The Minimal Norm Property for Quadratic Differentials in the Disk

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1. Introduction

Let Δ be the unit disk. We let $M(\Delta)$ be the open unit ball of $L^{\infty}(\Delta)$. For any μ in $M(\Delta)$, there exists a solution $f\colon \Delta \to \Delta$ of the Beltrami equation

$$f_{\bar{z}} = \mu f_z, \tag{1}$$

unique up to a postcomposition by a Möbius transformation. We call f a *quasiconformal* homeomorphism of a disk with the Beltrami coefficient μ, and we denote by f^{μ} the solution f of (1) normalized by $f(i) = i$, $f(1) = 1$, and $f(-1) = -1$. The solution f can be extended to a homeomorphism of the closure of Δ, and the restriction h of that extension to the boundary of Δ is called a *quasisymmetric* homeomorphism of a circle. The dilatation $K(h)$ of a quasisymmetric homeomorphism h is the infimum of all maximal dilatations of quasiconformal extensions of h to Δ. The boundary dilatation $H(h)$ of a quasisymmetric homeomorphism h is obtained by looking at the infimum of all maximal dilatations of quasiconformal extensions of h to a neighborhood U of the boundary and taking the limit of these dilatations as U shrinks to the boundary. We call a quasisymmetric homeomorphism h *symmetric* if $H(h) = 1$.

Let QC(Δ) be the space of all quasiconformal homeomorphisms of Δ. Two elements f_1, f_2 in QC(Δ) are *equivalent* if there exists a conformal homeomorphism α of Δ such that $f_1(t) = \alpha \circ f_2(t)$ for every $t \in \partial\Delta$. The Teichmüller space $T(\Delta)$ is QC(Δ) factored by this equivalence relation. The equivalence class of the identity mapping is called the *basepoint* of $T(\Delta)$.

We let $A(\Delta)$ be the Banach space of all holomorphic quadratic differentials φ on Δ satisfying $\|\varphi\| = \iint_{\Delta} |\varphi| < \infty$. One useful property of the Banach space $A(\Delta)$ is the following lemma, due to Strebel (see [S2]).

LEMMA 1. *Let φ be an arbitrary holomorphic quadratic differential of norm $\|\varphi\| \le M < \infty$ in the unit disk Δ. Let w be a boundary point of Δ. Then, for any $\varepsilon > 0$ and $\rho_2 > 0$, there exists a number ρ_1, $0 < \rho_1 < \rho_2$, such that*

$$\int_{\sigma_\rho} |\varphi(z)|^{1/2} |dz| < \varepsilon$$

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for some $\rho \in [\rho_1, \rho_2]$, *with* $\sigma_\rho = \{z \in \Delta : |z - w| = \rho\}$. *Whereas* ρ *depends on* φ, ρ_1 *does not.*

Every differential φ in $A(\Delta)$ defines two invariants, the area element $dA = |\varphi(z)|\,dx\,dy$ and the line element $ds = \sqrt{|\varphi(z)|}|dz|$. The φ-length of an arc γ in Δ is $\int_\gamma ds$, and the height of γ with respect to φ is $h_\varphi(\gamma) = \int_\gamma |\mathrm{Im}\sqrt{\varphi(z)}\,dz|$. The vertical distance between two points w_1, w_2 in the closure of Δ is the infimum of the heights of all curves γ in Δ with endpoints at w_1 and w_2. A vertical (horizontal) arc of φ is a smooth arc γ in Δ along which $\varphi(z)\,dz^2$ is less (greater) than 0. A vertical (horizontal) trajectory of φ is a maximal vertical (horizontal) arc. It is called *regular* if it does not tend to a zero of φ in either direction. A regular horizontal trajectory α is called *totally* regular, if for any sequence of points z_n converging to a point z on α and such that the horizontal trajectories of φ passing through points z_n are regular, $\alpha_n \to \alpha$ in the Euclidean metric. If γ is an open horizontal arc, then the subset of Δ covered by the vertical trajectories through the points of γ is called the *vertical strip* S determined by γ. There is a countable sequence of vertical strips determined by open horizontal arcs, which cover Δ up to a countable set of vertical trajectories and points (see [S3]).

If f is a quasiconformal homeomorphism of the unit disk Δ and φ is in $A(\Delta)$, then there exists the unique integrable holomorphic quadratic differential ψ such that the vertical φ-distance between any two boundary points r and s is equal to the vertical ψ-distance between $f(r)$ and $f(s)$ (see [S1]). We say that ψ is the *image* of φ under the mapping by heights induced by f, and we denote ψ by $H(f, \varphi)$. Notice that if $[f_1]$ and $[f_2]$ are the same points in $T(\Delta)$ then there exists a conformal homeomorphism α of Δ such that $f_1(t) = \alpha \circ f_2(t)$ for every $t \in \partial\Delta$. Therefore, $\psi_2 = H(f_2, \varphi)$ is a pullback of $\psi_1 = H(f_1, \varphi)$ by α:

$$\psi_2 = \psi_1(\alpha)\alpha'^2.$$

This yields $\|\psi_1\| = \|\psi_2\|$. We define a function from $T(\Delta) \times A(\Delta)$ onto $A(\Delta)$ by $(\tau, \varphi) \to H(f, \varphi)$, where f is normalized to fix 1, -1, and i, and where $[f] = \tau$. This function describes the mapping by heights up to a pullback by Möbius transformations, so we will also call it the mapping by heights and denote it by H.

In this article we show that the mapping by heights H is continuous. We do this by studying the minimal norm property for the measured foliations in the disk and developing the variation in the Dirichlet norm.

A *measured foliation* with measure $|dv|$ on Δ is given by an open cover U_i of a complement of a set of Lebesgue measure zero in Δ and C^1 functions v_i on each U_i such that

$$dv_i = \pm dv_j \quad \text{on } U_i \cap U_j. \tag{2}$$

The *leaves* of the foliation are curves along which v is constant. The height of an arc γ is defined by $h_v(\gamma) = \int_\gamma |dv|$. We will denote a measured foliation by the symbol $|dv|$. The norm $\|dv\|^2$ of a measured foliation $|dv|$ is an invariant defined by

$$\|dv\|^2 = \iint_\Delta \left(\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \right) dx\,dy.$$

An example of a measured foliation is $|dv| = |\mathrm{Im}\sqrt{\varphi}\,dz|$, where φ is a holomorphic quadratic differential on Δ and the corresponding set of measure zero is the set of zeros of φ.

2. A Minimal Norm Property

THEOREM 1. *Let φ be in $A(\Delta)$ and let ψ be a measurable quadratic differential in Δ. Suppose that $h_\varphi(\gamma) \le h_\psi(\gamma)$ for almost every regular vertical trajectory γ of φ. Then*

$$\|\varphi\| \le \iint_\Delta \sqrt{|\psi\varphi|} \le \|\psi\| \tag{3}$$

and $\|\varphi\| = \|\psi\|$ only if $\varphi = \psi$ a.e.

Proof. Decompose Δ into a disjoint union of a set of measure zero and vertical strips $S_1, S_2, S_3, \ldots$ determined by the horizontal arcs of φ. For every i there exists a conformal mapping h_i of S_i onto a plane vertical strip $V_i = \{\,(x, y) \mid 0 < x < b_i,\ c_i(x) < y < d_i(x)\,\}$ such that $c_i(x)$ is an upper and $d_i(x)$ is a lower semicontinuous function from $(0, b_i)$ into $[-\infty, \infty]$, $\varphi = 1$ on V_i, and for almost every $x \in (0, b_i)$, the vertical segment $\{\,(x, y) \mid c_i(x) < y < d_i(x)\,\}$ is mapped by h_i^{-1} onto a regular vertical trajectory γ_x in S_i. Therefore, for almost every $x \in (0, b_i)$,

$$h_\varphi(\gamma_x) \le h_\psi(\gamma_x) = \int_{c_i(x)}^{d_i(x)} |\mathrm{Im}\sqrt{\psi}\,dz| \le \int_{c_i(x)}^{d_i(x)} |\sqrt{\psi}|\,dy. \tag{4}$$

Integrating (4), we obtain

$$\iint_{S_i} |\varphi| \le \int_0^{b_i}\int_{c_i(x)}^{d_i(x)} |\sqrt{\psi}|\,dy\,dx = \iint_{S_i} |\sqrt{\psi\varphi}|.$$

Summing over all i yields

$$\|\varphi\| \le \iint_\Delta |\sqrt{\psi\varphi}|.$$

By Schwarz's inequality, $\left(\iint_\Delta |\sqrt{\psi\varphi}|\right)^2 \le \|\varphi\|\,\|\psi\|$. Therefore, $\|\varphi\| \le \|\psi\|$, and this yields (3).

If $\|\varphi\| = \|\psi\|$ then $|\mathrm{Im}\sqrt{\psi(z)}\,dz| = |\sqrt{\psi(z)}|\,dy$ for almost all $z \in V_i$; thus, $|\mathrm{Re}\sqrt{\psi(z)}| = |\sqrt{\psi(z)}|$ and $\mathrm{Im}\sqrt{\psi(z)} = 0$ for almost all $z \in V_i$. The equality in Schwarz's inequality implies that $|\sqrt{\psi(z)}| = C|\sqrt{\varphi(z)}|$ a.e. for some constant $C > 0$. Since $\|\varphi\| = \|\psi\|$ it follows that $C = 1$; thus $\psi(z) = 1 = \varphi(z)$ for a.e. $z \in V_i$. Therefore, $\psi = \varphi$ a.e. □

Theorem 1 also holds when $|\mathrm{Im}\sqrt{\psi}\,dz|$ is replaced by a measured foliation in Δ.

THEOREM 2. *Let $\varphi \in A(\Delta)$ and let $|dv|$ be a measured foliation on Δ satisfying the height condition $h_v(\gamma) \ge h_\varphi(\gamma)$ for almost every regular vertical trajectory γ induced by φ. Then the norm inequality $\|dv\|^2 \ge \|\varphi\|$ holds, with equality only for $|dv| = |\mathrm{Im}\sqrt{\varphi}\,dz|$.*

Proof. Consider a vertical strip S_i from the proof of Theorem 1. For almost every $x \in (0, b_i)$,

$$\int_{c_i(x)}^{d_i(x)} dy = h_\varphi(\gamma_x) \le h_v(\gamma_x) = \int_{c_i(x)}^{d_i(x)} \left|\frac{\partial v}{\partial y}\right| dy.$$

Integrating over $x \in (0, b_i)$ yields

$$\iint_{S_i} |\varphi| = \int_0^{b_i} \int_{c_i(x)}^{d_i(x)} dy\, dx \le \int_0^{b_i} \int_{c_i(x)}^{d_i(x)} \left|\frac{\partial v}{\partial y}\right| dy\, dx.$$

Therefore, by Schwarz's inequality,

$$\left(\iint_{S_i} |\varphi|\right)^2 \le \left(\int_0^{b_i} \int_{c_i(x)}^{d_i(x)} dy\, dx\right)\left(\int_0^{b_i} \int_{c_i(x)}^{d_i(x)} \left(\frac{\partial v}{\partial y}\right)^2 dy\, dx\right).$$

This yields

$$\iint_{S_i} |\varphi| \le \iint_{S_i} \left(\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right) dx\, dy. \tag{5}$$

Summing (5) over all vertical strips S_i, we obtain $\|\varphi\| \le \|dv\|^2$.

Let equality hold. Then $|\frac{\partial v}{\partial y}(z)| = C \ge 0$ and $\frac{\partial v}{\partial x}(z) = 0$ for $z \in V_i$. Since

$$\iint_{S_i} dx\, dy = \iint_{S_i} |\varphi| = \iint_{S_i} \left(\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right) dx\, dy,$$

$C = 1$. Therefore, for all z in V_i, $v(z) = \pm y +$ constant; hence $dv = \pm dy = \pm\mathrm{Im}(\sqrt{\varphi(z)}\, dz)$. □

3. Weak Continuity of the Mapping by Heights

DEFINITION 1. A sequence φ_n in $A(\Delta)$ weakly converges to $\varphi \in A(\Delta)$ if φ_n converges to φ locally uniformly on compact sets and the norms $\|\varphi_n\|$ are uniformly bounded above. A sequence φ_n is *degenerating* if it weakly converges to zero. A degenerating sequence φ_n is called *strictly* degenerating if the norms $\|\varphi_n\|$ are uniformly bounded below away from zero.

DEFINITION 2. A sequence $[f_n]$ of elements in $T(\Delta)$ induced by normalized quasisymmetric homeomorphisms f_n of the circle $\partial\Delta$ converges weakly to $[f]$ if $f_n(t)$ converges to $f(t)$ for every $t \in \partial\Delta$ and if the Teichmüller distances from $[f_n]$ to the basepoint are uniformly bounded.

In [S1], Strebel proved that if φ_n converges weakly to φ then $H(f, \varphi_n)$ converges weakly to $H(f, \varphi)$. Here, using the same method of proof, we slightly generalize that result.

LEMMA 2. *If φ_n is a sequence in $A(\Delta)$ that converges weakly to $\varphi \in A(\Delta)$, and if f_n is a sequence of normalized quasisymmetric homeomorphisms of $\partial\Delta$ such that $[f_n]$ weakly converges to $[f]$ for some quasisymmetric homeomorphism f of $\partial\Delta$, then $H(f_n, \varphi_n)$ converges weakly to $H(f, \varphi)$.*

Proof. The proof of this lemma follows the same steps of the proof of [S1, Thm. 5.2]. Let $\psi_n = H(f_n, \varphi_n)$. Since the Teichmüller distances d_n from $[f_n]$ to the basepoint are uniformly bounded and f_n are normalized, $\|\psi_n\| \leq \|\varphi_n\|e^{2d_n} \leq$ constant and functions f_n are uniformly Hölder continuous. Therefore, f_n tends to f uniformly on $\partial\Delta$ and (ψ_n) is a normal family. By passing to a subsequence, we can assume that ψ_n converges locally uniformly to some integrable holomorphic quadratic differential ψ.

Suppose first that $\varphi \neq 0$. Take a totally regular horizontal trajectory α of φ. Then there exist totally regular trajectories α_n of φ_n such that α_n converges to α in the Euclidean metric. Let p and q be the endpoints of α, and let p_n and q_n be the endpoints of α_n on $\partial\Delta$. By [S1, Thm. 5.2], $f_n(p_n)$ and $f_n(q_n)$ are connected by a totally regular horizontal trajectory β_n of ψ_n. Furthermore, $f_n(p_n) \to f(p)$ and $f_n(q_n) \to f(q)$.

Step 1: Every t on $\partial\Delta - \{f(p), f(q)\}$ has an ε neighborhood U_ε that is free from β_n for all sufficiently large n.

Proof. Assume the contrary. Then there is a point t in $\partial\Delta - \{f(p), f(q)\}$ and a sequence of trajectories β_n (we avoid double indices) with $z_n \in \beta_n$ and $z_n \to t$. Since α is totally regular, we can choose a totally regular horizontal trajectory δ of φ separating $f^{-1}(t)$ from α such that $h_\varphi(\alpha, \delta) > 0$. Take a sequence of totally regular horizontal trajectories δ_n of φ_n such that δ_n converges to δ in the Euclidean metric. Then there are totally regular horizontal trajectories γ_n of ψ_n that separate β_n from t and such that $h_{\psi_n}(\beta_n, \gamma_n) = h_{\varphi_n}(\alpha_n, \delta_n)$. (See Figure 1.)

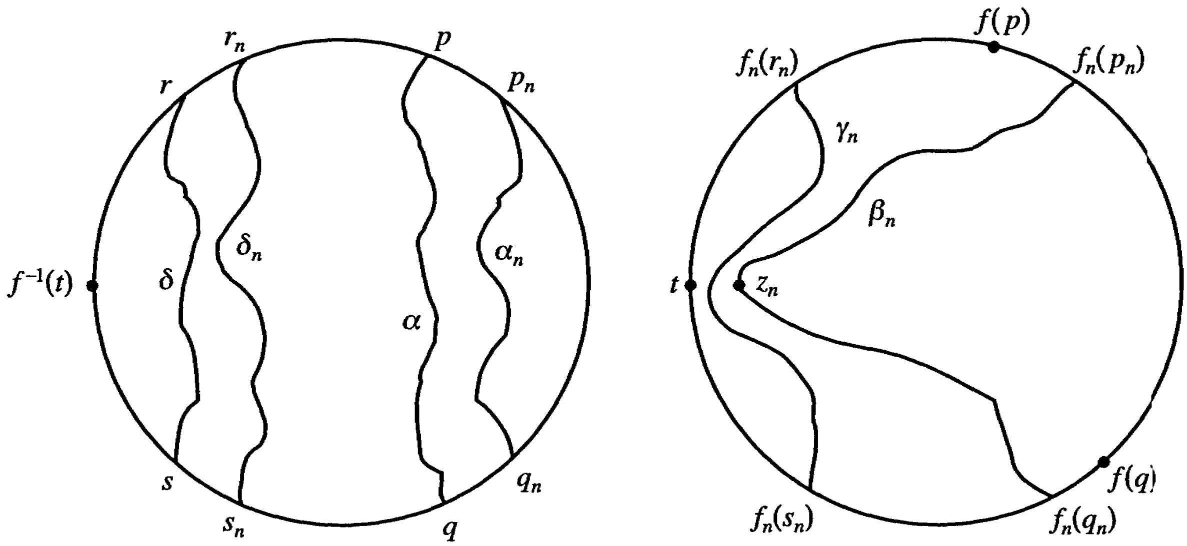

Figure 1

Now $h_{\psi_n}(\beta_n, \gamma_n) = h_{\varphi_n}(\alpha_n, \delta_n) \to h_\varphi(\alpha, \delta) > 0$ and $\|\psi_n\| \leq \|\varphi_n\|e^{2d_n} \leq$ constant. However, since $z_n \to t$, a semicircular arc with center at t must cross γ_n and β_n, which contradicts Lemma 1.

Step 2: Sequence ψ_n is not degenerate.

Proof. Take two totally regular horizontal trajectories α and δ of φ with $h_\varphi(\alpha, \delta) > 0$. Let p and q be the endpoints of α and let r and s be the endpoints of δ. Draw a diameter d that separates $f(p)$ and $f(r)$ from $f(q)$ and $f(s)$. There exist totally regular horizontal trajectories α_n and δ_n of φ_n with endpoints p_n, q_n and r_n, s_n (respectively) such that α_n converges to α and δ_n converges to δ in the Euclidean metric. (See Figure 2.)

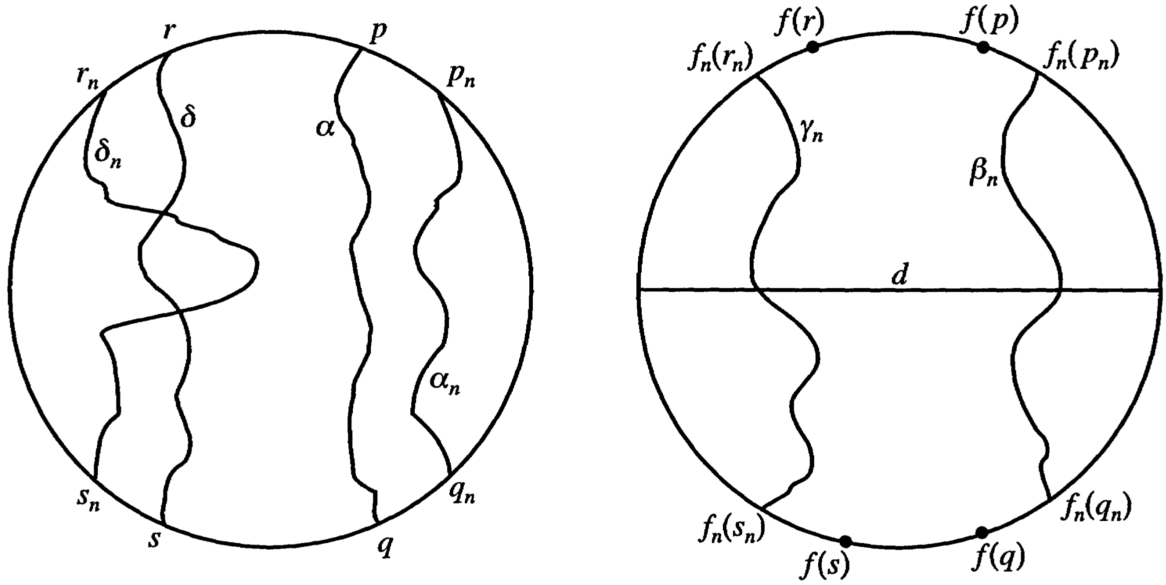

Figure 2

As a result, there exist totally regular horizontal trajectories β_n and γ_n of ψ_n that connect $f_n(p_n)$ and $f_n(q_n)$ and (respectively) $f_n(r_n)$ and $f_n(s_n)$. By Step 1, γ_n and β_n intersect d inside a compact subinterval. If $\|\psi\| = 0$, then $h_\varphi(\alpha, \delta) = \lim_{n\to\infty} h_{\varphi_n}(\alpha_n, \delta_n) = \lim_{n\to\infty} h_{\psi_n}(\beta_n, \gamma_n) = 0$, a contradiction.

Step 3: $f(p)$ and $f(q)$ are connected by a horizontal trajectory β of ψ.

Proof. Fix a double sequence of circles σ_m around $f(p)$ and $f(q)$ and restrict β_n to a part β'_n in the strip bounded by σ_n and σ_{-n}. (See Figure 3.)

By passing to a subsequence we may assume that every β'_n converges to a horizontal arc of ψ uniformly in the Euclidean metric. Taking a diagonal subsequence, we end up with a horizontal arc β of ψ. Since the endpoints $f_n(p_n)$ and $f_n(q_n)$ of β_n converge to $f(p)$ and $f(q)$, β is a horizontal trajectory that connects $f(p)$ and $f(q)$.

Step 4: β is totally regular.

Proof. If β goes through a zero z of ψ, then there is at least one more horizontal trajectory ray γ of ψ that starts at z. (See Figure 4).

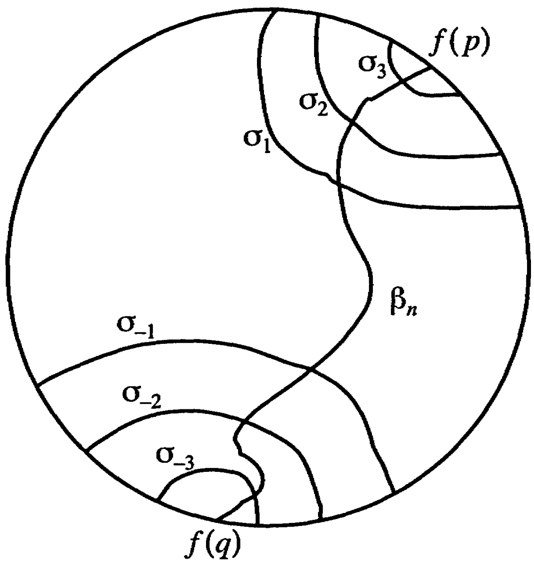

Figure 3

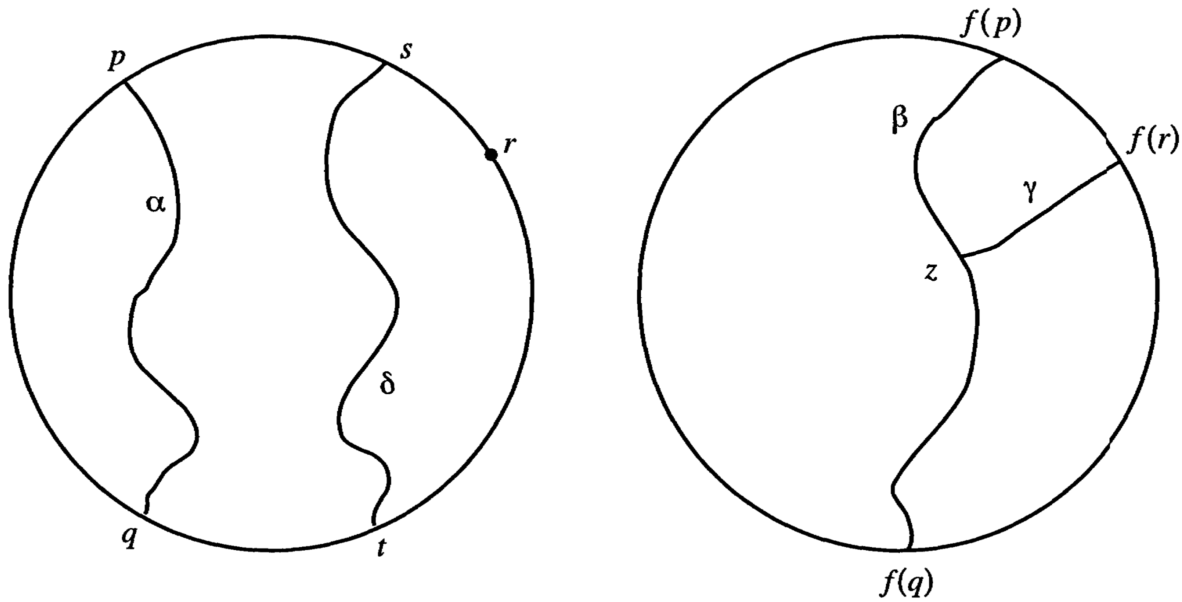

Figure 4

Let $f(r)$ be an endpoint of γ. Since there are only countably many endpoints of critical horizontal trajectories of ψ, we can choose a totally regular trajectory δ of φ with endpoints s and t such that (a) δ separates α from r and (b) $f(s)$ and $f(t)$ are not the endpoints of a critical horizontal trajectory of ψ. (Note that, by Lemma 1, if two horizontal trajectory rays end at the same point then the vertical distance between them is 0.) The points $f(s)$ and $f(t)$ are connected by a regular horizontal trajectory of ψ that must intersect $\beta \cup \gamma$. Therefore, β is not critical.

In order to show that β is totally regular, take totally regular trajectories α_n of φ that converge to α in the Euclidean norm. Let p_n and q_n be the endpoints of α_n. Then there exists a regular horizontal trajectory β_n of ψ that connects $f(p_n)$ and

$f(q_n)$. If β_n does not converge to β in the Euclidean metric, then there exists a point $a \in \bar{\Delta} - \bar{\beta}$ such that a is a limit point of some sequence $a_n \in \beta_n$. Since the endpoints of β_n converge to the endpoints of β, Lemma 1 implies that $h_\psi(\beta, \beta_n) \to 0$; therefore, $h_\psi(a, \beta) = 0$. Choose a point p on β, and let I be a vertical segment pointing to a with p as its initial point. Then there exists a regular horizontal trajectory of ψ that separates β from a and intersects $I - \{p\}$; this trajectory thus has a positive vertical distance from β, a contradiction. Since the approximation of α can be performed on both sides, β is totally regular.

Step 5: $h_\psi(f(p), f(q)) = h_\varphi(p, q)$ for every $p, q \in \partial\Delta$.

Proof. Let p and q be two distinct points on the boundary of the unit disk Δ. If α and δ are two totally regular horizontal trajectories of φ separating p and q, then there exist totally regular horizontal trajectories α_n and δ_n of φ_n such that α_n converges to α and δ_n converges to δ in the Euclidean metric. Therefore, there are totally regular horizontal trajectories β_n and γ_n of ψ_n such that $h_{\varphi_n}(\alpha_n, \delta_n) = h_{\psi_n}(\beta_n, \gamma_n)$. By Step 3, we can take points z_n from β_n and w_n from γ_n such that (a) z_n converges to a point z on the totally regular horizontal trajectory β of ψ and (b) w_n converges to a point w on the totally regular horizontal trajectory γ of ψ. Trajectories β and γ separate $f(p)$ from $f(q)$. By [S1], Thm. 4.3],

$$\begin{aligned} h_\varphi(\alpha, \delta) &= \lim_{n\to\infty} h_{\varphi_n}(\alpha_n, \delta_n) = \lim_{n\to\infty} h_{\psi_n}(\beta_n, \gamma_n) \\ &= \lim_{n\to\infty} h_{\psi_n}(z_n, w_n) = h_\psi(z, w) = h_\psi(\beta, \gamma). \end{aligned}$$

Because $h_\varphi(p, q) = \sup_{\alpha,\delta} h_\varphi(\alpha, \delta)$, where the supremum is over all pairs of totally regular horizontal trajectories α and δ that separate p from q, we have

$$h_\psi(f(p), f(q)) \ge h_\varphi(p, q) \quad \text{for every } p, q \in \partial\Delta.$$

Since $\varphi_n = H(f_n^{-1}, \psi_n)$ and $f_n^{-1}(t) \to f^{-1}(t)$ for every $t \in \partial\Delta$, and since the Teichmüller distance from $[f_n^{-1}]$ to the basepoint is the same as the Teichmüller distance from $[f_n]$ to the basepoint, we obtain

$$h_\varphi(f^{-1}(r), f^{-1}(s)) \ge h_\psi(r, s) \quad \text{for every } r, s \in \partial\Delta.$$

Step 6: $H(f_n, \varphi_n)$ converges weakly to $H(f, \varphi)$.

Proof. By the uniqueness theorem [S1, Thm. 5.6], $\psi = H(f, \varphi)$. Therefore, the limit ψ does not depend on which subsequence of ψ_n we take, which proves Step 6 and so finishes the proof of Lemma 2 when $\varphi \ne 0$.

Suppose now that $\varphi = 0$. If $\psi \ne 0$ then, by the previous discussion, $H(f_n^{-1}, \psi_n)$ converges weakly to $H(f^{-1}, \psi)$. But $H(f_n^{-1}, \psi_n) = \varphi_n$ converges weakly to $\varphi = 0$, a contradiction. □

COROLLARY 1. *H is continuous at (basepoint, φ) for every $\varphi \in A(\Delta)$.*

Proof. Suppose that a sequence φ_n in $A(\Delta)$ tends to φ in the L^1 norm, and that f_n is a sequence of quasiconformal homeomorphisms of Δ that fix i, 1, and -1 such that the Beltrami coefficients μ_n of f_n converge to 0 in the L^∞ norm. Let $\psi_n =$

$H(f_n, \varphi_n)$. From the theory of quasiconformal mappings we know that $f_n(z)$ converges to z uniformly on $\partial\Delta$ (see [L]) and thus, by Lemma 2, ψ_n converges weakly to φ. Since

$$\|\varphi_n\| \frac{1 - \|\mu_n\|_\infty}{1 + \|\mu_n\|_\infty} \le \|\psi_n\| \le \|\varphi_n\| \frac{1 + \|\mu_n\|_\infty}{1 - \|\mu_n\|_\infty},$$

it follows that

$$\|\psi_n\| \to \|\varphi\|.$$

Therefore, by the Lebesgue dominated convergence theorem,

$$\|\psi_n - \varphi\| - \|\psi_n\| \to -\|\varphi\|, \qquad \|\psi_n - \varphi\| \to 0. \qquad \square$$

The next corollary is the (strong) continuity of $[f] \to H([f], \varphi)$ for any fixed $\varphi \in A(\Delta)$.

COROLLARY 2. *Let f_n be a sequence of normalized quasisymmetric homeomorphisms of $\partial\Delta$ such that $[f_n] \to [f]$ for some quasisymmetric homeomorphism f of $\partial\Delta$. Then $H(f_n, \varphi) \to H(f, \varphi)$ for every $\varphi \in A(\Delta)$.*

Proof. Since $[f_n \circ f^{-1}]$ tends to the basepoint in the Teichmüller metric, Corollary 1 yields

$$H(f_n, \varphi) = H(f_n \circ f^{-1}, H(f, \varphi)) \to H(f, \varphi)). \qquad \square$$

4. Variation in the Dirichlet Norm

THEOREM 3. *Let $\varphi \neq 0$ be an integrable holomorphic quadratic differential in the unit disk Δ, and let f be a quasiconformal homeomorphism of Δ with the Beltrami coefficient μ. Let $\varphi_\mu = H(f^\mu, \varphi)$. Then*

$$\log\|\varphi_\mu\| = \log\|\varphi\| + 2\,\mathrm{Re}\,\frac{1}{\|\varphi\|}\iint_\Delta \mu\varphi + o(\|\mu\|_\infty). \tag{6}$$

Therefore, $F([f]) = \log\|H(f, \varphi)\|$ is a C^1 function on the universal Teichmüller space $T(\Delta)$.

Proof. Let $\psi(z) = \varphi_\mu(f(z))\, f_z^2(z)\big(1 - \mu(z)\frac{\varphi(z)}{|\varphi(z)|}\big)^2$. Then ψ is a measurable quadratic differential on Δ and, for almost every regular vertical trajectory β of φ,

$$\int_{f(\beta)} |\mathrm{Im}\sqrt{\varphi_\mu(f)}\, df| = \int_\beta |\mathrm{Im}\sqrt{\psi(z)}\, dz|.$$

Since every ray of a regular vertical trajectory of a differential in $A(\Delta)$ converges to a well-determined point on the boundary of Δ, both ends of β terminate at points of $\partial\Delta$. Let a and b be the endpoints of β on $\partial\Delta$; then $h_\varphi(a, b) = h_{\varphi_\mu}(f(a), f(b))$. Therefore,

$$\begin{aligned}\int_\beta |\mathrm{Im}\sqrt{\varphi(z)}\, dz| &= h_\varphi(a, b) = h_{\varphi_\mu}(f(a), f(b)) \\ &\le \int_{f(\beta)} |\mathrm{Im}\sqrt{\varphi_\mu(f)}\, df| = \int_\beta |\mathrm{Im}\sqrt{\psi(z)}\, dz|.\end{aligned}$$

Hence Theorem 1 implies that $\|\varphi\| \le \iint_\Delta |\sqrt{\psi\varphi}| \le \|\psi\|$, so

$$\|\varphi\| \le \iint_\Delta \sqrt{|\varphi|}\sqrt{\left|\varphi_\mu(f) f_z^2\left(1-\mu\frac{\varphi}{|\varphi|}\right)^2\right|\frac{1-|\mu|^2}{1-|\mu|^2}}.$$

Applying Schwarz's inequality, we obtain

$$\|\varphi\|^2 \le \iint_\Delta |\varphi_\mu(f) f_z^2|(1-|\mu|^2) \iint_\Delta |\varphi| \frac{\left|1-\mu\frac{\varphi}{|\varphi|}\right|^2}{1-|\mu|^2}.$$

Therefore,

$$\begin{aligned}\|\varphi\|^2 &\le \|\varphi_\mu\| \iint_\Delta |\varphi| \frac{\left|1-\mu\frac{\varphi}{|\varphi|}\right|^2}{1-|\mu|^2}\\ &= \|\varphi_\mu\|(\|\varphi\| - 2\operatorname{Re}\iint_\Delta \mu\varphi + O(\|\mu\|_\infty^2)).\end{aligned} \tag{7}$$

Hence

$$\frac{\|\varphi\|}{\|\varphi_\mu\|} \le 1 - \frac{2}{\|\varphi\|}\operatorname{Re}\iint_\Delta \mu\varphi + O(\|\mu\|_\infty^2),$$

and so

$$\log\|\varphi_\mu\| \ge \log\|\varphi\| + 2\operatorname{Re}\frac{1}{\|\varphi\|}\iint_\Delta \mu\varphi + O(\|\mu\|_\infty^2).$$

To get a reverse inequality we apply a similar argument to the inverse mapping f^{-1} of f. The Beltrami coefficient of f^{-1} is $\mu_1 = -\mu(f_z/\overline{f_z}) \circ (f^{-1})$. Thus,

$$\|\varphi_\mu\| \le \left\|\varphi \circ f^{-1}(f^{-1})_z^2\left(1-\mu_1\frac{\varphi_\mu}{|\varphi_\mu|}\right)^2\right\| \tag{8}$$

and

$$\|\varphi_\mu\|^2 \le \|\varphi\| \iint_\Delta |\varphi_\mu| \frac{\left|1-\mu_1\frac{\varphi_\mu}{|\varphi_\mu|}\right|^2}{1-|\mu_1|^2}. \tag{9}$$

Inequality (9) yields

$$\log\|\varphi\| \ge \log\|\varphi_\mu\| + 2\operatorname{Re}\frac{1}{\|\varphi_\mu\|}\iint_\Delta \mu_1\varphi_\mu + O(\|\mu\|_\infty^2).$$

Now, since

$$\frac{1}{K(\mu)} \le \frac{\|\varphi_\mu\|}{\|\varphi\|} \le K(\mu)$$

with $K(\mu) = (1+\|\mu\|_\infty)/(1-\|\mu\|_\infty) \to 1$, it is enough to prove that

$$\iint_\Delta \mu_1\varphi_\mu + \iint_\Delta \mu\varphi = o(\|\mu\|_\infty).$$

We have

$$\iint_\Delta \mu_1\varphi_\mu = -\iint_\Delta \mu\varphi_\mu(f) f_z^2(1-|\mu|^2) = -\iint_\Delta \mu\varphi_\mu(f) f_z^2 + O(\|\mu\|_\infty^3).$$

It is thus sufficient to prove that

$$\|\varphi_\mu(f) f_z^2 - \varphi\| \to 0. \tag{10}$$

From the theory of quasiconformal mappings we know that $f(z) \to z$ and $f_z^2(z) \to 1$ for almost every $z \in \Delta$ (see [A]). Furthermore, Lemma 2 implies that φ_μ converges weakly to φ. Therefore $\varphi_\mu(fz) f_z^2(z) \to \varphi(z)$ for almost all z in Δ. Since

$$\|\varphi_\mu\| \le \|\varphi_\mu(f) f_z^2\| \le \|\varphi_\mu\| \frac{1}{1 - \|\mu\|_\infty^2},$$

$\|\varphi_\mu(f) f_z^2\| \to \|\varphi\|$. Hence, by the Lebesgue dominated convergence theorem,

$$\lim \|\varphi_\mu(f) f_z^2 - \varphi\| - \|\varphi\| = \lim(\|\varphi_\mu(f) f_z^2 - \varphi\| - \|\varphi_\mu(f) f_z^2\|) = -\|\varphi\|.$$

Now we prove that F is C^1. Let $G\colon M(\Delta) \to (-\infty, \infty)$ be defined by $G(\mu) = \log\|\varphi_\mu\|$. Since the geometric mappings $[f^\nu] \to [f^\nu \circ (f^\mu)^{-1}]$ are biholomorphic and there is a holomorphic section from a neighborhood of the basepoint in $T(\Delta)$ into $M(\Delta)$, it is enough to prove that the derivative of G is continuous at $\mu = 0$. From (6) we see that

$$G'(0)(\nu) = 2 \operatorname{Re} \frac{1}{\|\varphi\|} \iint_\Delta \nu\varphi.$$

Since

$$H(f^{\mu+\nu}, \varphi) = H(f^{\mu+\nu} \circ (f^\mu)^{-1}, H(f^\mu, \varphi)) = H(f^{\mu+\nu} \circ (f^\mu)^{-1}, \varphi_\mu)$$

and since the Beltrami coefficient of $f^{\mu+\nu} \circ (f^\mu)^{-1}$ is

$$\frac{\nu}{1 - |\mu|^2} \frac{f_z^\mu}{\overline{f_z^\mu}} \circ (f^\mu)^{-1} + O(\|\nu\|_\infty^2),$$

we have

$$G'(\mu)(\nu) = 2 \operatorname{Re} \frac{1}{\|\varphi_\mu\|} \iint_\Delta \frac{\nu}{1 - |\mu|^2} \frac{f_z^\mu}{\overline{f_z^\mu}} \circ (f^\mu)^{-1} \varphi_\mu. \tag{11}$$

After a change of variable we obtain

$$G'(\mu)(\nu) = 2 \operatorname{Re} \frac{1}{\|\varphi_\mu\|} \iint_\Delta \varphi_\mu \circ f^\mu (f_z^\mu)^2 \nu.$$

Since $\|\varphi_\mu\| \to \|\varphi\|$, the continuity of the first derivative of G at $\mu = 0$ follows from (10). □

COROLLARY 3. *$M(\mu) = \|H(f^\mu, \varphi)\|$ is a C^1 function in the open unit ball $M(\Delta)$ of $L^\infty(\Delta)$, and*

$$M'(\mu)(\nu) = 2 \operatorname{Re} \iint_\Delta \frac{\nu}{1 - |\mu|^2} \frac{f_z^\mu}{\overline{f_z^\mu}} \circ (f^\mu)^{-1} \varphi_\mu,$$

where $\varphi_\mu = H(f^\mu, \varphi)$.

Proof. Since $M(\mu) = e^{G(\mu)}$ for every $\mu \in M(\Delta)$, Corollary 1 follows from (11). □

5. Strong Continuity of the Mapping by Heights

In [S1], Strebel asked whether the mapping by heights $\varphi \to H(f, \varphi)$ is continuous. The following lemma shows that the answer is affirmative.

LEMMA 3. *If f is a quasiconformal homeomorphism of the unit disk, then the mapping $\varphi \to H(f, \varphi)$ is continuous on $A(\Delta)$.*

Proof. Notice that it is enough to prove that $\|H(f, \varphi_n)\| \to \|H(f, \varphi)\|$ for every sequence φ_n in $A(\Delta)$ that converges to $\varphi \in A(\Delta)$. For, if $\|H(f, \varphi_n)\| \to \|H(f, \varphi)\|$ then, by Lemma 2 and the Lebesgue dominated convergence theorem,

$$\|H(f, \varphi_n) - H(f, \varphi)\| - \|H(f, \varphi_n)\| \to -\|H(f, \varphi)\|$$

and hence $H(f, \varphi_n)$ tends to $H(f, \varphi)$ in the L^1 metric. By Lemma 2, $H(f, \varphi_n)$ converges weakly to $H(f, \varphi)$; thus, by Fatou's lemma, $\liminf_{n\to\infty} \|H(f, \varphi_n)\| \geq \|H(f, \varphi)\|$. It is therefore enough to prove that

$$\limsup_{n\to\infty} \|H(f, \varphi_n)\| \leq \|H(f, \varphi)\|.$$

Suppose that f is a quasiconformal homeomorphism of Δ with Beltrami coefficient μ and that the sequence φ_n in $A(\Delta)$ tends to φ in the L^1 norm, so that $\limsup_{n\to\infty} \|H(f, \varphi_n)\| - \|H(f, \varphi)\| = A \neq 0$. Let

$$A(t) = \limsup_{n\to\infty} \|H(f^{t\mu}, \varphi_n)\| - \|H(f^{t\mu}, \varphi)\| \quad \text{for } t \in [0, 1].$$

Then:

(i) A is a nonnegative and bounded function and

$$S = \sup_{t\in[0,1]} A(t) \leq \|\varphi\| \frac{1 + \|\mu\|_\infty}{1 - \|\mu\|_\infty};$$

(ii) $A(1) = A > 0$ and $A(0) = \limsup_{n\to\infty} \|\varphi_n\| - \|\varphi\| = 0$; and
(iii) by Corollary 1, $\lim_{t\to 0} A(t) = 0$.

By (ii), $S > 0$. Hence there exists $s \in (0, 1]$ so that

$$A(s) > S/2. \tag{12}$$

Define the real-valued functions $h, h_1, h_2, \dots$ on $(-1, 1/\|\mu\|_\infty)$ by

$$h(t) = \|H(f^{t\mu}, \varphi)\| \quad \text{and}$$
$$h_n(t) = \|H(f^{t\mu}, \varphi_n)\| \quad \text{for } n = 1, 2, \dots.$$

Since the L^1 norms of quadratic differentials φ_n are uniformly bounded, and since

$$\|H(f^\nu, q)\| \leq \|q\| \frac{1 + \|\nu\|_\infty}{1 - \|\nu\|_\infty} \quad \text{for every } q \in A(\Delta) \text{ and } \nu \in M(\Delta),$$

there exists a constant C such that $\|H(f^{t\mu}, \varphi)\| \leq C$ and $\|H(f^{t\mu}, \varphi_n)\| \leq C$ for every $t \in (-1, 1]$ and every n. By Corollary 3, functions $h, h_1, h_2, \dots$ are C^1 on $(-1, 1/\|\mu\|_\infty)$ and

$$h'(t) = 2\,\mathrm{Re}\iint_{\Delta} \frac{\mu}{1-|t\mu|^2}\frac{\overline{f_z^{t\mu}}}{f_z^{t\mu}} \circ (f^{t\mu})^{-1} H(f^{t\mu}, \varphi),$$

$$h_n'(t) = 2\,\mathrm{Re}\iint_{\Delta} \frac{\mu}{1-|t\mu|^2}\frac{\overline{f_z^{t\mu}}}{f_z^{t\mu}} \circ (f^{t\mu})^{-1} H(f^{t\mu}, \varphi_n).$$

Furthermore,

$$\begin{aligned} A(s) &= \limsup_{n\to\infty}[h_n(s) - h(s)] = \limsup_{n\to\infty}\left(\int_0^s [h_n'(t) - h'(t)]\,dt\right) \\ &\le 2\limsup_{n\to\infty}\int_0^s \left|\iint_{\Delta} \frac{\mu}{1-|t\mu|^2}\frac{\overline{f_z^{t\mu}}}{f_z^{t\mu}} \circ (f^{t\mu})^{-1}(H(f^{t\mu}, \varphi_n) \right. \\ &\qquad\qquad \left. - H(f^{t\mu}, \varphi))\,dx\,dy\right| dt \\ &\le 2\frac{\|\mu\|_\infty}{1-\|\mu\|_\infty^2}\limsup_{n\to\infty}\int_0^s\iint_{\Delta} |H(f^{t\mu}, \varphi_n) - H(f^{t\mu}, \varphi)|\,dx\,dy\,dt. \end{aligned}$$

Since $B_n(t) = \iint_{\Delta}|H(f^{t\mu}, \varphi_n) - H(f^{t\mu}, \varphi)|\,dx\,dy \le 2C$, Fatou's lemma implies that

$$\begin{aligned} \limsup_{n\to\infty}\int_0^s B_n(t)\,dt &= 2Cs - \liminf_{n\to\infty}\int_0^s (2C - B_n(t))\,dt \\ &\le 2Cs - \int_0^s \liminf_{n\to\infty}(2C - B_n(t))\,dt \\ &= \int_0^s \limsup_{n\to\infty} B_n(t)\,dt. \end{aligned}$$

Hence,

$$\begin{aligned} &A(s) \\ &\le \frac{2\|\mu\|_\infty}{1-\|\mu\|_\infty^2}\int_0^s \limsup_{n\to\infty}\left(\iint_{\Delta}|H(f^{t\mu}, \varphi_n) - H(f^{t\mu}, \varphi)|\,dx\,dy\right) dt \\ &\le \frac{2\|\mu\|_\infty}{1-\|\mu\|_\infty^2}\int_0^s \left(\limsup_{n\to\infty}\iint_{\Delta}(|H(f^{t\mu}, \varphi_n) - H(f^{t\mu}, \varphi)| - |H(f^{t\mu}, \varphi_n)|)\,dt \right. \\ &\qquad \left. + \limsup_{n\to\infty}\iint_{\Delta}|H(f^{t\mu}, \varphi_n)|\right) dt. \end{aligned}$$

Therefore, by Lemma 2 and the Lebesgue dominated convergence theorem,

$$\begin{aligned} A(s) &\le \frac{2\|\mu\|_\infty}{1-\|\mu\|_\infty^2}\int_0^s\left(\iint_{\Delta} -|H(f^{t\mu}, \varphi)| + \limsup_{n\to\infty}\|H(f^{t\mu}, \varphi_n)\|\right) dt \\ &= \frac{2\|\mu\|_\infty}{1-\|\mu\|_\infty^2}\int_0^s A(t)\,dt \le \frac{2\|\mu\|_\infty}{1-\|\mu\|_\infty^2}Ss \le \frac{2\|\mu\|_\infty}{1-\|\mu\|_\infty^2}S. \end{aligned}$$

By (12),

$$\frac{S}{2} < \frac{2\|\mu\|_\infty}{1-\|\mu\|_\infty^2}S, \qquad 1 - \|\mu\|_\infty^2 < 4\|\mu\|_\infty.$$

If $\|\mu\|_\infty \leq \frac{1}{8}$ then $1 - \|\mu\|_\infty^2 \geq \frac{63}{64} > \frac{1}{2} \geq 4\|\mu\|_\infty$, a contradiction. Therefore, for every quasiconformal homeomorphism f of Δ with the Beltrami differential of L^∞ norm $\leq \frac{1}{8}$, the mapping by heights $\varphi \to H(f, \varphi)$ is continuous in the L^1 norm. Now let g be any quasiconformal homeomorphism of the unit disk Δ, and let ψ_n be a sequence of integrable holomorphic quadratic differentials converging to ψ in the L^1 norm. Then, by the theory of quasiconformal mappings, there exist an integer k and quasiconformal homeomorphisms $f_1, f_2, \dots, f_k$ of Δ so that $g = f_1 \circ f_2 \circ \dots \circ f_k$, and the Beltrami differentials of $f_1, f_2, \dots, f_k$ are less than $\frac{1}{8}$ in the L^∞ norm (see [L]). Hence,

$$H(f_k, \psi_n) \to H(f_k, \psi),$$

$$\begin{aligned} &H(f_{k-1} \circ f_k, \psi_n) \\ &\quad = H(f_{k-1}, H(f_k, \psi_n)) \to H(f_{k-1}, H(f_k, \psi)) = H(f_{k-1} \circ f_k, \psi), \end{aligned}$$

$$\vdots$$

$$\begin{aligned} &H(g, \psi_n) \\ &\quad = H(f_1, H(f_2 \circ f_3 \circ \dots \circ f_k, \psi_n)) \to H(f_1, H(f_2 \circ f_3 \circ \dots \circ f_k, \psi)) \\ &\quad = H(g, \psi). \end{aligned}$$

□

Now we are ready to prove that the mapping by heights $H([f], \varphi)$ is continuous on $T(\Delta) \times A(\Delta)$.

THEOREM 4. *The mapping by heights H is continuous.*

Proof. Let f be a normalized quasisymmetric homeomorphism of Δ, and let $\varphi \in A(\Delta)$. Let f_n be a sequence of normalized quasisymmetric homeomorphisms of $\partial\Delta$ such that $[f_n]$ converges to $[f]$ in the Teichmüller metric, and let φ_n be a sequence in $A(\Delta)$ that converges to φ in the L^1 norm. Then

$$H([f_n], \varphi_n) = H([f_n \circ f^{-1}], H([f], \varphi_n)).$$

By Lemma 3, $H([f], \varphi_n)$ tends to $H([f], \varphi)$ in the L^1 norm. Since $[f_n \circ f^{-1}]$ tends to the basepoint in the Teichmüller metric, Corollary 1 yields

$$H([f_n \circ f^{-1}], H([f], \varphi_n)) \to H([f], \varphi).$$

□

6. The Extremal Norm Properties of the Mapping by Heights

THEOREM 5. *Let f be a quasisymmetric homeomorphism of a circle with dilatation $K = (1 + k)/(1 - k)$ and boundary dilatation H. Then:*

(a) $\sup_{\varphi \neq 0}(\|H(f, \varphi)\| / \|\varphi\|) = K$;

(b) (Strebel) *the supremum in* (a) *is achieved at φ if and only if f has a representative that is a Teichmüller mapping with the Beltrami differential $k(|\varphi|/\varphi)$;*

(c) *if* φ_n *is a strictly degenerating sequence in* $A(\Delta)$, *then* $H(f, \varphi_n)$ *is strictly degenerating*; *and*

(d) $\sup_{(\varphi_n)} \limsup_{n\to\infty}(\|H(f,\varphi_n)\|/\|\varphi_n\|) = H$. *Here the supremum is over all strictly degenerating sequences* φ_n.

REMARK 1. Part (b) is proved by Strebel in [S1]. Here we present a different proof based on the minimal norm property.

REMARK 2. Note that it is necessary to take only *strictly* degenerating sequences φ_n in the supremum in (d). For, if f is such that $H < K$, then by the frame mapping condition and part (b) there exists $\varphi \in A(\Delta)$ such that $\|H(f,\varphi)\| = K\|\varphi\|$. Then $\varphi/n \to 0$ and $H(f, \varphi/n) = (1/n)H(f,\varphi)$; therefore,

$$\frac{\|H(f, \frac{\varphi}{n})\|}{\|\frac{\varphi}{n}\|} = \frac{\|H(f,\varphi)\|}{\|\varphi\|} = K.$$

Proof of Theorem 5. (a) Since $\|H(f,\varphi)\| \le K\|\varphi\|$ for every $\varphi \in A(\Delta)$, to prove (a) it is enough to find a sequence φ_n in $A(\Delta)$ such that $\|\varphi_n\| = 1$ and $\|H(f,\varphi_n)\| \to K$. Let μ be an extremal Beltrami differential in the Teichmüller class of f. Then $\|\mu\|_\infty = k$, and there exists a sequence φ_n in $A(\Delta)$ such that $\|\varphi_n\| = 1$ and

$$\iint_\Delta \frac{\varphi_n\mu}{1-|\mu|^2} \to \frac{k}{1-k^2}$$

(see [G3, Lemma 2, p. 124]). Let $\psi_n = H(f,\varphi_n)$. By inequality (7),

$$\begin{aligned} 1 &\le \|\psi_n\| \iint_\Delta |\varphi_n| \frac{\left|1-\mu\frac{\varphi_n}{|\varphi_n|}\right|^2}{1-|\mu|^2} \\ &\le \|\psi_n\|\left(1 + 2\iint_\Delta |\varphi_n|\frac{|\mu|^2}{1-|\mu|^2} - 2\,\mathrm{Re}\iint_\Delta \frac{\varphi_n\mu}{1-|\mu|^2}\right), \\ 1 &\le (\liminf\|\psi_n\|)\left(1 + \frac{2k^2}{1-k^2} - \frac{2k}{1-k^2}\right) \\ &\le (\liminf\|\psi_n\|)\frac{1}{K}. \end{aligned}$$

(b) If f is a Teichmüller mapping associated with the differentials φ and ψ in $A(\Delta)$, then $\psi = H(f,\varphi)$ and $\|\psi\| = K\|\varphi\|$.

To prove the converse, we assume that $K = \|\psi\|/\|\varphi\|$ with $\psi = H(f,\varphi)$. Let μ_1 be an extremal Beltrami differential in the equivalence class of f^{-1}. Let g be a quasiconformal homeomorphism of Δ with the Beltrami coefficient μ_1. Inequality (8) yields

$$\|\psi\| \le \left\|\varphi(g)g_z^2\left(1-\mu_1\frac{\psi}{|\psi|}\right)^2\right\|. \tag{13}$$

Hence

$$\begin{aligned}\|\psi\| &\le \left\|\varphi(g)g_z^2(1-|\mu_1|^2)\frac{\left(1-\mu_1\frac{\psi}{|\psi|}\right)^2}{1-|\mu_1|^2}\right\| \\ &\le \|\varphi\|\frac{(1+\|\mu_1\|_\infty)^2}{1-\|\mu_1\|_\infty^2} \\ &\le K\|\varphi\|.\end{aligned}$$

Therefore, we have an equality in (13). Hence $\mu_1 = -k(|\psi|/\psi)$ and, by the uniqueness part of Theorem 1,

$$\psi = \varphi(g)g_z^2\left(1-\mu_1\frac{\psi}{|\psi|}\right)^2.$$

Therefore,

$$\psi = \varphi(g)g_z^2(1+k)^2.$$

The quasiconformal homeomorphism g^{-1} is in the equivalence class of f, and its Beltrami coefficient μ satisfies

$$\begin{aligned}\mu(g) &= -\mu_1\frac{g_z}{\overline{g_z}} = k\frac{|\psi|}{\psi}\frac{g_z}{\overline{g_z}} \\ &= k\frac{|\varphi(g)||g_z|^2}{\varphi(g)g_z^2}\frac{g_z}{\overline{g_z}} = k\frac{|\varphi(g)|}{\varphi(g)},\end{aligned}$$

which proves (b).

(c) Let (φ_n) be a strictly degenerating sequence in $A(\Delta)$. Then $1/C \le \|\varphi_n\| \le C$ for some positive constant C. Let $\psi_n = H(f, \varphi_n)$. By Lemma 2, ψ_n is degenerating. Since

$$\|\psi_n\| \ge \frac{1}{K}\|\varphi_n\| \ge \frac{1}{KC},$$

ψ_n is strictly degenerating.

(d) Fix $\varepsilon > 0$, and let φ_n be a strictly degenerating sequence in $A(\Delta)$. By part (c), $\psi_n = H(f, \varphi_n)$ is strictly degenerating. There exists a compact set $F \subset \Delta$ and a Beltrami differential μ such that f^μ and f represent the same element in Teich(Δ) and

$$\frac{1+|\mu(z)|}{1-|\mu(z)|} \le H+\varepsilon \quad \text{for every } z \in F^c.$$

Let $K_1 = (1+\|\mu\|_\infty)/(1-\|\mu\|_\infty)$. Since ψ_n is degenerating, there exists a positive integer n_0 such that

$$\iint_{f^\mu(F)} |\psi_n| \le \frac{\varepsilon}{K_1}$$

for every $n > n_0$. Inequality (9) in the proof of Theorem 3 implies

$$\|\psi_n\|^2 \le \|\varphi_n\| \iint_\Delta |\psi_n|\frac{1+|\mu_1|}{1-|\mu_1|},$$

where μ_1 is the Beltrami differential of $(f^\mu)^{-1}$. Therefore, for all $n > n_0$,

$$\|\psi_n\|^2 \le \|\varphi_n\|\left(K_1\frac{\varepsilon}{K_1} + \iint_{f^\mu(F)^c} |\psi_n| \frac{1+|\mu((f^\mu)^{-1}(z))|}{1-|\mu((f^\mu)^{-1}(z))|}\right)$$
$$\le \|\varphi_n\|(\varepsilon + (H+\varepsilon)\|\psi_n\|).$$

Hence

$$\frac{\|\psi_n\|}{\|\varphi_n\|} \le \frac{\varepsilon}{\|\psi_n\|} + \varepsilon + H.$$

Since ψ_n is strictly degenerating, letting $\varepsilon \to 0$ proves

$$\sup_{(\varphi_n)} \limsup \frac{\|H(f,\varphi_n)\|}{\|\varphi_n\|} \le H.$$

To obtain a reverse inequality, fix $\varepsilon > 0$ and let ν be a Beltrami differential such that f^ν is in the Teichmüller class of f and $H^* - H \le \varepsilon$, where $H^* = (h^*+1)/(h^*-1)$ is the maximal dilatation of f^ν outside some compact subset of Δ. By the fundamental inequalities for boundary dilatation (see [G1] or [EGL]), there exists a degenerating sequence φ_n in $A(\Delta)$ such that $\|\varphi_n\| = 1$ and

$$\lim_{n\to\infty} \iint_\Delta \frac{\varphi_n \nu}{1-|\nu|^2} = \alpha \ge \frac{h^*}{1-h^{*2}} - \varepsilon.$$

Let $\psi_n = H(f,\varphi_n)$. Inequality (7) from the proof of Theorem 3 implies

$$\|\varphi_n\|^2 \le \|\psi_n\| \iint_\Delta |\varphi_n| \frac{\left|1-\nu\frac{\varphi_n}{|\varphi_n|}\right|^2}{1-|\nu|^2};$$
$$1 \le \liminf \|\psi_n\|\left(1 + 2\frac{h^{*2}}{1-h^{*2}} - 2\alpha\right)$$
$$\le \liminf \|\psi_n\|\left(1 + 2\frac{h^{*2}}{1-h^{*2}} - 2\frac{h^*}{1-h^{*2}} + 2\varepsilon\right)$$
$$\le \liminf \|\psi_n\|\left(\frac{1}{H^*} + 2\varepsilon\right).$$

Letting $\varepsilon \to 0$, we obtain

$$\liminf \frac{\|\psi_n\|}{\|\varphi_n\|} \ge H,$$

and this proves part (d). □

COROLLARY 4. *If f is a quasisymmetric homeomorphism of a circle, then:*

(i) *f is Möbius if and only if $\|H(f,\varphi)\| = \|\varphi\|$ for every φ in $A(\Delta)$; and*

(ii) *f is symmetric if and only if $\|H(f,\varphi_n)\| \to 1$ for every degenerating sequence of unit vectors in $A(\Delta)$.*

Proof. (i) follows immediately from part (a) of Theorem 5. Since

$$\varphi_n = H(f^{-1}, H(f,\varphi_n)),$$

and since f is symmetric if and only if f^{-1} is symmetric, (ii) is an easy consequence of part (d) of Theorem 5. □

References

[A] L. V. Ahlfors, *Lectures on quasiconformal mappings,* Wadsworth, Monterey, CA, 1987.

[EGL] C. J. Earle, F. P. Gardiner, and N. Lakic, *Teichmüller spaces with asymptotic conformal equivalence,* preprint.

[G1] F. P. Gardiner, *On Teichmüller contraction,* Proc. Amer. Math. Soc. 118 (1993), 865–875.

[G2] ———, *Measured foliations and the minimal norm property for quadratic differentials,* Acta Math. 152 (1984), 57–76.

[G3] ———, *Teichmüller theory and quadratic differentials,* Wiley, New York, 1987.

[GS] F. P. Gardiner and D. P. Sullivan, *Symmetric structures on a closed curve,* Amer. J. Math. 114 (1992), 683–736.

[L] O. Lehto, *Univalent functions and Teichmüller spaces,* Springer, New York, 1987.

[S1] K. Strebel, *The mapping by heights for quadratic differentials in the disk,* Ann. Acad. Sci. Fenn. Ser. A I Math. 18 (1993), 155–190.

[S2] ———, *On the geometry of quadratic differentials in the disk,* Results Math. 22 (1992), 799–816.

[S3] ———, *Quadratic differentials,* Springer, New York, 1984.

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