E^3-Complete Timelike Surfaces in E_1^3 Are Globally Hyperbolic

SENCHUN LIN & TILLA WEINSTEIN

1. Introduction

A C^∞ immersion $\mathcal{Z}$ of a surface S into Minkowski 3-space E_1^3 can also be viewed as a C^∞ immersion of S into Euclidean 3-space E^3. If the metric I induced on S by $\mathcal{Z}: S \to E_1^3$ is Lorentzian, then $\mathcal{Z}: S \to E_1^3$ is called *timelike.* If the metric I_ε induced on S by $\mathcal{Z}: S \to E^3$ is complete, then $\mathcal{Z}: S \to E_1^3$ is said to be *E^3-complete.*

In all results below, S is a surface provided with the Lorentzian metric I and the time orientation induced by a timelike C^∞ immersion $\mathcal{Z}: S \to E_1^3$. Among the conformally invariant properties definable on a time-oriented Lorentzian surface are the causality conditions of interest in general relativity. Two of these conditions, stable causality and global hyperbolicity (defined in [1, pp. 63–65]) are dealt with in this paper.

Theorem 1 makes the elementary observation that S must be stably causal. In case $\mathcal{Z}$ is E^3-complete, Theorem 2 states that S is globally hyperbolic. If $\mathcal{Z}: S \to E_1^3$ is E^3-complete from a simply connected S, Theorem 3 states that S is C^∞-conformally diffeomorphic to a subset of the Minkowski 2-plane E_1^2, and places strong restrictions on the conformal boundary $\partial_0 S$, which was defined by Kulkarni in [3] and studied in [7]. (This strengthens slightly a result announced without proof on p. 196 in [7].)

Under the hypotheses of Theorem 3, Theorem 4 states that S is C^∞-conformally diffeomorphic to E_1^2 provided that the mean curvature H for $\mathcal{Z}: S \to E_1^3$ vanishes outside a compact set on S. This generalizes the conformal Bernstein theorem from [4], which states that any entire timelike minimal surface in E_1^3 is C^∞-conformally diffeomorphic to E_1^2.

Because global hyperbolicity is the most restrictive of the causality conditions on space-times discussed in [1], it may appear that the surface S in Theorems 2 and 3 has little room for variation in its global Lorentzian structure. To show that this need not be the case, we describe in Section 4 uncountably many C^0- (and thereby C^∞-) conformally distinct simply connected, globally hyperbolic subsurfaces S_r of E_1^2. Like all subsurfaces of E_1^2, the S_r are C^∞ isometrically imbedded in E_1^3

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when E_1^2 is identified with the (x, z) plane in E_1^3. However, these are not E^3 complete imbeddings of the S_r. We do not yet know whether every globally hyperbolic time-oriented Lorentzian surface (even if simply connected) has an E^3-complete conformal immersion in E_1^3, or a C^∞-inextendible conformal imbedding in E_1^3 (as defined in [7, Sec. 8.1]).

Familiarity with basic Lorentzian properties of E_1^2 and E_1^3 will be assumed throughout. In E_1^3, we choose to make the vertical coordinate axis timelike. For background, see [7, Secs. 1.5 and 7.2].

2. The Main Result

Throughout this paper, a surface S is a connected (not necessarily oriented) C^∞ 2-manifold. Suppose that $\mathcal{Z}: S \to E_1^3$ is a timelike C^∞ immersion. Then the metric I induced on S by $\mathcal{Z}$ is Lorentzian. For some neighborhood V of any point p on S, $\mathcal{Z}: V \to E_1^3$ is a timelike C^∞ imbedding, with $\mathcal{Z}_*|_p$ a linear map from the tangent plane S_p to S at p onto the plane tangent to $\mathcal{Z}(V)$ at $\mathcal{Z}(p)$. A vector in S_p, or a 1-dimensional linear subspace of S_p, is given the causal type (spacelike, timelike, or null) of its image under $\mathcal{Z}_*|_p$. The two distinct 1-dimensional linear subspaces of S_p that are null are called *null directions* in S_p.

A C^∞ curve $\lambda: (a, b) \to S$ is *spacelike* (resp. *timelike* or *null*) provided that its tangent vector $\lambda'(t)$ is spacelike (resp. timelike or null) for each t in (a, b). A C^∞ curve $\lambda: (a, b) \to S$ is *causal* provided that $\lambda'(t)$ is either timelike or null for each t in (a, b). Since the zero vector is always spacelike, all C^∞ causal curves on S are regular.

The usual time orientation is taken on E_1^3, which prefers the upper half of the time cone at every point. Thus, any timelike C^∞ immersion $\mathcal{Z}: S \to E_1^3$ induces a time orientation on S, making a timelike or null vector v in S_p *future* (resp. *past*) *directed* if and only if $\mathcal{Z}_*|_p(v)$ is future (resp. past) directed in $\mathcal{Z}_*|_p(S_p)$. A causal C^∞ curve $\lambda: (a, b) \to S$ is *future* (resp. *past*) *directed* provided that $\lambda'(t)$ is future (resp. past) directed for one t value (and thereby for all t values) in (a, b).

The following elementary observation yields Theorem 1 as an obvious consequence.

LEMMA 1. *If $\lambda: (a, b) \to S$ is a future directed C^∞ causal curve for a timelike C^∞ immersion $\mathcal{Z}: S \to E_1^3$ with*

$$(\mathcal{Z} \circ \lambda)(t) = (x(t), y(t), z(t)),$$

then $z'(t) > 0$ on (a, b), so that $z(t)$ is strictly increasing, λ is a simple nonclosed curve on S, and $\mathcal{Z}$ imbeds λ in E_1^3.

Proof. The Euclidean angle $\theta(t)$ between $(\mathcal{Z} \circ \lambda)'(t) = (x'(t), y'(t), z'(t))$ and the positive z axis satisfies $0 \le \theta(t) \le \pi/4$, so that $z'(t) > 0$ for all t in (a, b). The remaining claims of the lemma follow directly from this fact. □

A continuous function $f: S \to \mathbb{R}$ on a time-oriented Lorentzian surface S is a *global time function* if and only if f is strictly increasing on each future directed causal curve. A time-oriented Lorentzian surface admits a global time function if and only if it is stably causal. (See [1, pp. 63-64] for references and for the definition of stable causality.) Thus Lemma 1 gives the following result.

THEOREM 1. *If* $\mathcal{Z}: S \to E_1^3$ *is a timelike* C^∞ *immersion, then* S *provided with the Lorentzian metric and time orientation induced by* $\mathcal{Z}$ *is stably causal.*

Proof. By Lemma 1, the C^∞ function $z: S \to \mathbb{R}$ given by

$$\mathcal{Z}(p) = (x(p), y(p), z(p))$$

for any point p on S is a global time function on the time-oriented Lorentzian surface S. □

Given a C^∞ immersion $\mathcal{Z}: S \to E_1^3$, the Riemannian metric I_ε induced on S by $\mathcal{Z}: S \to E^3$ can be used to parameterize any regular C^∞ curve on S by its E^3 arclength. The parameterization of a future (resp. past) directed C^∞ causal curve λ on S by E^3 arclength is always taken so that $\lambda'(s)$ is future (resp. past) directed.

A C^∞ causal curve $\lambda: (a, b) \to S$ for a timelike C^∞ immersion $\mathcal{Z}: S \to E_1^3$ is *extendible* provided that $\lambda(t)$ converges to a point on S as $t \to a^+$ or as $t \to b^-$. If λ is not extendible then it is *inextendible* (see [1, p. 61]). Note that the interval (a, b) on which λ is defined can be bounded or unbounded, so that a or b may each be finite or infinite.

LEMMA 2. *Let* $\lambda: (a, b) \to S$ *be a future directed inextendible* C^∞ *causal curve parameterized by* E^3 *arclength for an* E^3*-complete timelike* C^∞ *immersion* $\mathcal{Z}: S \to E_1^3$. *If*

$$(\mathcal{Z} \circ \lambda)(s) = (x(s), y(s), z(s)),$$

then $a = -\infty$, $b = \infty$ *and*

$$\lim_{s \to -\infty} z(s) = -\infty, \qquad \lim_{s \to \infty} z(s) = \infty.$$

Proof. Suppose that b is finite. Then, for a fixed s_0 in (a, b), the I_ε length of $\lambda: (s_0, b) \to S$ is finite. Given any sequence of values s_n in (s_0, b) with $s_n \to b$, the points $\lambda(s_n)$ lie in a bounded set for the distance function d_ε defined on S by I_ε. Since I_ε is complete, so is d_ε. Thus, some subsequence $\lambda(s_{n_j})$ converges to a point q on S as $n_j \to \infty$. Since the curve $\lambda: (s_0, b) \to S$ has finite I_ε length, $\lambda(s) \to q$ as $s \to b^-$. This means $\lambda: (a, b) \to S$ is extendible, a contradiction. We conclude that $b = \infty$. But then

$$\begin{aligned} \infty = \int_{s_0}^{b} ds &= \int_{s_0}^{b} \sqrt{x'(s)^2 + y'(s)^2 + z'(s)^2}\, ds \\ &\leq \sqrt{2} \int_{s_0}^{b} z'(s)\, ds = \sqrt{2}\Big(\lim_{s \to \infty} z(s) - z(s_0) \Big), \end{aligned}$$

so that

$$\lim_{s \to \infty} z(s) = \infty.$$

A similar argument shows that $a = -\infty$ with $\lim_{s \to -\infty} z(s) = -\infty$. □

On a Lorentzian surface S, a *Cauchy surface* is a subset Γ that intersects each inextendible causal curve exactly once. A time-oriented Lorentzian surface S is *globally hyperbolic* if and only if it admits a Cauchy surface. (See [1, p. 65] for references.) Thus, Lemma 2 gives the following result.

THEOREM 2. *If $\mathcal{Z}: S \to E_1^2$ is an E^3-complete, timelike C^∞ immersion, then S provided with the Lorentzian metric and time orientation induced by $\mathcal{Z}$ is globally hyperbolic.*

Proof. For each real constant c, let Γ_c be the set of all points on S that $\mathcal{Z}$ maps to the horizontal plane $z = c$ in E_1^3. By Lemma 2, each inextendible causal curve on S intersects Γ_c in exactly one point. It follows that Γ_c is a Cauchy surface and that S is globally hyperbolic. □

Under the hypotheses of Theorem 2, the C^∞-global time function $z: S \to \mathbb{R}$ is a *Cauchy time function,* which means that the preimage under z of any real value c is a Cauchy surface Γ_c on S. In Section 3 we study the Cauchy surfaces Γ_c under the additional assumption that S is simply connected.

3. Extending the Conformal Bernstein Theorem

Fix a timelike C^∞ immersion $\mathcal{Z}: S \to E_1^3$ and work with the Lorentzian metric I on S induced by $\mathcal{Z}$. Assume that S is oriented. Use on S only those C^∞ coordinate pairs whose chart maps are orientation preserving. In each tangent plane S_p, label by X_p (resp. Y_p) the null direction that coincides with a spacelike (resp. timelike) 1-dimensional linear subspace when rotated in the positive sense by an arbitrarily small amount. A naturally ordered pair X, Y of C^∞ null direction fields is obtained on S by having X (resp. Y) assign the value X_p (resp. Y_p) to any point p on S. (See [7, Lemma 1].) An *X-line* (resp. *Y-line*) on S is a maximal C^∞ integral curve of X (resp. Y). A *null line* on S is either an X-line or a Y-line. Note that any null line is an inextendible C^∞ causal curve on S.

As in [3] and [7], the *Minkowski 2-space* E_1^2 is taken to be the (u, v) plane with the metric $du\, dv$. A C^∞ orientation preserving diffeomorphism $\chi: V \to \chi(V) \subset E_1^2$ from an open set V on S is *conformal* if and only if $\chi^*(du\, dv) = \mu I$ for some C^∞ function $\mu > 0$ on V. Every point p on S lies in the domain V of such a conformal C^∞ orientation preserving diffeomorphism $\chi: V \to \chi(V) \subset E_1^2$, which is called a *proper null chart* on S. (See [7, Lemma 2].) One easily checks that a proper null chart $\chi: V \to E_1^2$ takes connected portions of X-lines (resp. Y-lines) onto horizontal (resp. vertical) line segments in $\chi(V)$.

A *box* U on S is the preimage under a proper null chart $\chi: V \to E_1^2$ of a bounded rectangular region $(a, b) \times (c, d)$ lying in $\chi(V)$. Since any point p on S lies in a box, a compactness argument shows that the image of any simple, nonclosed arc $\lambda: [\alpha, \beta] \to S$ along a null line lies in a box on S. If S is simply connected, the intersection of an X-line (resp. Y-line) with a box U is a single curve $\lambda: (\alpha, \beta) \to U$ that any proper null chart χ from U takes to a maximal horizontal (resp. vertical) line segment in $\chi(U) = (a, b) \times (c, d)$. (See [7, Lemma 13].)

If a regular C^∞ spacelike curve $\lambda: (\alpha, \beta) \to S$ lies in the domain V of a proper null chart χ, then $(\chi \circ \lambda): (\alpha, \beta) \to E_1^2$ is a curve in $\chi(V)$ with positive slope. For any point p on λ, there is a box U in V containing p such that the portion of λ in U is connected, with $\chi(\lambda \cap U)$ crossing each maximal horizontal and vertical segment in $\chi(U)$ exactly once. Thus the set $sp_X(\lambda)$ (resp. $sp_Y(\lambda)$) containing all points on all X-lines (resp. Y-lines) on S that intersect λ contains U, as does $sp_X(\hat{\lambda})$ (resp. $sp_Y(\hat{\lambda})$) for any C^∞ extension $\hat{\lambda}$ of λ on S.

Suppose now that S is simply connected. The existence of the Lorentzian metric I on S guarantees that S is C^∞ diffeomorphic to $\mathbb{R}^2$. (See [7, p. 53] or [1, p. 86].) Moreover, it is well known that any regular spacelike C^∞ curve $\lambda: (a, b) \to S$ must be simple and nonclosed. (See [1, p. 90].)

The next result looks more closely at the Cauchy surface Γ_c defined in the proof of Theorem 2 in case S is simply connected. Because $\mathcal{Z}$ is an immersion rather than an imbedding, the set $\mathcal{Z}(\Gamma_c)$ can have self-intersections. Nonetheless, one has the following description of the set Γ_c of all points on S that $\mathcal{Z}$ maps to the plane $z = c$.

LEMMA 3. *Suppose $\mathcal{Z}: S \to E_1^3$ is an E^3 complete timelike C^∞ immersion from a simply connected surface S. Then, for any real constant c, the Cauchy surface Γ_c is a simple, nonclosed spacelike curve that has a C^∞ parameterization $\Gamma_c: (-\infty, \infty) \to S$ by E^3 arclength.*

Proof. Fix c, and write $\Gamma = \Gamma_c$ for convenience. Choose a point p on Γ. Because the C^∞ immersion $\mathcal{Z}$ is timelike, $\mathcal{Z}$ imbeds some neighborhood V of p as the graph $\mathcal{Z}(V)$ of a C^∞ real-valued function over an open set in the (x, z) plane or the (y, z) plane. The horizontal plane $z = c$ cuts $\mathcal{Z}(V)$ in a simple, nonclosed, C^∞ spacelike curve in E_1^3 that can be parameterized by E^3 arclength. Thus, for some neighborhood V of any point p on Γ, the portion of Γ in V is a simple, nonclosed regular C^∞ spacelike curve γ_p.

The connected component Γ_p of Γ containing γ_p is simple and nonclosed, since it is a regular C^∞ spacelike curve on S. Fix an orientation on the simply connected surface S. Let $sp(\Gamma_p)$ be the set of all points on all X-lines on S that intersect Γ_p. If a point q in $sp_X(\Gamma_p)$ lies on Γ_p, then a box containing q lies in $sp_X(\Gamma_q)$, making q an interior point of $sp_X(\Gamma_p)$. If a point q in $sp_X(\Gamma_p)$ lies off Γ_p, then there is an arc $\lambda: [\alpha, \beta] \to S$ in $sp_X(\Gamma_p)$ along an X-line with $\lambda(\alpha) = q$ and $\lambda(\beta)$ on Γ_p. Let U be a box containing $\lambda([\alpha, \beta])$. If $sp_X(\gamma_{\lambda(\beta)})$ is the set of all points on all X-lines on S that intersect $\gamma_{\lambda(\beta)}$, then the intersection of $sp(\gamma_{\lambda(\beta)})$ with U is a (possibly smaller) box U_q that contains $\lambda([\alpha, \beta])$. Every X-line that intersects U_q lies in $sp_X(\Gamma_p)$, making q an interior point of $sp_X(\Gamma_p)$. It follows that $sp_X(\Gamma_p)$ is open.

By Lemma 2, every X-line on S intersects Γ. Thus, S is the union of the sets $sp_X(\Gamma_q)$ for all points q on Γ. The complement of $sp_X(\Gamma_p)$ in S is open, since it is the union of the sets $sp_X(\Gamma_q)$ for all points q on Γ off Γ_p. If S has a point outside of $sp_X(\Gamma_p)$ then S is disconnected, a contradiction. Thus $\Gamma = \Gamma_p$.

The simple, nonclosed, regular spacelike curve Γ has a C^∞ parameterization $\Gamma: (a, b) \to S$ by E^3 arclength. We must show that $(a, b) = \mathbb{R}$. Suppose b is finite, so that the points $\Gamma(s_n)$ for any sequence s_n in (a, b) with $s_n \to b^-$ lie in a bounded set for the distance function d_ε defined by I_ε on S. Since I_ε is complete, so is d_ε. Thus, for some subsequence s_{n_j}, the points $\Gamma(s_{n_j})$ converge to a point q on S as $n_j \to \infty$. Since $z = c$ at $\mathcal{Z}(\Gamma(s_{n_j}))$ for all n_j, it follows that $z = c$ at $\mathcal{Z}(q)$, putting q on Γ. But then some neighborhood of q contains just the portion γ_q of Γ on which q is an interior point, contradicting the definition of q. Thus $b = \infty$, and a similar argument shows that $a = -\infty$. □

When defining the conformal boundary of a simply connected, oriented, and time-oriented Lorentzian surface in [3], Kulkarni placed a *natural orientation* on null lines, making X-lines past directed and Y-lines future directed. The same convention is followed in [7]. This natural orientation parameterizes X-lines to the right and Y-lines upward in E_1^2 if the timelike vector field $\partial/\partial v - \partial/\partial u$ is used to define the time orientation on E_1^2. This is the time orientation taken on E_1^2 in what follows.

Given a timelike C^∞ immersion $\mathcal{Z}: S \to E_1^3$ from a simply connected, oriented surface S, use the time orientation induced on S by $\mathcal{Z}$ to naturally orient all null lines. For any point p on S, use the naturally oriented null directions X_p and Y_p as coordinate axes in S_p. Naturally orient each Cauchy surface Γ_c so that $\Gamma_c'(s)$ lies in the first quadrant of S_p at each point $p = \Gamma_c(s)$.

LEMMA 4. *Suppose $\mathcal{Z}: S \to E_1^3$ is a timelike C^∞ immersion from a simply connected, oriented surface S, provided with the Lorentzian metric and time orientation induced by $\mathcal{Z}$. Then there is an orientation preserving C^∞ diffeomorphism $\phi: \mathbb{R}^2 \to S$ such that:*

(i) *for each fixed ζ_0, $\phi(\zeta_0, s)$ is a regular C^∞ parameterization of the Cauchy surface $\Gamma_{-\zeta_0}$ which respects the natural orientation on $\Gamma_{-\zeta_0}$;*
(ii) *$\phi(0, s)$ parameterizes Γ_0 by E^3 arclength; and*
(iii) *for each fixed s_0, $\phi(\zeta, s_0)$ parameterizes the X-line that passes through $\phi(0, s_0)$ by the negative ζ of the global time function z on S.*

Proof. Let $\Gamma_0: (-\infty, \infty) \to S$ parameterize Γ_0 by E^3 arclength so as to respect natural orientation on Γ_0, with $s = 0$ at some point p_0 on Γ_0. If $\phi(0, s) = \Gamma_0(s)$, then all other values of $\phi(\zeta, s)$ are determined by (iii). Because $S = sp_X(\Gamma_0)$, ϕ is onto S. Since distinct X-lines never intersect on S and distinct Cauchy surfaces Γ_c never intersect on S, ϕ is one–one. The field $\vec{\zeta}$ of tangent vectors to the X-lines $\phi(\zeta, s_0)$ for all fixed values s_0 is C^∞ on S. Since $\phi(\zeta, s_0): \mathbb{R} \to S$ is the maximal integral curve of $\vec{\zeta}$ with initial value $\phi(0, s_0)$ on the C^∞ curve $\phi(0, s): \mathbb{R} \to S$, standard results in ordinary differential equations guarantee that $\phi: \mathbb{R}^2 \to S$ is C^∞.

To show that the Jacobian matrix of ϕ has rank 2 everywhere, it suffices to prove that the Jacobian matrix of $\mathcal{Z} \circ \phi$ has rank 2 everywhere. Here $(\mathcal{Z} \circ \phi)(\zeta, s) = (f(\zeta, s), g(\zeta, s), -\zeta)$ for C^∞ functions $f: \mathbb{R}^2 \to \mathbb{R}$ and $g: \mathbb{R}^2 \to \mathbb{R}$. Given a fixed ζ_0, the curve $\phi(\zeta_0, s): \mathbb{R} \to S$ describes the Cauchy surface $\Gamma_{-\zeta_0}$. If σ is E^3 arclength on $(\mathcal{Z} \circ \phi)(\zeta_0, s)$, then

$$(d\sigma/ds)^2 = f_s(\zeta_0, s)^2 + g_s(\zeta_0, s)^2.$$

To see that $d\sigma/ds \neq 0$, suppose ϕ had been defined with (ii) replaced by the requirement that $\phi(\zeta_0, s)$ is parameterized by its E^3 arclength σ for the one fixed value ζ_0. Arguing as before with the roles of Γ_0 and $\Gamma_{-\zeta_0}$ reversed, one would see that s is a C^∞ function of σ, which means that $d\sigma/ds \neq 0$. Since ζ_0 is arbitrary, the lemma is proved. □

Our next result refers to the conformal boundary $\partial_0 S$ constructed by Kulkarni in [3] for any simply connected, oriented, and time-oriented surface S. Only the briefest description of $\partial_0 S$ and its properties is given here. For further background and details, see [7, Chap. 4].

Assign to each null line γ on the simply connected, oriented, and time-oriented surface S a pair of ideal endpoints γ^- and γ^+. The natural orientation on γ is thought of as defining motion "toward γ^+" and "away from γ^-". An *end* $\gamma\uparrow$ (resp. $\gamma\downarrow$) of γ is the portion of γ beyond (resp. preceding) any one fixed point on γ. Thus an end $\gamma\uparrow$ (resp. $\gamma\downarrow$) is given the ideal endpoint γ^+ (resp. γ^-). If γ is an X-line (resp. Y-line), we write l, l^+, l^-, $l\uparrow$, and $l\downarrow$ (resp. m, m^+, m^-, $m\uparrow$, and $m\downarrow$) for γ, γ^+, γ^-, $\gamma\uparrow$, and $\gamma\downarrow$ in that order.

Suppose there is a proper null chart $\chi\colon V \to E_1^2$ containing a box U for which all of the closure $\bar{R}$ of $R = \chi(U)$ except one vertex r lies in $\chi(V)$. If, for some X-line l and Y-line m on S, one of the four situations (i)–(iv) listed next applies to an ordered pair ξ, η of ideal endpoints, then (and only then) we write $\xi \nearrow \eta$.

(i) The chart χ maps an end $l\uparrow$ (resp. $m\uparrow$) onto the top (resp. right) edge of R with $\xi = l^+$, $\eta = m^+$, and r the upper right vertex of R. (See Figure 1a.)

(ii) The chart χ maps an end $m\uparrow$ (resp. $l\downarrow$) onto the left (resp. top) edge of R with $\xi = m^+$, $\eta = l^-$, and r the upper right vertex of R. (See Figure 1b.)

(iii) The chart χ maps an end $l\downarrow$ (resp. $m\downarrow$) onto the bottom (resp. left) edge of R with $\xi = l^-$, $\eta = m^-$, and r the lower left vertex of R. (See Figure 1c.)

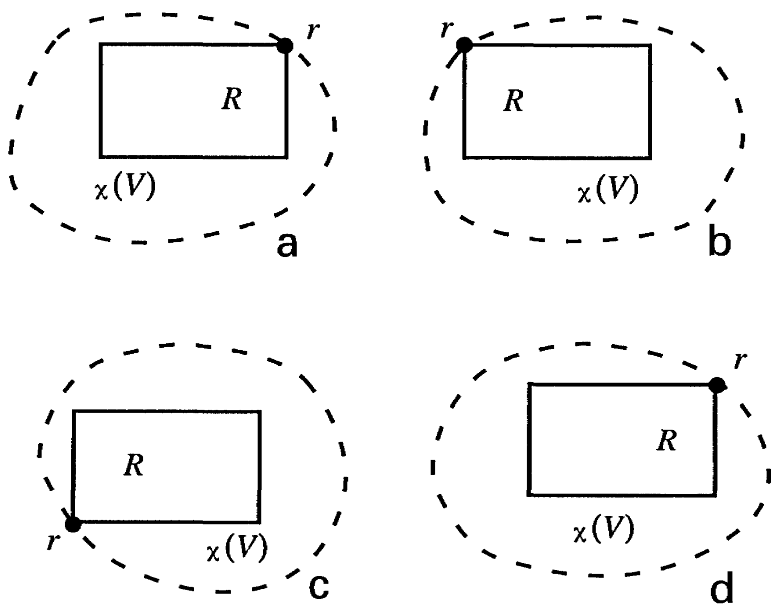

Figure 1

(iv) The chart χ maps an end $m\downarrow$ (resp. $l\uparrow$) onto the right (resp. bottom) edge of R with $\xi = m^-$, $\eta = l^+$, and r the lower right vertex of R. (See Figure 1d.)

We write $\xi \nearrow\!\!\nearrow \eta$ for an ordered pair ξ, η of ideal endpoints if and only if there are ideal endpoints $\xi_1, \xi_2, \ldots, \xi_m$ for some $m = 2, 3, \ldots$ such that $\xi = \xi_1 \nearrow \xi_2 \nearrow \cdots \nearrow \xi_m = \eta$. One never has $\xi \nearrow\!\!\nearrow \xi$, nor can one have $\xi \nearrow\!\!\nearrow \eta$ for ideal endpoints ξ and η of a single null line. In addition, one can never have both $\xi \nearrow\!\!\nearrow \eta$ and $\eta \nearrow\!\!\nearrow \xi$ for a particular pair ξ, η of ideal endpoints. (See [7, Sec. 4.1].)

Let Σ be the set of all ideal endpoints on S. Write $\xi \cong \eta$ for ξ and η in Σ if and only if $\xi = \eta$ or $\xi \nearrow\!\!\nearrow \eta$ or $\eta \nearrow\!\!\nearrow \xi$. Then $\cong$ is an equivalence relation on Σ, and Kulkarni defines $\partial_0 S$ as $\Sigma/\cong$. Thus each point on $\partial_0 S$ is an equivalence class of ideal endpoints on S. The ideal endpoints in any one point p on $\partial_0 S$ are linearly ordered by $\nearrow\!\!\nearrow$. Hence we can define the rank $\sigma(p)$ of p as the number of ideal endpoints in p if that is a finite number, as $+\infty$ if p has a first but no last entry, as $-\infty$ if p has a last but no first entry, and as $\pm\infty$ if p has neither a first nor a last entry. (See [7, Sec. 4.1] for examples that display all possible values for $\sigma(p)$.) If $\sigma(p) = 2$ (resp. $\sigma(p) = 1$), then the *type* of p is the linearly ordered set of its entries, so that any such p has type $\{l^+, m^+\}$, $\{m^+, l^-\}$, $\{l^-, m^-\}$, or $\{m^-, l^+\}$ (resp. $\{l^+\}$, $\{m^+\}$, $\{l^-\}$, or $\{m^-\}$).

The next result follows directly from Theorem 2 if one uses Higgins's theorem, stated in [7, p. 134] and proved in [2]. Since [2] has not yet appeared, an independent argument using Lemma 4 to establish Theorem 3 is sketched below.

THEOREM 3. *Suppose $\mathcal{Z}\colon S \to E^3_1$ is an E^3-complete, timelike C^∞ immersion of a simply connected, oriented surface S. Then, using the Lorentzian metric and time orientation induced on S by $\mathcal{Z}$,*

(i) *S is C^∞-conformally diffeomorphic to a subset of E^2_1 so as to preserve time orientation, and*
(ii) *$\sigma(p) \leq 2$ for each p on $\partial_0 S$, with p of type $\{m^+, l^-\}$ or $\{m^-, l^+\}$ if $\sigma(p) = 2$.*

Proof. Use the orientation preserving C^∞ diffeomorphism $\phi\colon \mathbb{R}^2 \to S$ provided by Lemma 4 to pull the time-oriented Lorentzian metric on S back to the (ζ, s) plane. Then ζ is a regular C^∞ parameter that respects natural orientation on each X-line $s = s_0$. Similarly, s is a regular C^∞ parameter that respects natural orientation on each Cauchy surface $\zeta = \zeta_0$.

Every Cauchy surface Γ_c on S is naturally oriented, so that $\Gamma_c'(s_0)$ lies in the first quadrant of S_p for $p = \Gamma_c(s_0)$ when the naturally oriented null directions X_p and Y_p are taken as coordinate axes in S_p. By Lemma 2, it follows that each Y-line in the (ζ, s) plane coincides with the graph of a C^∞ function $s = f(\zeta)$, with $f'(\zeta) < 0$ for all real values of ζ. The global time function $z = -\zeta$ is a regular C^∞ parameter on each Y-line $s = f(\zeta)$ that respects natural orientation. Over the interval $f(\mathbb{R})$, s is also a regular C^∞ parameter on each Y-line $s = f(\zeta)$ that respects natural orientation.

Given a point (ζ_0, s_0), let $(0, u(\zeta_0, s_0))$ be the point where the Y-line through (ζ_0, s_0) crosses the Cauchy surface $\zeta = 0$. Thus, for each real constant c, the equation $u(\zeta, s) = c$ describes a Y-line $s = f(\zeta)$. We leave it to the reader to check that $u(\zeta, s)$ is C^∞ with $u_\zeta(\zeta, s) > 0$ and $u_s(\zeta, s) > 0$. Hence, the map $\psi : \mathbb{R}^2 \to E_1^2$ given by

$$\psi(\zeta, s) = (u(\zeta, s), s)$$

is C^∞ and one–one onto an open set Ω in E_1^2. The Jacobian matrix

$$\begin{pmatrix} u_\zeta & 0 \\ u_s & 1 \end{pmatrix}$$

of ψ has positive determinant, so that ψ preserves orientation.

Since ψ takes X-lines (resp. Y-lines) in the (ζ, s) plane onto X-lines (resp. Y-lines) in Ω, it follows that $\psi : \mathbb{R}^2 \to \Omega$ is a C^∞-conformal diffeomorphism. (See [7, p. 37].) Thus $\psi \circ \phi^{-1}$ is a C^∞-conformal diffeomorphism taking S onto $\Omega \subset E_1^2$. Because natural orientation on X-lines and Y-lines is preserved by ψ and ϕ, $\psi \circ \phi^{-1}$ is also time-orientation preserving. This establishes (i).

Because Ω is not a bounded set in the (u, v) plane, some points on $\partial_0\Omega$ are hard to visualize. Thus we replace Ω by its image $\hat{\Omega} \subset (-1, 1) \times (-1, 1) \subset E_1^2$ under the orientation and time-orientation preserving C^∞-conformal diffeomorphism of E_1^2 taking u to $\tanh u$ and v to $\tanh v$. The points on $\partial_0\hat{\Omega}$ are represented by points on the topological boundary of $\hat{\Omega}$ in E_1^2 that are endpoints of maximal horizontal or vertical line segments in $\hat{\Omega}$. To study $\partial_0 S$, we study $\partial_0\hat{\Omega}$.

If $\sigma(p) \geq 3$ at a point p on $\partial_0\hat{\Omega}$, then p contains three ideal endpoints satisfying $m_1^- \nearrow l^+ \nearrow m_2^+$, $l_1^+ \nearrow m^+ \nearrow l_2^-$, $m_1^+ \nearrow l^- \nearrow m_2^-$, or $l_1^- \nearrow m^- \nearrow l_2^+$. (See Figure 2.) In all four cases, two different null lines (l_1 and l_2 or m_1 and m_2) in $\hat{\Omega}$

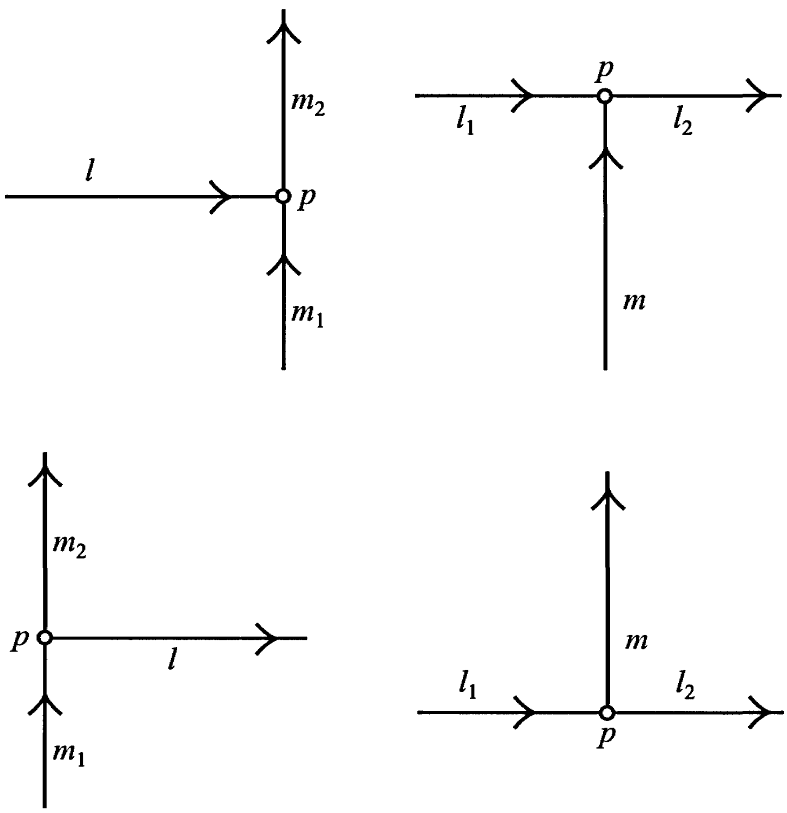

Figure 2

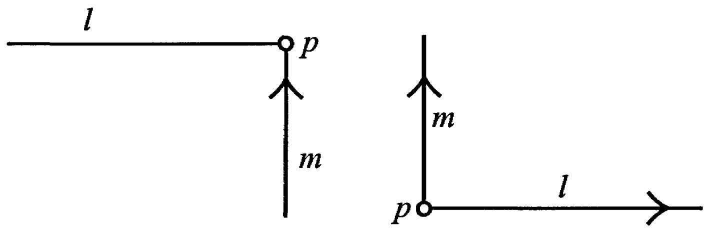

Figure 3

lie to opposite sides of p along the same horizontal or vertical line in the (u, v) plane. Since each Cauchy surface Γ_c on S is taken to a curve of positive slope in $\hat{\Omega}$, the image of Γ_c cannot meet both l_1 and l_2 or both m_1 and m_2. Since Γ_c crosses every null line in S, its image must do the same in $\hat{\Omega}$. This contradiction gives $\sigma(p) \leq 2$. Suppose $\sigma(p) = 2$ for p on $\partial_0\hat{\Omega}$, with p of type $\{l^+, m^+\}$ or $\{l^-, m^-\}$. (See Figure 3.) In either case, the image of a Cauchy surface Γ_c is a curve of positive slope in $\hat{\Omega}$ that cannot meet both l and m. Thus p has type $\{m^+, l^-\}$ or $\{m^-, l^+\}$, which completes the proof of (ii). □

Theorem 3 can be used to obtain the following extension of Theorem 1 from [4].

THEOREM 4. *Suppose $\mathcal{Z}: S \to E_1^3$ is an E^3-complete C^∞ timelike immersion from a simply connected S. If $\mathcal{Z}$ is minimal outside a compact subset S_0 of S, then S with the Lorentzian metric induced by $\mathcal{Z}$ is C^∞-conformally diffeomorphic to E_1^2.*

Proof. Orient S and use $\mathcal{Z}$ to time-orient S. Then S can be replaced by the subset $\hat{\Omega}$ of E_1^2 described in the proof of Theorem 3. Let $\hat{\Omega}_0$ be the compact subset of $\hat{\Omega}$ corresponding to S_0, and think of $\mathcal{Z}$ as an immersion $\mathcal{Z}: \hat{\Omega} \to E_1^3$. By Theorem 3, $\sigma(p) \leq 2$ for any p on $\partial_0\hat{\Omega}$ with p of type $\{m^+, l^-\}$ or $\{m^-, l^+\}$ in case $\sigma(p) = 2$. Suppose there is a point p on $\partial_0\hat{\Omega}$ with $\sigma(p) = 2$, so that one of the diagrams in Figure 4 applies. There is a rectangle $R = (a, b) \times (c, d)$ whose closure in the (u, v) plane less p lies in $\hat{\Omega}$, with one side along l and another side along m. Here R can be made so small that its closure in $\hat{\Omega}$ lies in an open set $\mathcal{U} \subset \hat{\Omega}$, with $\mathcal{U} \cap \hat{\Omega}_0 = \emptyset$, whose intersection with any horizontal or vertical line in the (u, v) plane is either empty or a single line segment. Since $\mathcal{Z}$ is minimal outside of Ω_0, mean curvature H for $\mathcal{Z}$ vanishes on $\mathcal{U}$. Since

$$I = \mu\, du\, dv$$

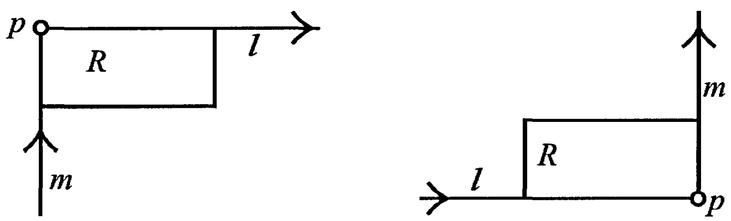

Figure 4

for some C^∞ function $\mu > 0$, the middle coefficient of II must vanish on $\mathcal{U}$, which means that $\mathcal{Z}_{uv} \equiv 0$ on $\mathcal{U}$. (See [7, p. 176].) It follows from the shape of $\mathcal{U}$ that

$$\mathcal{Z}(u, v) = \mathcal{X}(u) + \mathcal{Y}(v)$$

on $\mathcal{U}$. Since $\mathcal{Z}$ is E^3-complete, the E^3 length of the portion l_0 of l bordering R is infinite. Since the values of $\mathcal{Z}$ along any maximal horizontal line interval $\mathcal{I}_0$ differ from those along l_0 by a constant vector in E_1^3, the E^3 length of $\mathcal{I}_0$ must be infinite. But this is impossible, since the closure of $\mathcal{I}_0$ is the compact image of an arc $\lambda\colon [a, b] \to \hat{\Omega}$ along an X-line in $\hat{\Omega}$. This contradiction shows that $\sigma(p) \equiv 1$ on $\partial_0\hat{\Omega}$. By [7, Lemma 27, p. 94], it follows that $\hat{\Omega}$ (and thereby S) is C^∞-conformally diffeomorphic to E_1^2. □

A corollary of Theorem 4 is the conformal Bernstein theorem from [4], which states that any entire timelike minimal surface in E_1^3 must be C^∞-conformally diffeomorphic to E_1^2. By Theorem 4, we now know that any entire timelike surface in E_1^3 is C^∞-conformally diffeomorphic to E_1^2 provided it is minimal outside a compact set.

4. Some Examples

In this section we construct uncountably many simply connected globally hyperbolic subsurfaces of E_1^2, any two of which are C^0-conformally distinct and thereby C^∞-conformally distinct. For the definition of C^j-conformal equivalence for any $j = 0, 1, \ldots; \infty$, see [7, Sec. 3.1]. To see that these sorts of conformal equivalence are different from one another, see [6].

Given a real number r with $0 < r < 1$, let $r = .r_1r_2r_3\ldots$ be the binary expansion of r, chosen so that

$$r = \sum_{j=1}^{\infty} \frac{r_j}{2^j}$$

without allowing $r_j = 1$ when $j \geq j_0$ for any fixed $j_0 = 2, 3, \ldots$. Associate to $r = .r_1r_2r_3\ldots$ two sequences $\{p_k\}$ and $\{q_k\}$ of points in E_1^2 with

$$p_k = \left(\sum_{j=1}^{k} \frac{1 + r_j}{2^j}, \sum_{j=1}^{k} \frac{1}{2^j}\right),$$

$$q_k = \left(\sum_{j=1}^{k+1} \frac{1}{2^j} + \sum_{j=1}^{k} \frac{r_j}{2^j}, \sum_{j=1}^{k} \frac{1}{2^j}\right),$$

so that both p_k and q_k converge to the point $(1 + r, 1)$ as $k \to \infty$.

Construct a simple closed arc Λ_r in E_1^2 beginning and ending at the point $(1 + r, 1)$ by following in order the line segments joining $(1 + r, 1)$ to $(0, 1)$, $(0, 1)$ to $(0, 0)$, $(0, 0)$ to $(1/2, 0)$, $(1/2, 0)$ to p_1, p_1 to q_1, q_1 to q_2, $\ldots$, p_k to q_k, and q_k to $p_{k+1}, \ldots$. Let S_r be the interior of Λ_r provided with the Lorentzian metric, orientation and time orientation induced from E_1^2.

To see that S_r is globally hyperbolic, let $\gamma: [0, 1] \to S_r \cup \Lambda_r$ be any C^∞ arc with $\gamma(0) = (0, 0)$, $\gamma(1) = (1+r, 1)$, and $\gamma(t)$ in S_r with $\gamma'(t)$ spacelike for $0 < t < 1$. Then $\gamma: (0, 1) \to S_r$ is a Cauchy surface on S_r.

To show that S_r and $S_{r'}$ are C^∞-conformally distinct when $r \neq r'$, look at the conformal boundary $\partial_0 S_r$ for any r with $0 < r < 1$. (Claims made about $\partial_0 S_r$ can be checked in [7, Secs. 4.2-4.5].) Each point on $\partial_0 S_r$ is represented by the end-point of a maximal horizontal and/or vertical line segment in S_r. Thus every point on Λ_r represents a point on $\partial_0 S_r$, except for the points $(1 + r, 1)$, $(0, 1)$, $(0, 0)$, and $(1/2, 0)$ in case $r_1 = 0$ and for the points q_k in case $r_{k+1} = 0$ for a given $k = 1, 2, \ldots$. The topology on $\partial_0 S_r$ as defined in [7, Sec. 4.3] coincides with the topology induced upon it as a subset of Λ_r by E_1^2. There is also a cyclic order on $\partial_0 S_r$ (defined by Smyth in [5] and described in [7, Sec. 4.5]) that coincides with the usual counterclockwise cyclic order of points on Λ_r. This cyclic order puts a linear order on the points lying along any one line segment on $\partial_0 S_r$.

Each point on the open line segment from $(1 + r, 1)$ to $(0, 1)$ (resp. from $(0, 1)$ to $(0, 0)$ or from $(0, 0)$ to $(1/2, 0)$) has rank 1 and type $\{m^+\}$ (resp. type $\{l^-\}$ or $\{m^-\}$). If $r_1 = 1$, then $(1/2, 0)$ is also a rank-1 point of type $\{m^-\}$ on $\partial_0 S_r$. Similarly, if $r_{k+1} = 1$ for some $k = 1, 2, \ldots$, then q_k is a rank-1 point of type $\{m^-\}$ on $\partial_0 S_r$. For every $k = 1, 2, \ldots$, p_k is a rank-2 point of type $\{m^-, l^+\}$. Each point p on the open line segment from $(1/2, 0)$ to p_1 (resp. from q_k to p_{k+1} for $k = 1, 2, \ldots$) has rank $r_1 + 1$ (resp. $r_{k+1} + 1$) and type $\{l^+\}$ if $\sigma(p) = 1$ and type $\{m^-, l^+\}$ if $\sigma(p) = 2$. Finally, every point on the open line segment from p_k to q_k for any $k = 1, 2, \ldots$ has rank 1 and type $\{m^-\}$.

Fix r and r' with $0 < r < 1$ and $0 < r' < 1$. Any orientation and time-orientation preserving C^∞-conformal diffeomorphism F from S_r onto $S_{r'}$ is automatically an orientation and time-orientation preserving conformal homeomorphism from S_r onto $S_{r'}$ (as defined in [7, Sec. 3.1]). Any orientation and time-orientation preserving conformal homeomorphism F from S_r to $S_{r'}$ extends uniquely to a homeomorphism $\bar{F}$ from $S_r \cup \partial_0 S_r$ onto $S_{r'} \cup \partial_0 S_{r'}$, which restricts to a homeomorphism ∂F from $\partial_0 S_r$ to $\partial_0 S_{r'}$. The map ∂F preserves rank as well as the type of rank-1 and rank-2 points. Finally, $\partial_0 S_r$ preserves cyclic order, so that $p \hookrightarrow q \hookrightarrow r$ on $\partial_0 S_r$ implies that $F(p) \hookrightarrow F(q) \hookrightarrow F(r)$ on $\partial_0 S_{r'}$.

It is now easy to check that ∂F takes each maximal line segment on $\partial_0 S_r$ onto the corresponding maximal line segment on $\partial_0 S_{r'}$ so as to preserve the linear order induced by counterclockwise cyclic order. If $r_1 = 0$, then $(1/2, 0)$ does not lie on $\partial_0 S_r$ and therefore cannot lie on $\partial_0 S_{r'}$, which gives $r' = 0$. If $r_1 = 1$, then $(1/2, 0)$ lies on $\partial_0 S_r$, is fixed by ∂F, and lies on $\partial_0 S_{r'}$; this gives $r' = 1$. Assume that $r_j = r_j'$ for $j = 1, 2, \ldots, k$ for any $k = 1, 2, \ldots$. Then each point p_j and q_j on Λ_r coincides with the corresponding point p_j and q_j on $\Lambda_{r'}$ for $j = 1, 2, \ldots, k$. If $r_{k+1} = 0$, then q_k does not lie on $\partial_0 S_r$ and therefore cannot lie on $\partial_0 S_{r'}$, which gives $r_{k+1}' = 0$. If $r_{k+1} = 1$, then q_k lies on $\partial_0 S_r$ and therefore lies on $\partial_0 S_{r'}$, which gives $r_{k+1}' = 1$. The existence of an orientation and time-orientation preserving conformal homeomorphism F from S_r onto $S_{r'}$ thus implies that $r = r'$. We conclude that there are uncountably many C^0- (and thereby C^∞-) conformally distinct simply connected subsurfaces S_r of E_1^2, one for each r with $0 < r < 1$.

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S. Lin
Mathematics Department
Rutgers University
New Brunswick, NJ 08903

senchun@math.rutgers.edu

T. Weinstein
Mathematics Department
Rutgers University
New Brunswick, NJ 08903

tilla@math.rutgers.edu