Capacity Distortion by Inner Functions in the Unit Ball of $\mathbf{C}^n$

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1. Introduction

An inner function is a bounded holomorphic function from the unit ball $\mathbf{B}_n$ of $\mathbf{C}^n$ into the unit disk Δ of the complex plane such that the radial boundary values have modulus 1 almost everywhere. If E is a nonempty Borel subset of $\partial\Delta$, we denote by $f^{-1}(E)$ the following subset of the unit sphere $\mathbf{S}_n$ of $\mathbf{C}^n$:

$$f^{-1}(E) = \{\xi \in \mathbf{S}_n : \lim_{r\to 1} f(r\xi) \text{ exists and belongs to } E\}.$$

There is a classical lemma of Löwner (see e.g. [R, p. 405; T, p. 322]), about the distortion of boundary sets under inner functions.

LÖWNER'S LEMMA. *An inner function f, with $f(0) = 0$, is a measure-preserving transformation when viewed as a mapping from $\mathbf{S}_n$ to $\partial\Delta$. That is, if E is a Borel subset of $\partial\Delta$ then $|f^{-1}(E)| = |E|$, where in each case $|\cdot|$ denotes the corresponding normalized Lebesgue measure.*

Here we extend this result to fractional dimensions as follows.

THEOREM 1. *Let f be inner in the unit ball of $\mathbf{C}^n$ $(n \geq 1)$, set $f(0) = 0$, and let E be a Borel subset of $\partial\Delta$. Then:*

(i) *if $0 < \alpha < 2$ (and also $\alpha = 0$ if $n = 1$), then*

$$\operatorname{cap}_{2n-2+\alpha}(f^{-1}(E)) \geq C(n,\alpha)\operatorname{cap}_\alpha(E); \tag{1.1}$$

(ii) *if $\alpha = 0$ and $n > 1$, then*

$$\frac{1}{\operatorname{cap}_{2n-2}(f^{-1}(E))} \leq C(n)\left(1 + \log\frac{1}{\operatorname{cap}_0(E)}\right). \tag{1.2}$$

Here $\operatorname{cap}_\alpha$ and cap_0 denote (respectively) α-dimensional Riesz capacity and logarithmic capacity with respect to the distance in $\mathbf{S}_n$ given by

$$d(a,b) = |1 - \langle a,b\rangle|^{1/2},$$

where

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$$\langle a, b\rangle = \sum_{j=1}^{n} a_j \bar{b}_j$$

is the usual inner product in $\mathbf{C}^n$. This nonisotropic distance is the natural one in the analysis of problems concerning $\mathbf{S}_n$. Also, this distance is equivalent to the Carnot–Carathéodory distance in the Heisenberg group model for $\mathbf{S}_n$. We refer to [R] for details about this distance. Also we refer to [C], [KS], and [L] for definitions and basic background on capacity.

Observe that, as a consequence of Theorem 1, one obtains the following.

COROLLARY. *If $f\colon \mathbf{B}_n \to \Delta$ is inner and E is a Borel subset of $\partial\Delta$, then*

$$\operatorname{Dim}(f^{-1}(E)) \geq 2n - 2 + \operatorname{Dim}(E), \tag{1.3}$$

where Dim *denotes Hausdorff dimension with respect to the distance d.*

Analogous results with the Euclidean distance instead of d were obtained in [FPR]. Also, for some applications of these results we refer to [FP1], [FP2], and [FPR].

The basic tool that we use to prove (1.1) is a formula relating the α-energy J_α of a complex measure μ (see [L] for basic background on this subject) with its invariant Poisson extension $\mathcal{P}_\mu$. This approach is due to Beurling [B].

THEOREM 2. *If μ is a complex measure supported on $\mathbf{S}_n$, the unit sphere of $\mathbf{C}^n$, then for all $n \geq 1$ and $0 < \alpha < 2n$ we have that*

$$J_\alpha(\mu) \asymp \int_0^1 \left\{ \int_{\mathbf{S}_n} |\mathcal{P}_\mu(r\xi)|^2 \, d\xi \right\} r^{\alpha/2-1}(1-r^2)^{n-\alpha/2-1}\, dr. \tag{1.4}$$

By the expression $A \asymp B$ we mean that the quotient A/B is bounded above and below by constants that can depend at most on n and α.

Recall that the invariant Poisson extension $\mathcal{P}_\mu$ of a complex measure μ (supported in $\mathbf{S}_n$) is defined as

$$\mathcal{P}_\mu(z) = \int_{\mathbf{S}_n} \mathcal{P}(z, w)\, d\mu(w), \quad z \in \mathbf{B}_n,$$

where

$$\mathcal{P}(z, w) = \frac{(1-|z|^2)^n}{\omega_{2n}\, |1 - \langle z, w\rangle|^{2n}}, \quad z \in \mathbf{B}_n, \ \ w \in \mathbf{S}_n,$$

is the Poisson–Szegö kernel [R, p. 40; F] and ω_{2n} is the area of $\mathbf{S}_n$. Observe that if $n = 1$, the Poisson–Szegö kernel is simply the classical Poisson kernel.

Also, if μ is a complex measure on $\mathbf{S}_n$ and $0 \leq \alpha < 2n$, then the α-energy $J_\alpha(\mu)$ of μ is defined as

$$J_\alpha(\mu) = \iint_{\mathbf{S}_n \times \mathbf{S}_n} \Phi_\alpha(d(\xi, \eta))\, d\bar{\mu}(\xi)\, d\mu(\eta),$$

where

$$\Phi_\alpha(t) = \begin{cases} \log(1/t) & \text{if } \alpha = 0, \\ 1/t^\alpha & \text{if } 0 < \alpha < 2n. \end{cases}$$

If E is a closed subset of $\mathbf{S}_n$, then

$$(\mathrm{cap}_\alpha(E))^{-1} = \inf\{\, J_\alpha(\mu) : \mu \text{ a probability measure supported on } E\,\};$$

for $0 < \alpha < 2n$,

$$\log \frac{1}{\mathrm{cap}_0(E)} = \inf\{\, J_0(\mu) : \mu \text{ a probability measure supported on } E\,\},$$

and the infimum is attained by a unique probability measure μ_e, which is called the *equilibrium distribution* of E.

If E is any Borel subset of $\mathbf{S}_n$, then the α-capacity of E is defined as

$$\mathrm{cap}_\alpha(E) = \sup\{\,\mathrm{cap}_\alpha(K) : K \subset E,\ K \text{ compact}\,\}.$$

The analog of (1.4) with the Euclidean distance instead of d was obtained in [FPR]; it is remarkable that in [FPR] *equality* is obtained with an explicit constant (see Theorem B in the next section).

In order to prove Theorem 2, we will need a result (that appears in [PR]) about the integral of the square of a hypergeometric function.

THEOREM A [PR]. *For all nonnegative integers p, q, n ($n \geq 1$) and for all $\beta = \alpha/4$ ($0 < \beta < n/2$), we have*

$$\int_0^1 \left(\frac{F(t)}{F(1)}\right)^2 t^{p+q+\beta-1}(1-t)^{n-2\beta-1}\,dt \asymp \frac{\Gamma(p+\beta)\Gamma(q+\beta)}{\Gamma(p+n-\beta)\Gamma(q+n-\beta)},$$

where $F(t) = F(p, q; p+q+n; t)$.

By $F(a, b; c; t)$ we denote the usual Gauss hypergeometric function

$$F(a, b; c; t) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{t^k}{k!},$$

where $(u)_k$ is the Pochammer symbol,

$$(u)_k = u(u+1)\cdots(u+k-1) = \frac{\Gamma(u+k)}{\Gamma(u)},$$

and $\Gamma(\,\cdot\,)$ denotes the Gamma function.

The outline of this paper is as follows. In Section 2 we will prove Theorem 2. Theorem 1 will be proved in Section 3.

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NOTATION. By C we will denote a constant, depending at most on n and α, whose value can change from line to line and even within the same line.

2. Proof of Theorem 2

If we use the kernel $\Phi_\alpha(|\xi - \eta|)$ instead of $\Phi_\alpha(d(\xi, \eta))$, we obtain the classical α-dimensional Riesz energy that we will denote by $I_\alpha(\mu)$. Observe that, if $n = 1$, then $I_{\alpha/2}(\mu) = J_\alpha(\mu)$ for all $0 < \alpha < 2$ and also $I_0(\mu)/2 = J_0(\mu)$. This remark and the following theorem give the case $n = 1$ of Theorem 2, with equality for an appropriate constant instead of the symbol $\asymp$.

THEOREM B [FPR]. *If μ is a complex measure supported on Σ_{N-1}, the unit sphere of $\mathbf{R}^N$, and if P_μ is its classical Poisson extension ($\mathcal{P}_\mu = P_\mu$ if $N = 2$), then we have the following.*

(i) *If $0 < \alpha < N - 1$, then*

$$I_\alpha(\mu) = K(N,\alpha) \int_0^1 \left\{ \int_{\Sigma_{N-1}} |P_\mu(r\xi)|^2 \, d\xi \right\} r^{\alpha-1}(1-r^2)^{N-2-\alpha} \, dr,$$

with

$$K(N,\alpha) = \frac{4\pi^{N/2}}{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{N-\alpha}{2}\right)}.$$

(ii) *If $m = \mu(\Sigma_{N-1})$, then*

$$I_0(\mu) = \omega_N \int_0^1 \int_{\Sigma_{N-1}} \left| P_\mu(r\xi) - \frac{m}{\omega_N} \right|^2 d\xi (1-r^2)^{N-2} \frac{dr}{r} + \frac{|m|^2}{2} \left[\frac{\Gamma'}{\Gamma}\left(\frac{N}{2}\right) - \frac{\Gamma'}{\Gamma}(N-1) \right],$$

where ω_N denotes the area of Σ_{N-1}. In particular, if $N = 2$,

$$I_0(\mu) = 2\pi \int_0^1 \int_0^{2\pi} \left| P_\mu(re^{i\theta}) - \frac{m}{2\pi} \right|^2 d\theta \, \frac{dr}{r}.$$

Observe that in [FPR] Theorem B was proved only for signed measures, but the result, stated in the actual form, follows simply by splitting μ into real and imaginary parts.

In [F], Folland obtained an expansion in spherical harmonics of the Poisson–Szegö kernel for the unit ball $\mathbf{B}_n$ in $\mathbf{C}^n$. Let $\Delta_{\mathbf{B}_n}$ be the Laplace–Beltrami operator associated to the Bergman metric on $\mathbf{B}_n$,

$$\Delta_{\mathbf{B}_n} = \frac{4}{n+1}(1-|z|^2) \sum_{i,j=1}^{n} (\delta_{ij} - z_i \bar{z}_j) \frac{\partial^2}{\partial z_i \partial \bar{z}_j}.$$

Here $\Delta_{\mathbf{B}_n}$ is the basic invariant differential operator on the symmetric space $SU(n,1)/U(n) \approx \mathbf{B}_n$. The solution of the Dirichlet problem

$$\begin{cases} \Delta_{\mathbf{B}_n} u = 0 & \text{in } \mathbf{B}_n, \\ u = f & \text{in } \partial\mathbf{B}_n, \end{cases} \tag{2.1}$$

with continuous boundary data f, is given by the representation formula

$$u(z) = \int_{\mathbf{S}_n} \mathcal{P}(z,w) f(w) \, dw.$$

If $\mathcal{H}^{p,q}$ denotes the linear space of restrictions to $\mathbf{S}_n$ of harmonic polynomials $g(z,\bar{z})$ on $\mathbf{C}^n$ that are homogeneous of degree p in z and of degree q in $\bar{z}$, then the solution of the Dirichlet problem (2.1) with $f \in \mathcal{H}^{p,q}$ is given by

$$u(r\xi) = S^{p,q}(r) f(\xi), \quad 0 \le r \le 1, \ \xi \in \mathbf{S}_n, \tag{2.2}$$

where

$$S^{p,q}(r) = r^{p+q} \frac{F(p, q; p+q+n; r^2)}{F(p, q; p+q+n; 1)}.$$

Formula (2.2) gives to $S^{p,q}(r)$ a crucial role in obtaining the expansion of the Poisson–Szegö kernel in spherical harmonics.

In [PR] we give uniform asymptotic estimates of these functions when p, q grow to infinity.

THEOREM C [PR]. *There exists a universal constant C, not depending on n, q, m, z, such that—for all real numbers $m, n \ge 1$, $q \ge 1/m$, and $0 \le z < 1$—if we denote*

$$G = F(mq, q; mq + q + n; z) B(mq, q + n),$$

where $B(x, y)$ is the usual Euler beta function, then

$$G \ge CL,$$

where

$$L = t_0^{mq}(1 - m(1 - t_0))^q (1 - t_0)^{n-1} \left(\frac{1 - z}{a^2 - b^2 z} \right)^{1/4} \frac{1}{m\sqrt{q+1}}$$

and

$$t_0 = \frac{a + bz - \sqrt{(1 - z)(a^2 - b^2 z)}}{2z} = \frac{2}{a + bz + \sqrt{(1 - z)(a^2 - b^2 z)}},$$

$$a = 1 + \frac{1}{m}, \quad b = 1 - \frac{1}{m}.$$

Moreover, the inequality is sharp in the sense that

$$\lim_{q \to \infty} \frac{G}{L} = \sqrt{2\pi}.$$

Observe that, without loss of generality, we can suppose $m \ge 1$ because of the symmetry of the hypergeometric function in the two first parameters.

We summarize the results about these spherical harmonics (see e.g. [F]) in the following result. This theorem generalizes the properties of classical spherical harmonics, which are described in [SW].

THEOREM D [F]. *The following statements obtain for all $n \ge 2$.*

(i) *$L^2(\mathbf{S}_n)$ is the orthogonal sum $L^2(\mathbf{S}_n) = \bigoplus_{p,q=0}^{\infty} \mathcal{H}^{p,q}$, and the dimension of $\mathcal{H}^{p,q}$ is*

$$D = D(p, q; n) = \frac{(p + q + n - 1)\,(p + n - 2)!\,(q + n - 2)!}{p!\,q!\,(n - 1)!\,(n - 2)!}. \tag{2.3}$$

(ii) *If $f_1^{p,q}, f_2^{p,q}, \ldots, f_D^{p,q}$ is any orthonormal basis for $\mathcal{H}^{p,q}$, then*

$$\sum_{j=1}^{D} f_j^{p,q}(\xi) \overline{f_j^{p,q}(\eta)} = H^{p,q}(\langle \xi, \eta \rangle), \quad \xi, \eta \in \mathbf{S}_n, \tag{2.4}$$

where $H^{p,q}(\langle \cdot, \eta \rangle)$ is the zonal harmonic of degrees p and q and pole η.

(iii) *The L^2-norm of the function $H^{p,q}(\langle\xi, \cdot\rangle)$ is*

$$\int_{\mathbf{S}_n} |H^{p,q}(\langle\xi,\eta\rangle)|^2\, d\eta = \frac{D}{\omega_{2n}}, \quad \text{for all } \xi \in \mathbf{S}_n, \tag{2.5}$$

where $d\eta$ denotes the usual Lebesgue surface measure in $\mathbf{S}_n$ (not normalized).

(iv) *The function $H^{p,q}$ has the following explicit expression in terms of the Jacobi polynomials:*

$$H^{p,q}(z) = \frac{D}{\omega_{2n}} \rho^{u-v} e^{i(p-q)\theta} \frac{P_v^{(n-2,u-v)}(2\rho^2-1)}{P_v^{(n-2,u-v)}(1)}, \tag{2.6}$$

where $z = \rho e^{i\theta}$, $u = \max\{p,q\}$, $v = \min\{p,q\}$, and

$$P_m^{(a,b)}(t) = \frac{(-1)^m}{m!\,2^m(1-t)^a(1+t)^b} \frac{d^m}{dt^m}\left[(1-t)^{a+m}(1+t)^{b+m}\right]$$

is the Jacobi polynomial of degree m and parameters a, b ([L1, p. 275; AS, p. 785]; observe that in [F] there is a typographical error in the definition of these polynomials). *Moreover, $\{H^{p,q}(z)\}_{p,q=0}^{\infty}$ is an orthogonal basis of $L^2(\{|z|<1\})$ with respect to the measure $(1-|z|^2)^{n-2}\,dx\,dy$ (since every polynomial in the variables z and $\bar{z}$ can be expressed as a finite linear combination of $\{H^{p,q}(z)\}_{p,q=0}^{\infty}$).*

(v) *For $0 \le r < 1$ and $\xi, \eta \in \mathbf{S}_n$, we have that*

$$\mathcal{P}(r\xi,\eta) = \sum_{p,q=0}^{\infty} S^{p,q}(r) H^{p,q}(\langle\xi,\eta\rangle). \tag{2.7}$$

We need to obtain the expansion of the integral kernel $\Phi_\alpha(d(\xi,\eta))$ in terms of these spherical harmonics.

First, fix α, with $0 < \alpha < 2n$, and let $\beta = \alpha/4$. If we denote by $g(z)$ the function of one complex variable,

$$g(z) = \frac{1}{|1-z|^{\alpha/2}} = \frac{1}{|1-z|^{2\beta}}, \quad |z| < 1, \tag{2.8}$$

then we can express the kernel $\Phi_\alpha(d(\xi,\eta))$ in terms of g as $\Phi_\alpha(d(\xi,\eta)) = g(\langle\xi,\eta\rangle)$. Now, develop $g(z)$ as a Fourier series in the following way.

LEMMA 1. *For all $n \ge 2$ and $0 < \beta < n/2$, we have the Fourier expansion*

$$g(z) = \sum_{p,q=0}^{\infty} g^{p,q} H^{p,q}(z),$$

where $g^{p,q}$ has the expression

$$g^{p,q} = 2\pi^n \frac{\Gamma(n-2\beta)}{\Gamma(\beta)^2} \frac{\Gamma(p+\beta)\Gamma(q+\beta)}{\Gamma(p+n-\beta)\Gamma(q+n-\beta)}. \tag{2.9}$$

In order to prove this result, we will need the following lemma.

LEMMA 2. *For all* $0 \le \rho < 1$, $\beta > 0$, *and all integers* m, *we have that*

$$\int_0^{2\pi} \frac{e^{im\theta}}{|1-\rho e^{i\theta}|^{2\beta}}\, d\theta = 2\pi\rho^{|m|}\frac{\Gamma(|m|+\beta)}{|m|!\,\Gamma(\beta)} F(\beta, |m|+\beta; |m|+1; \rho^2). \quad (2.10)$$

Proof. Without loss of generality we can assume that $m \ge 0$, since the case $m < 0$ will follow by conjugation. We have that

$$\begin{aligned}&\int_0^{2\pi} e^{im\theta}(1-\rho e^{i\theta})^{-\beta}(1-\rho e^{-i\theta})^{-\beta}\, d\theta \\ &\quad = \int_0^{2\pi} e^{im\theta}\left(\sum_{k=0}^{\infty}\frac{(\beta)_k}{k!}\rho^k e^{ik\theta}\right)\left(\sum_{j=0}^{\infty}\frac{(\beta)_j}{j!}\rho^j e^{-ij\theta}\right) d\theta \\ &\quad = 2\pi\sum_{k=0}^{\infty}\frac{(\beta)_k}{k!}\frac{(\beta)_{k+m}}{(k+m)!}\rho^{2k+m},\end{aligned}$$

and the lemma follows by substituting the definition of the Pochammer symbols and using the definition of the hypergeometric function. □

Proof of Lemma 1. We will use in this proof the notation $\langle\phi, \psi\rangle$ to denote the usual scalar product in $L^2(\{|z| < 1\})$ with respect to the measure $(1-|z|^2)^{n-2}\, dx\, dy$. This will not cause confusion since we will not use the inner product in $\mathbf{C}^n$ within this proof.

We have that

$$g^{p,q} = \frac{\langle g, H^{p,q}\rangle}{\langle H^{p,q}, H^{p,q}\rangle}. \quad (2.11)$$

We recall [R, p. 15] that

$$\int_{S_n}\varphi(\langle\xi, \eta\rangle)\, d\eta = \frac{(n-1)\omega_{2n}}{\pi}\int_0^{2\pi}\int_0^1 \varphi((\rho e^{i\theta})(1-\rho^2)^{n-2}\rho\, d\rho\, d\theta, \quad (2.12)$$

for all $\varphi \in L^1(\{|z| < 1\})$ with respect to the measure $(1-|z|^2)^{n-2}\, dx\, dy$. Using (2.5) and (2.12), we deduce that

$$\begin{aligned}\frac{D}{\omega_{2n}} &= \int_{S_n}|H^{p,q}(\langle\xi, \eta\rangle)|^2\, d\eta \\ &= \frac{(n-1)\omega_{2n}}{\pi}\int_0^{2\pi}\int_0^1 |H^{p,q}(\rho e^{i\theta})|^2(1-\rho^2)^{n-2}\rho\, d\rho\, d\theta;\end{aligned}$$

hence

$$\langle H^{p,q}, H^{p,q}\rangle = \frac{\pi D}{(n-1)(\omega_{2n})^2}. \quad (2.13)$$

On the other hand, since $|p-q| = u - v$,

$$\begin{aligned}\langle g, H^{p,q}\rangle = \frac{D}{\omega_{2n}P_v^{(n-2,u-v)}(1)}\cdot\int_0^1\int_0^{2\pi}\frac{e^{i(q-p)\theta}}{|1-\rho e^{i\theta}|^{2\beta}} \\ P_v^{(n-2,u-v)}(2\rho^2-1)\rho^{u-v}(1-\rho^2)^{n-2}\, d\theta\, \rho\, d\rho \quad \text{(by (2.6))}\end{aligned}$$

$$= \frac{2\pi D}{\omega_{2n} P_v^{(n-2,u-v)}(1)} \frac{\Gamma(u-v+\beta)}{(u-v)!\,\Gamma(\beta)} \cdot \int_0^1 F(\beta, u-v+\beta; u-v+1; \rho^2)$$
$$P_v^{(n-2,u-v)}(2\rho^2-1)\rho^{2(u-v)}(1-\rho^2)^{n-2}\rho\, d\rho \quad \text{(by (2.10))}.$$

By making the variable change $t = 2\rho^2 - 1$, we obtain

$$\langle g, H^{p,q}\rangle = \frac{2\pi D}{2^{n+u-v}\omega_{2n} P_v^{(n-2,u-v)}(1)} \frac{\Gamma(u-v+\beta)}{(u-v)!\,\Gamma(\beta)}$$
$$\cdot \int_{-1}^1 F(\beta, u-v+\beta; u-v+1; (1+t)/2)$$
$$P_v^{(n-2,u-v)}(t)(1-t)^{n-2}(1+t)^{u-v}\, dt.$$

If we denote by (ϕ, ψ) the scalar product in $L^2[-1,1]$ with respect to the measure $(1-t)^{n-2}(1+t)^{u-v}\, dt$, then the preceding formula can be written as

$$\langle g, H^{p,q}\rangle = \frac{2\pi D}{2^{n+u-v}\omega_{2n} P_v^{(n-2,u-v)}(1)} \frac{\Gamma(u-v+\beta)}{(u-v)!\,\Gamma(\beta)} (F, P_v^{(n-2,u-v)}), \tag{2.14}$$

where F denotes the hypergeometric function $F(\beta, u-v+\beta; u-v+1; (1+t)/2)$.

It is known [L2, p. 29] that:

if $a, b > -1$, $\lambda = a+b+1$, $-1 \le w \le 1$, *and* $t < 1/2$, *then*

$$G(w) \equiv F(a_1, a_2; a_3; (1+w)t) = \sum_{v=0}^{\infty} C_v P_v^{(a,b)}(w),$$

where

$$C_v = \frac{(a_1)_v (a_2)_v (2t)^v}{(a_3)_v (v+\lambda)_v} {}_3F_2(b+1+v, a_1+v, a_2+v; \lambda+1+2v, a_3+v; 2t).$$

Here ${}_3F_2$ is the generalized hypergeometric function

$${}_3F_2(\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2; t) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k(\alpha_2)_k(\alpha_3)_k}{(\beta_1)_k(\beta_2)_k} \frac{t^k}{k!}.$$

This gives

$$\frac{(G, P_v^{(a,b)})}{(P_v^{(a,b)}, P_v^{(a,b)})} = C_v \tag{2.15}$$

for all $t < 1/2$. Then, by making $t \to 1/2$, we obtain that (2.15) is also true for $t = 1/2$. Therefore,

$$(F, P_v^{(n-2,u-v)})$$
$$= C_v (P_v^{(n-2,u-v)}, P_v^{(n-2,u-v)})$$
$$= \frac{(\beta)_v (u-v+\beta)_v}{(u-v+1)_v (u+n-1)_v}$$
$$\cdot {}_3F_2(u+1, v+\beta, u+\beta; u+v+n, u+1; 1)(P_v^{(n-2,u-v)}, P_v^{(n-2,u-v)})$$
$$= \frac{(\beta)_v (u-v+\beta)_v}{(u-v+1)_v (u+n-1)_v}$$
$$\cdot F(v+\beta, u+\beta; u+v+n; 1) \frac{2^{u-v+n-1}}{u+v+n-1} \frac{(v+n-2)!\, u!}{(u+n-2)!\, v!},$$

where we have used the fact [L1, p. 276; AS, p. 774] that

$$(P_v^{(a,b)}, P_v^{(a,b)}) = \frac{2^{a+b+1}\Gamma(v+a+1)\Gamma(v+b+1)}{(2v+a+b+1)v!\,\Gamma(v+a+b+1)}.$$

Hence, using the Gauss summation formula

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad \text{if } c-a-b>0$$

[L1, p. 99; AS, p. 556], we obtain that

$$(F, P_v^{(n-2,u-v)}) = 2^{u-v+n-1}(\beta)_v(u-v+\beta)_v \frac{(v+n-2)!\,(u-v)!\,\Gamma(n-2\beta)}{v!\,\Gamma(u+n-\beta)\Gamma(v+n-\beta)}.$$

By substituting this formula (which makes sense since $n-2\beta>0$) in (2.14), we obtain that

$$\begin{aligned}\langle g, H^{p,q}\rangle &= \frac{(n-2)!\,\pi D}{\omega_{2n}} \frac{\Gamma(n-2\beta)}{\Gamma(\beta)^2} \frac{\Gamma(u+\beta)\Gamma(v+\beta)}{\Gamma(u+n-\beta)\Gamma(v+n-\beta)} \\ &= \frac{(n-2)!\,\pi D}{\omega_{2n}} \frac{\Gamma(n-2\beta)}{\Gamma(\beta)^2} \frac{\Gamma(p+\beta)\Gamma(q+\beta)}{\Gamma(p+n-\beta)\Gamma(q+n-\beta)},\end{aligned} \tag{2.16}$$

where we have used the fact [L1, p. 274; AS, p. 774] that

$$P_v^{(n-2,u-v)}(1) = \frac{(v+n-2)!}{v!\,(n-2)!}.$$

The lemma follows now by substituting (2.13) and (2.16) in (2.11), and by using that $\omega_{2n} = 2\pi^n/(n-1)!$. □

Proof of Theorem 2. We choose an orthonormal basis $\{f_j^{p,q}\}_{j=1}^D$ of $\mathcal{H}^{p,q}$, for each $p,q \geq 0$. Let $\{\mu_j^{p,q}\}$ $(p,q\geq 0,\ 1\leq j\leq D = D(p,q;n))$ be the Fourier coefficients of μ; that is,

$$\mu \sim \sum_{p,q=0}^{\infty} \sum_{j=1}^{D} \mu_j^{p,q} f_j^{p,q}.$$

Recall that $\mathcal{P}_\mu$ is defined by

$$\mathcal{P}_\mu(r\xi) = \int_{\mathbf{S}_n} \mathcal{P}(r\xi,\eta)\,d\mu(\eta),$$

where $\mathcal{P}(r\xi,\eta)$ is the Poisson–Szegö kernel

$$\mathcal{P}(r\xi,\eta) = \frac{1}{\omega_{2n}} \frac{(1-r^2)^n}{|1-r\langle\xi,\eta\rangle|^{2n}}, \quad 0\leq r<1, \quad \xi,\eta\in\mathbf{S}_n.$$

Recalling (2.4) and (2.7), we deduce that

$$\mathcal{P}(r\xi,\eta) = \sum_{p,q=0}^{\infty} S^{p,q}(r)H^{p,q}(\langle\xi,\eta\rangle) = \sum_{p,q,j} S^{p,q}(r) f_j^{p,q}(\xi)\overline{f_j^{p,q}(\eta)}.$$

Now, Plancherel's theorem gives that

$$\mathcal{P}_\mu(r\xi) = \sum_{p,q,j} S^{p,q}(r)\mu_j^{p,q} f_j^{p,q}(\xi).$$

Again using Plancherel's theorem, we obtain that

$$\int_{\mathbf{S}_n} |\mathcal{P}_\mu(r\xi)|^2 \, d\xi = \sum_{p,q,j} (S^{p,q}(r))^2 |\mu_j^{p,q}|^2,$$

and so, if we denote by Λ the right-hand side in (1.4), we have that (recall $\beta = \alpha/4$)

$$\Lambda = \sum_{p,q,j} |\mu_j^{p,q}|^2 \int_0^1 (S^{p,q}(r))^2 r^{2\beta-1}(1-r^2)^{n-2\beta-1} \, dr;$$

substituting $r^2 = t$, using the definition of $S^{p,q}(r)$ yields

$$\begin{aligned}\Lambda &= \frac{1}{2}\sum_{p,q,j} |\mu_j^{p,q}|^2 \int_0^1 \left(\frac{F(p,q;p+q+n;t)}{F(p,q;p+q+n;1)}\right)^2 t^{p+q+\beta-1}(1-t)^{n-2\beta-1} \, dt \\ &\asymp \sum_{p,q=0}^{\infty} \frac{\Gamma(p+\beta)\Gamma(q+\beta)}{\Gamma(p+n-\beta)\Gamma(q+n-\beta)} \sum_{j=1}^{D} |\mu_j^{p,q}|^2 \\ &\asymp \sum_{p,q=0}^{\infty} g^{p,q} \sum_{j=1}^{D} |\mu_j^{p,q}|^2,\end{aligned} \tag{2.17}$$

where we have used Theorem A and Lemma 1.

On the other hand, using again Lemma 1 and (2.4),

$$\Phi_\alpha(d(\xi,\eta)) = g(\langle \xi,\eta\rangle) = \sum_{p,q=0}^{\infty} g^{p,q} H^{p,q}(\langle \xi,\eta\rangle) = \sum_{p,q,j} g^{p,q} f_j^{p,q}(\xi)\overline{f_j^{p,q}(\eta)};$$

using Plancherel's theorem and (2.17), we obtain that

$$\int_{\mathbf{S}_n} \Phi_\alpha(d(\xi,\eta))\, d\mu(\eta) = \sum_{p,q,j} g^{p,q}\mu_j^{p,q} f_j^{p,q}(\xi),$$

$$J_\alpha(\mu) = \sum_{p,q,j} g^{p,q} |\mu_j^{p,q}|^2 \asymp \Lambda.$$

This finishes the proof of Theorem 2. □

3. Proof of Theorem 1

We need the following lemmas.

LEMMA 3 [FPR]. *Let μ be a finite positive measure in $\partial\Delta$, and let f be an inner function. Then there exists a unique positive measure $\tilde{\nu}$ in $\mathbf{S}_n$ such that $\mathcal{P}_\mu \circ f = \mathcal{P}_{\tilde{\nu}}$ and*

$$\tilde{\nu}(f^{-1}(\text{support } \mu)) = \tilde{\nu}(\mathbf{S}_n).$$

Moreover, if $f(0) = 0$, *then*

$$\frac{1}{\omega_{2n}}\tilde{\nu}(\mathbf{S}_n) = \frac{1}{2\pi}\mu(\partial\Delta).$$

A different normalization is useful; choosing $\nu = (2\pi/\omega_{2n})\tilde{\nu}$, one obtains

$$\mathcal{P}_\nu = \frac{2\pi}{\omega_{2n}}\mathcal{P}_\mu \circ f \quad \text{and} \quad \nu(\mathbf{S}_n) = \mu(\partial\Delta).$$

The following lemma is well known.

LEMMA 4 (Subordination principle). *Let* $f\colon \mathbf{B}_n \to \Delta$ *be a holomorphic function such that* $f(0) = 0$, *and let* $v\colon \Delta \to \mathbf{R}$ *be a subharmonic function. Then, for all* $0 \le r < 1$,

$$\frac{1}{\omega_{2n}}\int_{\mathbf{S}_n} v(f(r\xi))\,d\xi \le \frac{1}{2\pi}\int_0^{2\pi} v(re^{i\theta})\,d\theta.$$

LEMMA 5. *Let* μ *be a complex measure on* $\partial\Delta$, *let* f *be an inner function with* $f(0) = 0$, *and let* ν *be a complex measure on* $\mathbf{S}_n$ *such that* $\mathcal{P}_\nu = (2\pi/\omega_{2n})\mathcal{P}_\mu \circ f$. *Then:*

(i) *If* $n \ge 1$ *and* $0 < \alpha < 2$ *or if* $n = 1$ *and* $\alpha = 0$, *then there exists a constant* C *depending at most on* n *and* α *such that*

$$J_{2n-2+\alpha}(\nu) \le CJ_\alpha(\mu).$$

(ii) *If* $\alpha = 0$, $n \ge 2$, *and* $m = \mu(\partial\Delta)$, *then there exists a constant* C *depending at most on* n *such that*

$$J_{2n-2}(\nu) \le C(|m|^2 + J_0(\mu)).$$

Proof. Since $v = |\mathcal{P}_\mu|^2$ is subharmonic (in the Euclidean sense), we obtain by subordination (Lemma 4) that

$$\int_{\mathbf{S}_n} |\mathcal{P}_\nu|^2\,d\xi = \left(\frac{2\pi}{\omega_{2n}}\right)^2 \int_{\mathbf{S}_n} |\mathcal{P}_\mu(f)|^2\,d\xi \le \frac{2\pi}{\omega_{2n}}\int_0^{2\pi} |\mathcal{P}_\mu|^2\,d\theta. \tag{3.1}$$

Using Theorem 2 twice, the inequality (3.1), and the fact that $n \ge 1$, we have that

$$\begin{aligned}
J_{2n-2+\alpha}(\nu) &\asymp \int_0^1 \left\{\int_{\mathbf{S}_n} |\mathcal{P}_\nu(r\xi)|^2\,d\xi\right\} r^{n-2+\alpha/2}\,\frac{dr}{(1-r^2)^{\alpha/2}} \\
&\le C\int_0^1 \left\{\int_0^{2\pi} |\mathcal{P}_\mu(re^{i\theta})|^2\,d\theta\right\} r^{n-1+\alpha/2-1}\,\frac{dr}{(1-r^2)^{\alpha/2}} \\
&\le C\int_0^1 \left\{\int_0^{2\pi} |\mathcal{P}_\mu(re^{i\theta})|^2\,d\theta\right\} r^{\alpha/2-1}\,\frac{dr}{(1-r^2)^{\alpha/2}} \\
&\le CJ_\alpha(\mu).
\end{aligned}$$

This finishes the proof of part (i) in the case $n \ge 1$, $0 < \alpha < 2$. The other case follows from [FPR, Lemma 5], since $J_0(\nu) = I_0(\nu)/2$.

In order to prove (ii), using that $m = \mu(\partial\Delta) = \nu(\mathbf{S}_n)$ we obtain that

$$\int_{\mathbf{S}_n} \left| \mathcal{P}_\nu(r\xi) - \frac{m}{\omega_{2n}} \right|^2 d\xi + \frac{|m|^2}{\omega_{2n}} = \int_{\mathbf{S}_n} |\mathcal{P}_\nu(r\xi)|^2 \, d\xi.$$

Integrating this equality and using Theorem 2, we have that

$$\begin{aligned} J_{2n-2}(\nu) &\asymp \int_0^1 \int_{\mathbf{S}_n} \left| \mathcal{P}_\nu(r\xi) - \frac{m}{\omega_{2n}} \right|^2 d\xi \, r^{n-2} \, dr + \frac{|m|^2}{(n-1)\omega_{2n}} \\ &= \int_0^1 \int_{\mathbf{S}_n} \left| \frac{2\pi}{\omega_{2n}} (\mathcal{P}_\mu \circ f)(r\xi) - \frac{m}{\omega_{2n}} \right|^2 d\xi \, r^{n-2} \, dr + \frac{|m|^2}{(n-1)\omega_{2n}} \\ &\le C \int_0^1 \int_0^{2\pi} \left| \mathcal{P}_\mu(re^{i\theta}) - \frac{m}{2\pi} \right|^2 d\theta \, \frac{dr}{r} + \frac{|m|^2}{(n-1)\omega_{2n}} \\ &\le C(|m|^2 + I_0(\mu)) \le C(|m|^2 + J_0(\mu)), \end{aligned}$$

where we have used subordination (Lemma 4) with $v = |\mathcal{P}_\mu - m/(2\pi)|^2$ and Theorem B. □

Finally we can finish the proof of Theorem 1. We may assume that E is closed. In order to prove (1.1), let us denote by μ_e the α-equilibrium probability distribution of E, and let ν be the probability measure in $\mathbf{S}_n$ such that $\mathcal{P}_\nu = (2\pi/\omega_{2n})\mathcal{P}_{\mu_e} \circ f$. By Lemma 5,

$$J_{2n-2+\alpha}(\nu) \le C J_\alpha(\mu_e) = C(\mathrm{cap}_\alpha(E))^{-1}. \tag{3.2}$$

But, from Lemma 3, $\nu\big(f^{-1}(E)\big) = 1$, and so

$$J_{2n-2+\alpha}(\nu) = \iint_{f^{-1}(E)\times f^{-1}(E)} \Phi_{2n-2+\alpha}(d(\xi,\eta)) \, d\nu(\xi) \, d\nu(\eta).$$

Now let $\{K_j\}$ be an increasing sequence of compact subsets of $f^{-1}(E)$ such that $\nu(K_j) \nearrow 1$. Then, for each j,

$$\begin{aligned} J_{2n-2+\alpha}(\nu) &= \iint_{f^{-1}(E)\times f^{-1}(E)} \Phi_{2n-2+\alpha}(d(\xi,\eta)) \, d\nu(\xi) \, d\nu(\eta) \\ &\ge \nu(K_j)^2 \iint_{K_j \times K_j} \Phi_{2n-2+\alpha}(d(\xi,\eta)) \, \frac{d\nu(\xi)}{\nu(K_j)} \, \frac{d\nu(\eta)}{\nu(K_j)} \\ &\ge \nu(K_j)^2 (\mathrm{cap}_{2n-2+\alpha}(K_j))^{-1} \\ &\ge \nu(K_j)^2 (\mathrm{cap}_{2n-2+\alpha}(f^{-1}(E)))^{-1}. \end{aligned}$$

Consequently, if we let $j \to \infty$, we obtain that

$$J_{2n-2+\alpha}(\nu) \ge (\mathrm{cap}_{2n-2+\alpha}(f^{-1}(E)))^{-1}. \tag{3.3}$$

Therefore, in the case $0 < \alpha < 2$, $n \ge 1$, (1.1) now follows from (3.2) and (3.3). The case $\alpha = 0$, $n = 1$, follows from [FPR, Thm. 1].

In order to prove (1.2) we proceed as follows. Let μ_e be the equilibrium distribution of E for the logarithmic capacity, and let ν be the measure supported on $\mathbf{S}_n$ such that $\mathcal{P}_\nu = (2\pi/\omega_{2n})\mathcal{P}_{\mu_e} \circ f$. Using Lemma 5,

$$J_{2n-2}(\nu) \le C(1 + J_0(\mu_e)) = C\left(1 + \log \frac{1}{\mathrm{cap}_0(E)}\right).$$

Now, to finish the proof, one need only follow the same approach used to prove part (i).

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