Generalized Roundness and Negative Type

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1. Introduction

In this paper we exhibit the equivalence of Enflo's nonlinear notion of generalized roundness and the classical embedding notion of negative type. This enables us to develop a rudimentary theory of generalized roundness and to give applications to the L_p-spaces. In particular, we show that for $p > 2$ and $n \geq 3$, the n-dimensional l_p spaces fail to have generalized roundness q for all $q > 0$.

The notions of roundness and generalized roundness were introduced by Enflo in [E1], [E2], and [E3] to study the uniform structure of metric spaces. We begin by recalling some material from these papers. However, we make some slight alterations to Enflo's original definitions to allow easier exposition later.

1.1. Definition. (a) We say that a metric space (X, d) has *roundness* q, written $q \in \mathrm{r}(X, d)$, if whenever a_1, a_2, b_1, b_2 are in X we have

$$d(a_1, a_2)^q + d(b_1, b_2)^q \leq \sum_{1 \leq i,j \leq 2} d(a_i, b_j)^q. \tag{1}$$

(b) A pair $(a_1, \dots, a_n), (b_1, \dots, b_n)$ of n-tuples in a metric space is called a *double-n-simplex*. Such a double-n-simplex will be denoted $[a_i; b_i]_{i=1}^n$. We call a pair of points (a_i, a_j) or (b_i, b_j) an *edge*, and a pair of points (a_i, b_j) a *connecting line*.

We say that a metric space (X, d) has *generalized roundness* q, written $q \in \mathrm{gr}(X, d)$, if for every $n \geq 2$ and every double-n-simplex $[a_i; b_i]_{i=1}^n$ in X we have

$$\sum_{1 \leq i < j \leq n} \big(d(a_i, a_j)^q + d(b_i, b_j)^q\big) \leq \sum_{1 \leq i,j \leq n} d(a_i, b_j)^q. \tag{2}$$

When (2) holds for a specific double-n-simplex $[a_i; b_i]_{i=1}^n$ in X we will write $q \in \mathrm{gr}[a_i; b_i]_{i=1}^n$.

1.2. Remark. In [E1] and [E2] Enflo defined the roundness of a metric space (X, d) to be $\sup\{\, q \mid q \in \mathrm{r}(X, d)\,\}$ and the generalized roundness to be $\sup\{\, q \mid q \in \mathrm{gr}(X, d)\,\}$. It is easy to check that $\{\, q \mid q \in \mathrm{r}(X, d)\,\}$ and $\{\, q \mid q \in \mathrm{gr}(X, d)\,\}$ are

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both closed subsets of $[0, \infty)$; hence the above suprema are maxima whenever they are finite.

If $[a_i; b_i]_{i=1}^n$ is a double-n-simplex with $a_1 = b_1$ then, due to cancellation of like terms, $q \in \mathrm{gr}[a_i; b_i]_{i=1}^n$ if and only if $q \in \mathrm{gr}[a_i; b_i]_{i=2}^n$. So in computing the generalized roundness of a metric space (X, d) it suffices to consider only those double-n-simplexes $[a_i; b_i]_{i=1}^n$ for which $\{a_i\}_{i=1}^n \cap \{b_j\}_{j=1}^n = \emptyset$.

Note that the roundness of a metric space (X, d) is computed using (2) for double-2-simplexes. It is easy to see that every metric space has roundness one and generalized roundness zero. It is clear that, for a metric space (X, d), generalized roundness q implies roundness q.

In [E1] Enflo proved that, for all p such that $1 \leq p \leq 2$ and for any positive Borel measure μ, $\max\{ q \mid q \in \mathrm{r}(L_p(\mu)) \} = p$ and, as a consequence, showed that an infinite-dimensional $L_{p_1}(\mu_1)$ is not uniformly homeomorphic to $L_{p_2}(\mu_2)$ if $p_1 \neq p_2$, $1 \leq p_1, p_2 \leq 2$. It is known from Enflo [E2] that $\max\{ q \mid q \in \mathrm{gr}(L_2(\mu)) \} = 2$. In lectures given at Kent State University, Enflo indicated that, for all p such that $1 \leq p \leq 2$, $\max\{ q \mid q \in \mathrm{gr}(L_p(\mu)) \} = p$.

In [E2] Enflo used generalized roundness to construct a countable metric space that is not uniformly homeomorphic to any subset of $L_2[0, 1]$. This gave a negative answer to a question asked by Smirnov : "Is every separable metric space uniformly homeomorphic to a subset of $L_2[0, 1]$?" By the previous paragraph, the analogous question with L_2 replaced by L_p for $1 \leq p < 2$ also has a negative answer.

Beyond Enflo's rather elegant application of generalized roundness in [E2], there has been no systematic study of the notion. In this paper we develop the rudiments of a general theory. One of our main tools is the existence of a strong link between the notions of generalized roundness and negative type. The notion of negative type emerged from investigations by Menger [Me], and Schoenberg [S1; S2] into the nature of isometric embeddings of metric spaces into Hilbert spaces.

Our main result, established in the next section, is the following.

1.3. THEOREM. *For $2 < p \leq \infty$, if $L_p(\Omega, \Sigma, \mu)$ is at least three-dimensional then it fails to have generalized roundness q for any $q > 0$.*

Prior to connecting with negative type, we give direct computations of the roundness of infinite-dimensional $L_p(\mu)$s for the case $p > 2$. It is known from Enflo [E3] that, for a normed space $(X, \|\cdot\|)$,

(i) $q \in \mathrm{r}(X, \|\cdot\|)$ if and only if $\|x + y\|^q + \|x - y\|^q \leq 2\,(\|x\|^q + \|y\|^q)$ for all $x, y \in X$, and

(ii) $q \in \mathrm{r}(X, \|\cdot\|)$ implies $q_1 \in \mathrm{r}(X, \|\cdot\|)$ for all q_1 such that $1 \leq q_1 \leq q$.

Indeed, (ii) follows easily using vector-valued interpolation via the complex method (see e.g. [BL, 5.1.2]). We use these facts in the proof of our next claim.

1.4. PROPOSITION. *For $2 \leq p \leq \infty$, $\mathrm{r}(L_p[0, 1]) = [1, p']$ where $1/p + 1/p' = 1$.*

Proof. Recall this inequality of Clarkson [C]: For all $x, y \in L_p$,

$$2^{1/p'}(\|x\|_{L_p}^p + \|y\|_{L_p}^p)^{1/p} \leq \left(\|x+y\|_{L_p}^{p'} + \|x-y\|_{L_p}^{p'}\right)^{1/p'}.$$

Since $p' \leq 2 \leq p$, an application of Hölder's inequality gives

$$\begin{aligned}\left(\|x\|_{L_p}^{p'} + \|y\|_{L_p}^{p'}\right)^{1/p'} &\leq 2^{1/p'-1/p}\left(\|x\|_{L_p}^{p} + \|y\|_{L_p}^{p}\right)^{1/p} \\ &\leq 2^{-1/p}\left(\|x+y\|_{L_p}^{p'} + \|x-y\|_{L_p}^{p'}\right)^{1/p'}.\end{aligned}$$

Fix $f, g \in L_p$. Letting $x = f+g$ and $y = f-g$, we can rephrase this inequality to read

$$\|f+g\|_{L_p}^{p'} + \|f-g\|_{L_p}^{p'} \leq 2^{-p'/p}\left(\|2f\|_{L_p}^{p'} + \|2g\|_{L_p}^{p'}\right) = 2\left(\|f\|_{L_p}^{p'} + \|g\|_{L_p}^{p'}\right).$$

Hence $[1, p'] \subseteq \mathrm{r}(L_p[0,1])$. To show equality, consider any q satisfying (i) for the L_p-norm. Then choose $x = \frac{1}{2}\left[\chi_{(0,1/2)} + \chi_{(1/2,1)}\right]$ and $y = \frac{1}{2}\left[\chi_{(0,1/2)} - \chi_{(1/2,1)}\right]$. Note that $x+y = \chi_{(0,1/2)}$ and $x-y = \chi_{(1/2,1)}$. Thus $(\frac{1}{2})^{q/p} \leq 2(\frac{1}{2})^q$ and so $q \leq p'$. □

1.5. Remark. Proposition 1.4 is also clearly true for any $L_p(\Omega, \Sigma, \mu)$ of dimension at least 2. Also, Clarkson's inequalities hold for noncommutative L_p spaces (see e.g. [FK], which also contains further references to the many authors who have contributed to this inequality), including the Schatten classes C_p [Mc; To]. It follows that $\mathrm{r}(L_p) = [1, p']$ if $2 \leq p \leq \infty$, and $\mathrm{r}(L_p) = [1, p]$ if $1 \leq p \leq 2$, for any noncommutative L_p-space corresponding to a faithful, normal, semifinite trace; or, more generally, for any L_p-space corresponding to a von Neumann algebra in the sense of Haagerup (see e.g. [Te]). Finally, we remark that in Herz [H], Clarkson's inequality is also used to show that if an l^p-norm is of q-negative type (which is equivalent to generalized roundness q, by the next section), then $q \leq \min\{p, p'\}$. Though the proof details are not given, the result implicitly contains a significant part of our Proposition 1.4.

Let us note that Enflo [E1] introduced roundness to complete the result of Lindenstrauss [L] that L_{p_1} and L_{p_2} are not uniformly homeomorphic if $p_1 \neq p_2$. Enflo's method captured the result when $1 \leq p_1, p_2 \leq 2$. Now, calculating the roundness of L_p for $p > 2$ does not give Lindenstrauss' result, but only a part of it: if $p_1 \in [1, 2]$, $p_2 > 2$, and $p_1 < p_2'$, then L_{p_1} and L_{p_2} are not uniformly homeomorphic.

Concerning the general problem of the uniform and Lipschitz classification of Banach spaces, we refer the reader to [A1; A2; AMM; B; HM; R], as well as to a recent paper on uniform homeomorphisms [Cha]. The survey paper of Benyamini [B] contains many other references.

2. Generalized Roundness and Negative Type

Before beginning, we remark that some other references related to the theme of this section are [WW; H; Cho, Sec. 41] and (more recently) [GL].

2.1. DEFINITION. Let $q \in [0, \infty)$. A metric space (X, d) is said to have *q-negative type*, written $q \in \mathrm{n}(X, d)$, if, for all $n \in \mathbb{N}$, all finite sequences $\{x_1, \dots, x_n\} \subseteq X$, and all choices of real numbers $\xi_1, \dots, \xi_n$ with $\sum_{j=1}^{n} \xi_j = 0$, we have

$$\sum_{1 \le i,j \le n} d(x_i, x_j)^q \xi_i \xi_j \le 0. \tag{3}$$

In order to connect generalized roundness with negative type, we shall begin by reformulating it.

2.2. THEOREM. *For a metric space (X, d), the following are equivalent:*

(i) $q \in \mathrm{gr}(X, d)$,

(ii) *for all $n \in \mathbb{N}$, all finite sequences $\{x_1, \dots, x_n\} \subseteq X$, and all collections of weights $w_1, \dots, w_n, s_1, \dots, s_n \ge 0$ that satisfy $\sum_{j=1}^{n} w_j = \sum_{j=1}^{n} s_j$ (= 1 if one wishes to normalize), we have*

$$\sum_{1 \le i,j \le n} d(x_i, x_j)^q (w_i - s_i)(w_j - s_j) \le 0. \tag{4}$$

2.3. REMARK. The left-hand side of (4) is very easily seen to equal

$$\sum_{1 \le i,j \le n} [w_i w_j + s_i s_j] d(x_i, x_j)^q - 2 \sum_{1 \le i,j \le n} w_i s_j d(x_i, x_j)^q, \tag{5}$$

strongly hinting at generalized roundness.

Proof of Theorem 2.2. (ii) $\Rightarrow$ (i). Suppose that (ii) holds. Consider a given double-m-simplex $[a_i; b_i]_{i=1}^{m}$ in the metric space (X, d). Setting $n = 2m$, consider the points

$$x_i := \begin{cases} a_i, & 1 \le i \le m \\ b_{i-m}, & m+1 \le i \le n \end{cases}$$

and the weights

$$w_i := \begin{cases} 1, & 1 \le i \le m \\ 0, & m+1 \le i \le n \end{cases}, \qquad s_i := \begin{cases} 0, & 1 \le i \le m \\ 1, & m+1 \le i \le n \end{cases}.$$

Obviously, $\sum_{i=1}^{n} w_i = m = \sum_{i=1}^{n} s_i$. Applying (ii), we see that

$$\begin{aligned} 0 &\ge \sum_{1 \le i,j \le n} d(x_i, x_j)^q (w_i - s_i)(w_j - s_j) \\ &= \sum_{1 \le i,j \le m} d(a_i, a_j)^q + \sum_{m+1 \le i,j \le n} d(b_{i-m}, b_{j-m})^q \\ &\quad - \sum_{\substack{m+1 \le i \le n \\ 1 \le j \le m}} d(b_{i-m}, a_j)^q - \sum_{\substack{1 \le i \le m \\ m+1 \le j \le n}} d(a_i, b_{j-m})^q \\ &= \sum_{1 \le i,j \le m} \left[d(a_i, a_j)^q + d(b_i, b_j)^q \right] - 2 \sum_{1 \le i,j \le m} d(a_i, b_j)^q. \end{aligned}$$

In other words, we have shown that $\sum_{1 \le i < j \le m} \left[d(a_i, a_j)^q + d(b_i, b_j)^q \right] \le \sum_{1 \le i,j \le m} d(a_i, b_j)^q$, and hence $q \in \mathrm{gr}[a_i; b_i]_{i=1}^{m}$. It follows that X has generalized roundness q.

(i) $\Rightarrow$ (ii). Suppose that X has generalized roundness q. Consider a finite sequence $x_1, \dots, x_n$ in X. Suppose that $N \geq n$ and that $l_1, \dots, l_n, m_1, \dots, m_n$ are nonnegative integers satisfying $\sum_{j=1}^{n} l_j = N = \sum_{j=1}^{n} m_j$.

Construct a double-N-simplex as follows. Set

$$a_1 = a_2 = \cdots = a_{l_1} = x_1, \quad a_{l_1+1} = a_{l_1+2} = \cdots = a_{l_1+l_2} = x_2, \quad \text{and so on;}$$
$$b_1 = b_2 = \cdots = b_{m_1} = x_1, \quad b_{m_1+1} = b_{m_1+2} = \cdots = b_{m_1+m_2} = x_2, \quad \text{and so on.}$$

Since X has generalized roundness q, inequality (2) for this double-N-simplex can be manipulated to give

$$\sum_{1 \leq i,j \leq n} d(x_i, x_j)^q \left[\frac{l_i l_j + m_i m_j}{N^2}\right] \leq 2 \sum_{1 \leq i,j \leq n} d(x_i, x_j)^q \left[\frac{l_i m_j}{N^2}\right]. \tag{6}$$

Comparing (6) with (4) and (5), and noting that the set

$$\left\{ \frac{1}{N}(k_1, \dots, k_n) \;\middle|\; N, k_i \in \mathbb{N},\ N \geq n, \text{ and } \sum_{j=1}^{n} k_j = N \right\}$$

is a dense subset of

$$\left\{ (z_1, \dots, z_n) \;\middle|\; \text{each } z_i \geq 0 \text{ and } \sum_{j=1}^{n} z_j = 1 \right\},$$

an elementary continuity argument completes the proof that (i) $\Rightarrow$ (ii). □

From this reformulation of generalized roundness we derive the following theorem.

2.4. THEOREM. *A metric space (X, d) has q-negative type if and only if it has generalized roundness q.*

Proof. Suppose that (X, d) has q-negative type. Let $[a_i; b_i]_{i=1}^{n} \subseteq (X, d)$ be a double-n-simplex. Set

$$x_1 = a_1, x_3 = a_2, \dots, x_{2n-1} = a_n,$$
$$x_2 = b_1, x_4 = b_2, \dots, x_{2n} = b_n,$$

and set $\xi_j := (-1)^j$ for all $1 \leq j \leq 2n$. It is clear that we have $\sum_{j=1}^{2n} \xi_j = 0$. Hence, by our hypothesis, we have

$$\sum_{1 \leq i,j \leq 2n} d(x_i, x_j)^q \xi_i \xi_j \leq 0. \tag{7}$$

Summing over (i, j) both odd, (i, j) both even, i even and j odd, and i odd and j even, we see from (7) that

$$0 \geq \sum_{1 \leq i,j \leq 2n} d(x_i, x_j)^q \xi_i \xi_j = \sum_{1 \leq i,j \leq n} \left\{ d(a_i, a_j)^q + d(b_i, b_j)^q - 2d(a_i, b_j)^q \right\}.$$

It follows that $q \in \mathrm{gr}[a_i; b_i]_{i=1}^{n}$. We conclude that X has generalized roundness q.

On the other hand, suppose that X has generalized roundness q and consider $x_1, \ldots, x_n$ in X and real numbers $\xi_1, \ldots, \xi_n$ satisfying $\sum_{j=1}^{n} \xi_j = 0$.

If $\xi_j \geq 0$ set $w_j = \xi_j$ and $s_j = 0$. If $\xi_j < 0$ set $w_j = 0$ and $s_j = -\xi_j$. Then $\sum_{j=1}^{n} w_j = \sum_{j=1}^{n} s_j$ and so

$$\sum_{1 \leq i,j \leq n} d(x_i, x_j)^q (w_i - s_i)(w_j - s_j) \leq 0$$

by Theorem 2.2(ii). Since $w_j - s_j = \xi_j$ for all $1 \leq j \leq n$, we conclude that X has q-negative type. □

It is well known that if a metric space has q-negative type then it has q_1-negative type for all $0 \leq q_1 \leq q$ (see [WW, p. 11]). It follows immediately from Theorem 2.4 that generalized roundness shares this interval property.

2.5. COROLLARY. *If a metric space has generalized roundness q then it has generalized roundness q_1 for all $0 \leq q_1 \leq q$.*

We remark that such a statement does not hold true for roundness in a general metric space. An example of this is given in [E3, p. 254]. It does hold true for roundness in Banach spaces, however, by the interpolation technique mentioned in Section 1.

Using Corollary 2.5, we can give a short proof of the following result, known to Enflo for $1 \leq p \leq 2$ (though to our knowledge no proof has previously appeared in the literature). We remark that for $0 < p < 1$ we are using the usual quasinorm on L_p and the obvious extension of Definition 1.1(b) to the associated quasimetric.

2.6. COROLLARY. (a) *Let $0 < p \leq 2$. Let μ be a positive measure. Then* $\mathrm{gr}(L_p(\mu)) = [0, p]$.

(b) *Moreover, if $(X, \|\cdot\|_X)$ is a Banach space with generalized roundness q, $0 < p \leq q \leq 2$, then the Lebesgue–Bochner space $L_p(\mu, X)$ has* $\mathrm{gr}(L_p(\mu, X)) = [0, p]$. *In particular, this is the case for $X = L_q(\nu)$.*

Proof. (a) We begin by showing that $(\mathbb{R}, |\cdot|)$ has generalized roundness 2, as stated in [E2]. Let $[a_i; b_i]_{i=1}^{n}$ be a double-n-simplex of real numbers. Then clearly

$$\sum_{1 \leq i < j \leq n} \left[(a_i - a_j)^2 + (b_i - b_j)^2\right] = n \sum_{i=1}^{n} (a_i^2 + b_i^2) - \left(\sum_{i=1}^{n} a_i\right)^2 - \left(\sum_{i=1}^{n} b_i\right)^2,$$

and

$$\sum_{1 \leq i,j \leq n} (a_i - b_j)^2 = n \sum_{i=1}^{n} (a_i^2 + b_i^2) - 2\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{j=1}^{n} b_j\right).$$

So to prove that $(\mathbb{R}, |\cdot|)$ has generalized roundness 2, it is enough to show

$$2\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{j=1}^{n} b_j\right) \leq \left(\sum_{i=1}^{n} a_i\right)^2 + \left(\sum_{j=1}^{n} b_j\right)^2,$$

which is trivially true. Similarly, if we replace real by complex scalars, the immediately preceding result is still true.

By Corollary 2.5, we can replace 2 by p for all $0 < p \leq 2$ and then integrate the scalar inequality to obtain $p \in \mathrm{gr}(L_p(\mu))$.

To see that $L_p(\mu)$ does not have generalized roundness q for any $q > p$, simply consider the double-2-simplex $[a_1, a_2; b_1, b_2]$ where $a_1 \equiv 0$, $a_2 \equiv 1$, $b_1 = \chi_{(0,1/2)}$, and $b_2 = \chi_{(1/2,1)}$. Alternatively, note that $q \in \mathrm{gr}(L_p(\mu))$ implies that $q \in \mathrm{r}(L_p(\mu))$; hence $q \leq p$ by [E1].

(b) This is very similar to part (a). □

2.7. REMARK. In [E1], Enflo also integrates a scalar inequality to prove that for $1 \leq p \leq 2$, $L_p[0, 1]$ has roundness p. He derives his scalar inequality by using elementary calculus.

Another application of Theorem 2.4 is an indirect proof of Theorem 1.3 in the special case that $L_p(\mu)$ is infinite-dimensional. Since we are assuming $p > 2$, such spaces do not have q-negative type for any $q > 0$ (see e.g. [WW, p. 36]), and Theorem 1.3 follows.

The generalized roundness of $l_p^{(n)}$ in the case $p > 2$ can also be settled using Theorem 2.4 in conjunction with existing theory. To begin with, it is well known that L_1 contains a linear isometric copy of every two-dimensional normed space. This result was obtained independently by several authors in the early sixties; and in particular was proven by Herz [H]. (Also see, for example, Yost [Y] for a short proof and further references.) So, recalling that L_1 has generalized roundness 1, we see that for every two-dimensional normed space X, $\mathrm{gr}(X) \supseteq [0, 1]$. Such need not be true for higher (finite) dimensions. Two important theorems come into play. One is due to Bretagnolle, Dacunha-Castelle, and Krivine [BDK] (and may also be found in [WW, p. 23]). The other is due to Dor [D], Misiewicz [Mi], and Koldobsky [K].

THEOREM [BDK]. *If a finite-dimensional normed space X has q-negative type for some $0 < q \leq 2$, then there is a linear isometry from X into some L_q-space.*

THEOREM [D; Mi; K]. *For $2 < p \leq \infty$ and $n \geq 3$, $l_p^{(n)}$ is not linearly isometric to a subspace of any L_q-space with $0 < q \leq 2$.*

Theorem 2.4 and the above two results imply our final result.

2.8. THEOREM. *For $2 < p \leq \infty$ and $n \geq 3$, $l_p^{(n)}$ fails to have generalized roundness q for any $q > 0$.*

Of course, Theorem 2.8 is equivalent to Theorem 1.3.

OPEN QUESTION. What is the (maximal) generalized roundness of the Schatten class C_p (or more generally, of any noncommutative L_p-space) for $1 \leq p \leq 2$?

We remark that $\mathcal{C}_p$ has generalized roundness 0 for $p > 2$, since its subspace l_p has generalized roundness 0. We further remark that the "integration of a pointwise inequality" proof strategy of Corollary 2.6 does not work for $\mathcal{C}_p$, $1 \leq p \leq 2$.

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