small enough. By upper semicontinuity, there exists $r > 0$ such that $r < \operatorname{dist}(\bar{D}_\alpha, \partial D)$. Choose $m_0 > 0$ so large that $(1/m)\log(c_2/r) < \varepsilon$

and $(1/m)c_1 < \varepsilon$ for $m \geq m_0$. Then it follows from (2.13) that the following estimate holds:

$$u(z) - \varepsilon \leq u_m(z) \leq \varepsilon \quad \forall z \in D_\alpha. \tag{2.15}$$

Define the following two functions:

$$v_m(z) := u_m(z) + \frac{n}{m}\tilde{\varphi}(z) - \varepsilon, \quad z \in D; \tag{2.16}$$

$$w_m(z) := \begin{cases} \sup\{v_m(z), \tilde{\varphi}(z) + \alpha\}, & z \in D_\alpha, \\ \tilde{\varphi}(z) + \alpha, & z \in D \setminus D_\alpha. \end{cases} \tag{2.17}$$

Since $v_m$ is plurisubharmonic continuous on $D$ and negative on $D_\alpha$, it follows from (2.17) that $w_m$ is plurisubharmonic on $D$.

Therefore, by (2.14) and (2.15), $w_m - \alpha \in P_0^c(D; \varphi)$. Hence, by (2.15)–(2.17) we have $u + \frac{n}{m}\tilde{\varphi} - 2\varepsilon - \alpha \leq w_m - \alpha \leq G'$ on $D_\alpha$ for $m \geq m_0$. This yields the following estimates:

$$u(z) + \frac{n}{m}\tilde{\varphi}(z) \leq G'(z) + 2\varepsilon + \alpha \quad \forall z \in D_\alpha, \ \forall m \geq m_0.$$

As $m$ tends to infinity and $\alpha$, $\varepsilon$ tend to zero, we obtain the required inequality and the theorem is proved. □

REMARK 2.7. The continuity of the Green function on $D \setminus K$ ($K := S_\varphi$) has been proved using a deep theorem of Demailly. It is interesting to observe that, under the condition $\sum_{a \in A_\varphi} \nu(\varphi; a) < +\infty$ (which is always satisfied in one variable; see also Example 2.4), we can give an elementary proof of continuity using only the theorem of Demailly–Lelong [D2; L2] about the continuity of the Green function for a finite number of weighted poles.

Indeed, let $(A_j)_{j \geq 1}$ be an increasing sequence of finite nonempty subsets of $A_\varphi$ such that $A_\varphi = \bigcup_{j \geq 1} A_j$. For each $j \in \mathbb{N}$, let

$$\psi_j(z) := \sum_{a \in A_\varphi \setminus A_j} \nu(\varphi; a) \log|z - a|, \quad z \in D.$$

Then it follows from our hypothesis that $\psi_j$ is plurisubharmonic on $D$. Let us prove that the sequence $\{G_j\}_{j \geq 1}$ converges uniformly on each compact subset of $D \setminus K$. Indeed, let $E$ be a compact subset of $D \setminus K$. Modifying $\psi_j$ in a neighborhood of the boundary of $D$, as in (2.6), we get a plurisubharmonic function $\tilde{\psi}_j$ for which the following inequalities hold:

$$G_j(z) + \tilde{\psi}_j(z) \leq G(z) \leq G_j(z) \quad \forall z \in D, \ \forall j \geq 1. \tag{2.18}$$

Put $\rho := \operatorname{dist}(E; K)$. Then $\rho > 0$ and, by (2.18), we conclude that

$$0 \leq G_j(z) - G(z) \leq -\log \rho \sum_{a \in A_\varphi \setminus A_j} \nu(\varphi; a) \quad \forall z \in E, \ \forall j \geq 1.$$

From this estimate it follows that the sequence $\{G_j\}_{j \geq 1}$ converges uniformly on each compact subset of $D \setminus K$. Because each function $G_j$ is continuous on $D$, we deduce that $G$ is continuous on $D \setminus K$.

Theorem 2.6 and Examples 2.4 and 2.5 suggest the following question.

QUESTION 1. Let $D$ be a hyperconvex manifold, $K$ a complete pluripolar compact subset of $D$, and $\varphi$ an admissible plurisubharmonic function on $D$ such that $S_\varphi = K$. Under what extra condition on $\varphi$ is the weighted pluricomplex Green function $G := G_D(\cdot; \varphi)$ continuous on $D$?

Observe that (by Theorem 2.6) it is enough to have $S_G = S_\varphi$ and (by Theorem 2.1) this is always the case if the admissible function $\varphi$ satisfies the condition $A_\varphi = S_\varphi$; see Example 2.4.

When the weighted pluricomplex Green function is continuous on $D$, we will say that $G := G_D(\cdot; \varphi)$ is a *pluricomplex Green potential* for the pair $(K, D)$. In one complex variable, as Example 2.5 shows, such a function always exists and can be constructed from an Evans potential (see [T]) for any polar compact set $K \subset \mathbb{C}$.

In several complex variables, we do not know if an analog of the Evans potential exists. More precisely, we want to ask the following question.

QUESTION 2. Let $D$ be a Stein manifold and $K$ a complete pluripolar compact subset of $D$. Does there exists a continuous plurisubharmonic function $\psi$ on $D$ such that $S_\psi = K$ and $(dd^c\psi)^n = (2\pi)^n \sum_{a\in A_\psi} \nu(\psi; a)^n \delta_a$ in the sense of measures in $D$?

When such a function exists, we will call it a *pluricomplex Evans's potential* for $(K, D)$.

As in the one-dimensional case, we can always construct a pluricomplex Green potential for $(K, D)$ if it is known that a pluricomplex Evans potential exists. More precisely, we can use Theorem 3.3 from the next section to show the following result, which implies that the two problems are equivalent.

PROPOSITION 2.8. *Let $D$ be a hyperconvex manifold, and let $K \subset D$ be a complete pluripolar compact subset of $D$ such that the pair $(K, D)$ has an Evans potential $\psi$. Then the weighted pluricomplex Green function $G := G_D(\cdot; \psi)$ is continuous on $D$ and $S_G = K$.*

*Proof.* Consider for small $\eta > 0$ the open set $D_\eta := \{z \in D; \psi(z) < \eta\}$. Then $D_\eta$ is hyperconvex and, by Theorem 3.3 (to be proved in the next section), we have $G_{D_\eta}(z; \psi) = \psi(z) - \eta$ for $z \in D$. By Proposition 2.2, it follows that $G_D(z; \psi) \leq \psi(z) - \eta$ for $z \in D_\eta$. This implies that $G_D(z; \psi) = -\infty$ for $z \in K$ and proves that $G_D(\cdot; \psi)$ is continuous on each point of $K$. Therefore, the conclusion of the theorem follows from Theorem 2.6. □

Examples 2.4 and 2.5 and the following result will permit us to construct more examples of pairs $(K, D)$ having a pluricomplex Green potential.

PROPOSITION 2.9. *Let $\{K_j\}$ be a sequence of complete pluripolar compact subsets of a hyperconvex manifold $D$ such that $K := \bigcup_j K_j$ is compact. If each pair $(K_j, D)$ has a pluricomplex Green potential, then the pair $(K, D)$ has also a pluricomplex Green potential.*

*Proof.* We already know from [Z1] that $K$ is a complete pluripolar subset of $D$. For each $j \in \mathbb{N}$, let $G_j$ be a pluricomplex Green potential of the pair $(K_j, G_j)$. Choose an increasing sequence $\{E_j\}$ of compacts in $D$ such that $D \setminus K = \bigcup_j E_j$ and a decreasing sequence $\{s_j\}_j$ of real numbers decreasing to 0 such that $\sum_j s_j \inf_{E_j} G_j > -\infty$. Put $\varphi(z) = \sum_j s_j G_j(z)$, $z \in D$. Then $\varphi$ is an admissible plurisubharmonic function on $D$ and $S_\varphi = K$.

Now denote by $G$ the pluricomplex Green function of $D$ associated to the admissible function $\varphi$. Then, since $\varphi \le G \le s_j G_j$ on $D$, we have $S_G = K$, which implies by Theorem 2.6 that $G$ is a pluricomplex Green potential for the pair $(K, D)$. □

## 3. Comparison Theorems for a Class of Unbounded Functions

Comparison theorems involving various classes of unbounded PSH functions have been obtained in connection with the extension of the definition of the complex Monge–Ampère operator (see [B; C2; P]).

In this section we will prove a maximum principle for the following class of unbounded plurisubharmonic functions:

$$\tilde{P}(D) := \{u \in \mathrm{PSH}(D);\ \exists E \Subset D,\ u \in L^\infty(D \setminus E)\}, \tag{3.0}$$

where $D$ is an arbitary Stein manifold. It is well known that the complex Monge–Ampère operator is well-defined for the class (3.0) and is continuous under decreasing sequences of plurisubharmonic functions (see [D1; Si1]).

Before stating the maximum principle for the class (3.0), we need some preliminary results for this class. The following result is well known [BT2; C2] for bounded plurisubharmonic functions.

LEMMA 3.1. *Let $u, v \in \tilde{P}(D)$ such that* $\liminf_{z \to \partial D}(u(z) - v(z)) \ge 0$. *Then the following properties hold:*

(1) *if $u = v$ near the boundary of $D$ then*

$$\int_D (dd^c v)^n = \int_D (dd^c u)^n; \tag{3.1}$$

(2) *if $u \le v$ on $D$ then*

$$\int_D (dd^c v)^n \le \int_D (dd^c u)^n. \tag{3.2}$$

Recall that the hypothesis $\liminf_{z \to \partial D}(u(z) - v(z)) \ge 0$ means that, for any $\varepsilon > 0$, there exists $E \Subset D$ such that $u(z) - v(z) \ge -\varepsilon$ for all $z \in D \setminus E$.

*Proof.* The first assertion follows immediately, exactly as in the case of locally bounded plurisubharmonic functions, since the Monge–Ampère operator has the same inductive definition [C3].

To prove the second assertion, assume that $u \le v$ on $D$ and define the function $v_\varepsilon := \sup\{u + \varepsilon, v\}$ for $\varepsilon > 0$. Then $v_\varepsilon$ is plurisubharmonic on $D$ and satisfies $v_\varepsilon = u + \varepsilon$ in a neighborhood of the boundary of $D$; hence, by the first assertion,

$$\int_D (dd^c v_\varepsilon)^n = \int_D (dd^c u)^n. \tag{3.3}$$

Because $v_\varepsilon$ decreases to $v$ as $\varepsilon \to 0$, from (3.3) and the convergence theorem of Demailly we deduce that

$$\int_D (dd^c v)^n \leq \liminf_{\varepsilon \to 0} \int_D (dd^c v_\varepsilon)^n = \int_D (dd^c u)^n. \tag{3.4}$$

Inequality (3.4) implies (3.2), and thus the lemma is proved. □

The next lemma is also known [D1], but for completeness we give a direct proof of it based on Lemma 3.1.

LEMMA 3.2. *Let $u \in \tilde{P}(D)$. Then we have the estimate*

$$(2\pi)^n \sum_{a \in A_u} \nu(u; a)^n \leq \int_{S_u} (dd^c u)^n. \tag{3.5}$$

*In particular, $\sum_{a \in A_u} \nu(u; a)^n < +\infty$ and the set $A_u$ is countable.*

*Proof.* By modifying $u$ on the complement of a compact subset of $D$ containing $S_u$, we can always assume that there exists a domain $D' \Subset D$ such that $u$ tends to zero at the boundary of $D'$.

Let $A \subset A_u$ be a finite subset. Consider the function defined by

$$\varphi_A(z) := \sum_{a \in A} \nu(u; a) \log|z - a|, \quad z \in D.$$

For fixed $t > 0$, let $D_t := \{z \in D';\ u(z) < -t\}$. Then $D_t \Subset D$. Thus, setting $G_t := G_{D_t}(\cdot; \varphi_A)$ and applying Proposition 2.2, we see that $u + t \leq G_t$ on $D_t$. Therefore, by applying Lemma 3.1, we obtain the estimates

$$\int_A (dd^c G_t)^n \leq \int_{D_t} (dd^c G_t)^n \leq \int_{D_t} (dd^c u)^n. \tag{3.6}$$

By (2.11), the left-hand side of (3.6) is equal to $(2\pi)^n \sum_{a \in A} \nu(u; a)^n$. Since $A$ is an arbitrary finite subset of $A_u$, (3.5) of the lemma follows by passing to the limit in (3.6) when $t$ tends to $+\infty$. □

By Lemma 3.2, to each function $u \in \tilde{P}(D)$ we can associate an atomic Borel measure on $D$ defined by

$$\Theta_n(u) := (2\pi)^n \sum_{a \in A_u} \nu(u; a)^n \delta_a. \tag{3.7}$$

For any compact subset $E \Subset D$ we set

$$\tilde{P}(D; E) := \mathrm{PSH}(D) \cap L^\infty_{\mathrm{loc}}(D \setminus E). \tag{3.8}$$

It is well known that the Monge–Ampère mass and the boundary value do not characterize the functions from the class (3.8) (see Example 3.4). However, one can prove the following comparison theorem for this class, which is a generalization of the comparison theorem of Demailly [D1; D2].

THEOREM 3.3. *Let $E$ be a compact subset of Lebesgue measure zero in $D$, and let $u$ be a plurisubharmonic function on $D$ that satisfies*

(1) $u \in \tilde{P}(D; E)$ *and*
(2) $\int_E (dd^c u)^n = (2\pi)^n \sum_{a \in A_u} \nu(u; a)^n$.

*Let $v$ be a plurisubharmonic function on $D$ that satisfies the following properties:*

(3) $v \in \tilde{P}(D; E)$;
(4) $\liminf_{z \to \partial D}(u(z) - v(z)) \geq 0$;
(5) $(dd^c v)^n \geq (dd^c u)^n$ *in the sense of measures on $D \setminus E$; and*
(6) $\Theta_n(v) \geq \Theta_n(u)$ *in the sense of measures on $D$.*

*Then $v \leq u$ on $D$.*

*Proof.* Let $\gamma$ be a stictly plurisubharmonic exhaustion on $D$ (see [H]) and put $D_c := \{z \in D;\ \gamma(z) < c\}$. First fix positive numbers $\delta > 0$ and $c_0 > 0$ so large that $u \geq v - \delta$ on $D \setminus D_c$ for $c \geq c_0$. Now, for $\varepsilon > 0$ and $c \geq c_0$ fixed, consider the function $v_\varepsilon := \sup\{v + \varepsilon(\gamma - c),\ u + \delta\}$. Then $v_\varepsilon$ is plurisubharmonic on $D$, $u \leq v_\varepsilon$ on $D_c$, and—by (4)—we have $\liminf_{z \to \partial D_c}(u(z) - v_\varepsilon(z)) \geq 0$. Hence, by Lemma 3.1,

$$\int_{D_c} (dd^c v_\varepsilon)^n \leq \int_{D_c} (dd^c u)^n. \tag{3.9}$$

From (1) and (3) it follows that $u$ and $v_\varepsilon$ are locally bounded plurisubharmonic functions on the open set $\Omega := D_c \setminus E$. Then, by a result of Demailly [D3], the following inequality holds in the sense of measures on $\Omega$:

$$\begin{aligned}(dd^c v_\varepsilon)^n \geq\ & 1_{\{u+\delta \leq v + \varepsilon(\gamma - c)\}} (dd^c(v + \varepsilon\gamma - \varepsilon c))^n \\ & + 1_{\{u+\delta > v+\varepsilon(\gamma-c)\}} (dd^c u)^n.\end{aligned} \tag{3.10}$$

Because, in the sense of measures on $D$, we have

$$(dd^c(v + \varepsilon\gamma))^n \geq (dd^c v)^n + \varepsilon^n (dd^c \gamma)^n,$$

it follows from (3.10) and (5) that

$$\int_\Omega (dd^c v_\varepsilon)^n \geq \int_\Omega (dd^c u)^n + \varepsilon^n \int_{\Omega \cap \{u+\delta \leq v+\varepsilon(\gamma-c)\}} (dd^c\gamma)^n. \tag{3.11}$$

Now we claim that the inequality (3.11) holds with $\Omega$ replaced by $D_c$. Indeed, since the set $E$ is of Lebesgue measure zero, it is enough to compare the masses of the two measures $(dd^c v_\varepsilon)^n$ and $(dd^c u)^n$ on the set $E$ containing $S_u \cup S_v$. From (6) it is easy to deduce that $\Theta_n(v_\varepsilon) = \Theta_n(u)$ in the sense of measures on $D$. Then, applying Lemma 3.2 and taking (2) into account,

$$\int_E (dd^c v_\varepsilon)^n \geq \int_{S_{v_\varepsilon}} (dd^c v_\varepsilon)^n \geq \int_{A_{v_\varepsilon}} \Theta_n(v_\varepsilon) = \int_{A_u} \Theta_n(u) = \int_E (dd^c u)^n,$$

which proves our claim.

Therefore, from the previous claim and (3.9), we deduce the following estimates:

$$\int_{D_c} (dd^c u)^n + \varepsilon^n \int_{\{u+\delta \le v+\varepsilon(\gamma-c)\}\cap D_c} (dd^c\gamma)^n \le \int_{D_c} (dd^c u)^n. \tag{3.12}$$

Since $D_c \Subset D$, it follows from the Chern–Levine–Nirenberg inequalities that $\int_{D_c}(dd^c u)^n < +\infty$ (see [D2; Si1]). Then, from (3.12), it follows that the set $\{u+\delta \le v+\varepsilon(\gamma-c)\}\cap D_c$ is of Lebesgue measure zero in $D$ for any $\varepsilon > 0$, $\delta > 0$, and $c \ge c_0$, which implies that $v \le u$ on $D$. This proves the theorem. □

It is important to observe that—as the following example shows—the theorem is not true without condition (6), even if in condition (5) we assume that $(dd^c u)^n = (dd^c v)^n$ in the sense of measures in $D$.

Example 3.4. For $\lambda := (\lambda_1, \dots, \lambda_n)$ with $0 < \lambda_1 \le \lambda_2 \le \dots \le \lambda_n$, the function $\varphi_\lambda(z) := \log\max\{|z_j|^{\lambda_j};\ 1 \le j \le n\}$ is plurisubharmonic on the unit polydisc $U_n$. It is well known that $\varphi_\lambda$ satisfies the complex Monge–Ampère equation

$$(dd^c\varphi_\lambda)^n = (2\pi)^n \lambda_1 \cdots \lambda_n \delta_0$$

in the sense of currents on $U_n$ (see [D4]). Moreover, it is clear that $\nu(\varphi_\lambda; 0) = \lambda_1$. Thus, if $\lambda_1 \cdots \lambda_n = 1$ then all functions $\varphi_\lambda$ satisfy the same complex Monge–Ampère equation on $U_n$ and have the same boundary values. Therefore, when $n \ge 2$, if $u := \varphi_{(1,\dots,1)}$ and $v := \varphi_\lambda$ with $\lambda_1 < 1$ then all conditions of the theorem but (6) are satisfied and it is clear that $v \not\le u$ on $U_n$. Observe that in this case $G_{U_n}(\cdot; \varphi_\lambda) = \lambda_1 \varphi_{(1,\dots,1)}$ on $U_n$ for any $\lambda = (\lambda_1, \dots, \lambda_n)$ with $0 < \lambda_1 \le \dots \le \lambda_n$.

## 4. The Pluricomplex Green Function and the Dirichlet Problem

We are now interested in the following Dirichlet problem for the complex Monge–Ampère operator.

Dirichlet Problem. Let $D \Subset \mathbb{C}^n$ be a strictly pseudoconvex domain. Given a Borel measure $\mu$ on $D$, find a suitable class of plurisubharmonic functions $\mathcal{P}(D)$ on which the complex Monge–Ampère operator $(dd^c)^n$ is well-defined and such that, for any continuous function $h$ on $\partial D$, the following problem has a unique solution:

$$\begin{cases} u \in \mathcal{P}(D), \\ (dd^c u)^n = \mu, \\ \lim_{z\to\zeta} u(z) = h(\zeta)\ \forall \zeta \in \partial D. \end{cases} \tag{I}$$

The Dirichlet problem for plurisubharmonic functions was first considered by Bremermann [Br], who used the method of Perron to solve it. Bedford and Taylor [BT1] subsequently introduced the complex Monge–Ampère operator and solved the Dirichlet problem (I) when $\mathcal{P}(D) = \mathrm{PSH}(D) \cap L^\infty_{\mathrm{loc}}(D)$ and the measure $\mu$ is absolutely continuous (with continuous density) with respect to the Lebesgue measure. Since then, several authors [C1; CK; CP; P; Bl; Ko4] have endeavored to solve the problem without assuming the continuity of the density of $\mu$. Kolodziej [Ko1; Ko2; Ko4] has given interesting sufficient conditions on $\mu$ to belong to the range of the complex Monge–Ampère operator on the class $\mathrm{PSH}(D) \cap L^\infty_{\mathrm{loc}}(D)$ and has solved the Dirichlet problem for such measures. For a nice and complete survey on this problem, we refer to [Ko4].

For singular measures, much less is known about the solvability of the Dirichlet problem; see [Lm2; D2; L2].

In this section, we will solve the Dirichlet problem (I) for the singular measure $\mu := \Theta_n(\varphi)$ associated to an admissible plurisubharmonic function $\varphi$ on $D$. First we shall prove the following result for the pluricomplex Green function.

THEOREM 4.1. *Let $D$ be a hyperconvex open subset of $\mathbb{C}^n$ and $\varphi$ an admissible plurisubharmonic function on $D$. Then the Green function $G = G_D(\cdot;\varphi)$ associated to the admissible plurisubharmonic function $\varphi$ is the unique plurisubharmonic function on $D$ satisfying the following properties.*

(i) $G \in \mathrm{PSH}(D) \cap L^\infty_{\mathrm{loc}}(D \setminus K)$, *where* $K := S_\varphi$.
(ii) $G(z) \to 0$ *as* $z \to \partial D$.
(iii) $\nu(G; a) = \nu(\varphi; a)$ *for all* $a \in D$.
(iv) *$G$ satisfies the complex Monge–Ampère equation*

$$(dd^c G)^n = (2\pi)^n \sum_{a \in A_\varphi} \nu(\varphi; a)^n \delta_a \tag{4.1}$$

*in the sense of measures on $D$, where $\delta_a$ is the Dirac measure at the point $a$.*

*Proof.* From Theorem 2.1, it follows that $G$ satisfies properties (i), (ii), and (iii).

Let us prove (iv). Let $\{A_j\}_{j \ge 1}$ be an increasing sequence of nonempty finite subsets of $A_\varphi$ such that $A_\varphi = \bigcup_{j \ge 1} A_j$. For each $j \ge 1$, define the functions $\varphi_j(z) = \sum_{a \in A_j} \nu(\varphi; a) \log|z - a|$ $(z \in D)$ and $G_j := G_D(\cdot;\varphi_j)$. Then $\{G_j\}_{j \ge 1}$ is a decreasing sequence of plurisubharmonic functions satisfying $G \le G_j \le 0$ on $D$. The limit function $\tilde{G} := \lim_{j \to +\infty} G_j$ is hence a plurisubharmonic function on $D$, and $G \le \tilde{G} \le 0$ on $D$. Moreover, it is easy to see that $\nu(\tilde{G};\cdot) \ge \nu(\varphi;\cdot)$ on $D$ and so $\tilde{G} \le G$ on $D$. This proves that $G = \tilde{G}$ on $D$.

By the convergence theorem (see [D1; Si1]), we deduce that

$$(dd^c G)^n = \lim_{j \to +\infty} (dd^c G_j)^n$$

in the sense of currents on $D$; (iv) then follows from (2.11). The uniqueness of the pluricomplex Green function is an immediate consequence of Theorem 3.3.

We conclude this section by solving a more general Dirichlet problem for the complex Monge–Ampere equation.

THEOREM 4.2. *Let $D \Subset \mathbb{C}^n$ be a pseudoconvex open set with Lipschitz boundary, let $h$ be a continuous real function on $\bar{D}$ that is plurisubharmonic on $D$, and let $\varphi$ be a continuous admissible plurisubharmonic function on $D$. Then the following Dirichlet problem has a unique solution:*

$$\begin{cases} U \in \tilde{P}(D),\ \nu(U;\cdot) = \nu(\varphi;\cdot), \\ (dd^c U)^n = \Theta_n(\varphi), \\ \lim_{z\to\zeta} U(z) = h(\zeta)\ \forall\zeta \in \partial D. \end{cases} \tag{4.2}$$

*Moreover, the solution is continuous on $D \setminus S_\varphi$.*

*Proof.* Consider the following upper envelope on $D$:

$$U(z) := \sup\{u(z);\ u \in \tilde{P}_h(D;\varphi)\}, \quad z \in D, \tag{4.3}$$

where $\tilde{P}_h(D;\varphi)$ is the class of PSH functions $u \in \tilde{P}(D)$ such that $\nu(u;\cdot) \ge \nu(\varphi;\cdot)$ on $D$ and $\lim_{z\to\zeta} U(z) = h(\zeta)$ for all $\zeta \in \partial D$. It is well known that $D$ is regular with respect to the Dirichlet problem for the Laplace operator on $\mathbb{C}^n$. Hence there exists a real harmonic function $H$ on $D$ with boundary values $h$. By [D2], $D$ is hyperconvex; it then follows from Theorem 2.1 that $u_0 := G_D(\cdot;\varphi) + h \in \tilde{P}_h(D;\varphi)$ and $u_0 \le U \le H$ on $D$. By proceeding in the same way as in the proof of Theorem 2.1, we conclude that $U \in \tilde{P}_h(D;\varphi) \cap L^\infty_{\mathrm{loc}}(D \setminus S_\varphi)$. It remains to prove that $U$ satisfies the complex Monge–Ampère equation

$$(dd^c U)^n = \Theta_n(\varphi) \tag{4.4}$$

in the sense of measures on $D$. Toward this end, we proceed exactly as in the proof of Theorem 4.1. Let $(A_j)_{j\ge1}$ be an increasing sequence of finite subsets of $A_\varphi$ such that $A_\varphi = \bigcup_{j\ge1} A_j$ and set $\varphi_j(z) := \sum_{a\in A_j} \log|z-a|$. Let $U_j$ denote the upper envelope of the class $\tilde{P}_h(D;\varphi_j)$. Then it is clear from the definition that $(U_j)$ is a decreasing sequence of plurisubharmonic functions on $D$ satisfying $U \le U_j$ on $D$ for any $j \ge 1$. Thus the limit $V = \lim_{j\to+\infty} U_j$ is plurisubharmonic on $D$ and satisfies the inequality $U \le V$ on $D$. Moreover, it is easy to see that $V \in \tilde{P}_h(D;\varphi)$. This proves that $V = U$. It then follows from the convergence theorem of Demailly [D1] that (4.4) is a consequence of

$$(dd^c U_j)^n = (2\pi)^n \sum_{a\in A_j} \nu(\varphi;a)^n \delta_a \tag{4.5}$$

in the sense of measures in $D$.

Let us prove (4.5). Since $(dd^c U_j)^n = 0$ on $D \setminus A_j$, for any $j \ge 1$ and for any $a \in A_j$ we have $U_j \sim \nu(\varphi;a)\log|z-a|$. When $z \to a$, it follows from the comparison theorem [D1; K2] that

$$(dd^c U_j)^n(\{a\}) = \nu(\varphi;a)^n (dd^c \log|z-a|)^n(\{a\}) = (2\pi)^n \nu(\varphi;a)^n.$$

Thus equation (4.5) follows.

The continuity of the solution on $D \setminus S_\varphi$ can be proved in exactly the same way as the proof of Theorem 2.6. □

COROLLARY 4.3. *Let $D \Subset \mathbb{C}^n$ be a strictly pseudoconvex domain and let $h$ be a continuous real function on $\partial D$. Then the Dirichlet problem* (4.2) *has a unique solution. Moreover, this solution is continuous on $D \setminus S_\varphi$.*

*Proof.* It is well known [Br] that, if $D \Subset \mathbb{C}^n$ is a strictly pseudoconvex domain and $h$ is a continuous boundary data, then there exists a continuous function $H$

on $\bar{D}$ that is plurisubharmonic on $D$ and coincides with $h$ on $\partial D$. The corollary is then a consequence of Theorem 4.2. □

The Dirichlet problem (4.2) can be solved for arbitrary continuous boundary data on more general pseudoconvex domains, known as domains with B-regular boundary (see [Si2]).

In connection with Theorem 4.2, it is interesting to ask the following question.

QUESTION 3. Given a discrete measure $\mu$ with a complete pluripolar compact support $K \subset \mathbb{C}^n$, does there exist a plurisubharmonic function $\varphi$ on some open neighborhood $D$ of $K$ such that $\varphi \in L^\infty_{\text{loc}}(D \setminus K)$ and $\varphi$ satisfies

$$(2\pi)^n \sum_{a \in A_\varphi} \nu(\varphi; a)^n \delta_a = \mu$$

on $D$?

Because the support $K := \operatorname{Supp}(\mu)$ of $\mu$ is compact, the set $A := \{a \in K; \mu(\{a\}) > 0\}$ is countable. Let us order the points of $A$ into a sequence $(a_j)_{j \geq 1}$, and put $\mu_j := \mu(\{a_j\})$ so that $\mu = \sum_j \mu_j \delta_{a_j}$. If the series $\sum_j \mu_j^{1/n}$ converges, then the function $\sum_j \mu_j^{1/n} \log|z - a_j|$ is plurisubharmonic on $\mathbb{C}^n$ and satisfies the required properties (see the proof of Lemma 1.3).

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## References

[B] E. Bedford, *Survey of pluri-potential theory,* Several complex variables (Stockholm, 1987/1988), pp. 48–97, Princeton Univ. Press, Princeton, NJ, 1993.

[BT1] E. Bedford and B. A. Taylor, *The Dirichlet problem for a complex Monge–Ampère equation,* Invent. Math. 37 (1976), 1–44.

[BT2] ———, *A new capacity for plurisubharmonic functions,* Acta Math. 149 (1982), 1–40.

[Bl] Z. Blocki, *The complex Monge–Ampère operator in hyperconvex domains,* Doctoral thesis, Jagiellonian Univ., Krakow, Poland, 1995.

[Br] H. Bremermann, *On a generalized Dirichlet problem for plurisubharmonic functions and pseudoconvex domains. Characterization of Šhilov boundaries,* Trans. Amer. Math. Soc. 91 (1959), 246–276.

[C1] U. Cegrell, *On the Dirichlet problem for the complex Monge–Ampère operator,* Math. Z. 185 (1984), 247–251.

[C2] ———, *Sums of continuous plurisubharmonic functions and the complex Monge–Ampère operator in* $\mathbb{C}^n$, Math. Z. 193 (1986), 373–380.

[C3] ———, *Capacities in complex analysis,* Aspects of Math., E14, Vieweg, Braunschweig, 1988.

[CK] U. Cegrell and S. Kolodziej, *The Dirichlet problem for the complex Monge–Ampère operator: Perron classes and rotation invariant measures,* Michigan Math. J. 41 (1994), 563–569.

[CP] U. Cegrell and L. Persson, *The Dirichlet problem for the complex Monge–Ampère equation: Stability in* $L^2$, Michigan Math. J. 39 (1992), 145–151.

[CS] U. Cegrell and A. Sadullaev, *Approximation of plurisubharmonic functions and the Dirichlet problem for the complex Monge–Ampère operator,* Math. Scand. 71 (1992), 62–68.

[D1] J. P. Demailly, *Mesures de Monge–Ampère et caractérisation des variétés algébriques affines,* Mém. Soc. Math. France (N.S.) 19 (1985), 1–125.

[D2] ———, *Mesures de Monge–Ampère et mesures pluriharmoniques,* Math. Z. 194 (1987), 519–564.

[D3] ———, *Potential theory in several complex variables,* ICPAM Summer School on Complex Analysis, Nice, France, 1989.

[D4] ———, *Regularization of closed positive currents and intersection theory,* J. Algebraic Geom. 1 (1992), 361–409.

[H] L. Hörmander, *An introduction to complex analysis in several variables,* 3rd. ed., North-Holland, New York, 1990.

[JP] M. Jarnicki and P. Pflug, *Invariant distances and metrics in complex analysis,* de Gruyter, Berlin, 1993.

[KeR] N. Kerzman and J. P. Rosay, *Fonctions plurisousharmoniques d'exhaustion bornées et domaines taut,* Math. Ann. 257 (1981), 171–184.

[Ki1] C. O. Kiselman, *Densité des fonctions plurisousharmoniques,* Bull. Soc. Math. France 107 (1979), 295–304.

[Ki2] ———, *Attenuating the singularities of plurisubharmonic functions,* Ann. Polon. Math. 60 (1994), 173–197.

[K1] M. Klimek, *Extremal plurisubharmonic functions and invariant pseudodistances,* Bull. Soc. Math. France 113 (1985), 231–240.

[K2] ———, *Pluripotential theory,* Oxford University Press, New York, 1991.

[K3] ———, *Invariant pluricomplex Green functions,* Topics in complex analysis, vol. 31, pp. 207–226, Banach Center Publications, Institut of Mathematics, Polish Academy of Sciences, Warsaw, 1995.

[Ko1] S. Kolodziej, *The range of the complex Monge–Ampère operator,* Indiana Univ. Math. J. 43 (1994), 1321–1338.

[Ko2] ———, *The range of the complex Monge–Ampère operator II,* Indiana Univ. Math. J. 44 (1995), 765–782.

[Ko3] ———, *The complex Monge–Ampère equation,* preprint, Institut of Math. of Jagielonski Univ., Krakow, Poland, 1995.

[Ko4] ———, *Some sufficient conditions for solvability of the Dirichlet problem for the complex Monge–Ampère operator,* Ann. Polon. Math. 65 (1996), 11–21.

[L1] P. Lelong, *Fonctions plurisousharmoniques et formes différentielle positives,* Gordon & Breach, New York / Dunod, Paris, 1968.

[L2] ———, *Fonction de Green pluricomplexe et lemmes de Schwarz dans les espaces de Banach,* J. Math. Pures Appl. (9) 68 (1989), 319–347.

[Lm1] L. Lempert, *La métrique de Kobayashi et la représentation des domaines sur la boule,* Bull. Soc. Math. France 109 (1981), 427–474.

[Lm2] ———, *Solving the degenerate Monge–Ampère equation with one concentrated singularity,* Math. Ann. 263 (1983), 515–532.

[N] S. Nivoche, *Pluricomplex Green function, capacitative notions and approximation problems in $\mathbb{C}^n$*, Indiana Univ. Math. J. 44 (1995), 489–510.

[P] L. Persson, *On the Dirichlet problem for the complex Monge–Ampère operator,* Thesis, Univ. of Umea, 1992.

[Po] E. A. Poletsky, *Holomorphic currents,* Indiana Univ. Math. J. 42 (1993), 85–144.

[Si1] N. Sibony, *Quelques problèmes de prolongement de courants en analyse complexe,* Duke Math. J. 52 (1985), 157–197.

[Si2] ———, *Une classe de domaines pseudoconvexes,* Duke Math. J. 55 (1987), 299–319.

[S] Y. T. Siu, *Analyticity of sets associated to Lelong numbers and the extension of closed positive currents,* Invent. Math. 27 (1974), 53–156.

[St] J. L. Stehlé, *Fonctions plurisousharmoniques et convexité holomorphe de certains fibrés analytiques,* Séminaire P. Lelong, Lecture Notes in Math., 474, pp. 155–179, Springer, Berlin, 1975.

[T] M. Tsuji, *Potential theory in modern function theory,* Chelsa, New York, 1975.

[Za] V. P. Zahariuta, *Spaces of analytic functions and complex potential theory,* Linear Topol. Spaces Complex Anal. 1 (1994), 74–146.

[Z1] A. Zeriahi, *Ensembles pluripolaires exceptionnels pour la croissance partielle des fonctions holomorphes,* Ann. Polon. Math. 50 (1989), 81–91.

[Z2] ———, *Pluricomplex Green functions and approximation of holomorphic functions,* Complex analysis, harmonic analysis and applications, Pitman Res. Math. Notes Ser., 347, pp. 104–142, Longman, Harlow, U.K., 1996.

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