The Holonomy in Open Manifolds of Nonnegative Curvature

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According to the well-known result by Cheeger and Gromoll [CG], every open (i.e. complete noncompact) manifold V^n of nonnegative sectional curvature $K_\sigma \geq 0$ is diffeomorphic to the space of the normal bundle νS of some totally geodesic submanifold S, called the *soul* of V^n. In this article we consider some relations between the geometry of V^n and the holonomy of this bundle. If V^n is isometric to the direct product $V^n = S \times W$ (where W is an open manifold of nonnegative sectional curvature, diffeomorphic to the Euclidean space), then the holonomy operator is the identity; that is, for every closed curve $\omega(s) \subset S$, $0 \leq s \leq 1$, the parallel translation I_ω along this curve maps every vector of $\nu_p S$ for $p = \omega(0)$ into itself. So $I_\omega = \mathrm{id}$ for every closed curve ω on S, and we will say that νS has trivial holonomy. One of the main results of this article is that the converse is also true (see Section 1).

THEOREM 1. *If νS has trivial holonomy, then V^n is isometric to the direct product: $V^n = S \times W$.*

This theorem was announced in [M1].

In Section 2 we find some conditions on the behavior of the curvature near S for a trivial holonomy that, according to Theorem 1, lead to the metric splitting. Originally these conditions (Theorems 2, 3, and 4 herein) were found with the help of some geometric construction and received rather long but straightforward proofs; see [M3]. Then a very short proof of Theorem 2 was presented to the author by G. Perelman, who suggested the possibility of finding a similar short and analytic proof for Theorem 4 also. That is done at the end of Section 2.

THEOREM 2. *For every point p on S and every 2-dimensional direction σ that is normal to S at this point (i.e., $\sigma \subset \nu_p S$), if*

$$K_\sigma = 0$$

then $I_\omega = \mathrm{id}$ for every contractible curve ω on S and the universal cover $\tilde{V}^n$ of V^n is isometric to the direct product.

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If S is flat, then the universal cover $\tilde{V}^n$ of V^n is isometric to the direct product and the holonomy along any contractible curve in S vanishes. The next theorem is a local version of this statement.

THEOREM 3. *For every closed curve ω contractible in some flat domain D in S, the holonomy along ω vanishes. That is, for every point p of ω and every v of $\nu_p S$,*

$$I_\omega v = v.$$

One of the most important examples of an open manifold of nonegative curvature is the total space TS^n of the tangent bundle of the sphere S^n; see [CG]. When $O(n+1)$ has a bi-invariant metric of nonnegative curvature and $O(n)$ acts on flat Euclidean space R^n by rotation, the map

$$\pi: O(n+1)\times R^n \to O(n+1)\times R^n/O(n) = TS^n$$

is a Riemannian submersion. Therefore, according to [O], TS^n admits a metric of nonnegative curvature. It turns out that the soul S of TS^n is unique, the holonomy of the soul's normal bundle is nontrivial, and all mixed curvatures vanish. It was shown in [M2] that for $n=4$ these are the only directions of zero curvature. For instance, let $\sigma(p,v,e)$ be the 2-dimensional direction at the point p generated by the vector e tangent to S, v being normal to S, and let $l_w(\rho)$ be the geodesic issuing from p in a direction $w \neq v$ normal to S. Then for the 2-dimensional direction $\sigma(p,e,v,w,\rho)$ obtained by the parallel translation of $\sigma(p,e,v)$ along $l_w(\rho)$ we have

$$K_{\sigma(p,e,v,w,\rho)} \geq k\rho^2$$

for some $k>0$. For arbitrary open manifold V^n of nonnegative curvature, according to [CG, Thm. 3.1] we have

$$K_{\sigma(p,e,v,w,0)} = 0.$$

Thus, in general

$$K_{\sigma(p,e,v,w,\rho)} = O(\rho^2).$$

Is it possible for an arbitrary manifold that the curvatures of this type be of greater order in ρ? The next theorem shows that, in some sense, this is impossible.

THEOREM 4. *If for every point p on S and every e, v, and w*

$$K_{\sigma(p,e,v,w,\rho)} = o(\rho^2)$$

as $\rho\to 0$, then $I_\omega = \mathrm{id}$ for every contractible curve ω on S, the universal cover $\tilde{V}^n$ of V^n is isometric to the direct product, and

$$K_{\sigma(p,e,v,w,\rho)} \equiv 0.$$

1. The Holonomy Operator and Short Maps on the Soul S

Recall that the soul S of the open manifold V^n of nonnegative sectional curvature is the limit of an equidistant family of compact totally convex sets C_t, $0\leq t\leq T$, with the following properties:

(1) $\operatorname{int} C_T \neq \emptyset$;
(2) for some $0 = t_0 < t_1 < \cdots < t_m = T$ and every $t_{i-1} \leq t < t_i$,

$$C_t = \{p \in C_{t_i} \mid \rho(p, \partial C_{t_i}) \geq t_i - t\};$$

(3) $\dim C_{t_{i-1}} < \dim C_{t_i}$; and
(4) $S = C_{t_0}$.

From the fact that every C_t is totally convex one can deduce the existence of short maps, that is, of distance-nonincreasing maps $\phi_t \colon C_t \to S$; see [S]. With the help of these maps we can prove the following theorem.

THEOREM 1. *If νS has trivial holonomy then V^n is isometric to the direct product $V^n = S \times W$, where W is an open manifold diffeomorphic to the Euclidean space of the corresponding dimension.*

Proof. Let p be some point of S. Choose any vector $v(p)$ of $\nu_p S$ and define the vector field v on S in the following way:

$$v(q) = I_{\omega_q}(v(p)),$$

where $\omega_q \subset S$ is any curve from the point p to the point q, and I_{ω_q} is the parallel translation along ω_q. Since the holonomy is trivial it follows that the field v is well-defined. Let $\psi_\theta \colon S \to V^n$ be the family of maps:

$$\psi_\theta(q) = \exp(\theta v(q)).$$

The vector field v is parallel along every curve on S. Therefore, from the Berger version of the Rauch comparison theorem (see [B]), it follows that there exists a θ_0 such that for all $0 < \theta < \theta_0$ the maps ψ_θ are short (i.e., distance-nonincreasing) maps, and that ψ_θ is an isometry if and only if for every geodesic $\gamma(t) \subset S$ the Synge film $\pi(s, t) = \psi_s(\gamma(t))$ is totally geodesic and flat. We reformulate the last property in the following way: Let p be a point on S, $e \in T_p S$, $v \in \nu_p S$, and $l_v(\theta) = \exp(\theta v)$. Let $e(\theta), v(\theta)$ be two parallel vector fields along $l_v(\theta)$ such that $e(0) = e$ and $v(0) = v$. Moreover, let $\sigma(p, v, e, \theta)$ be a 2-dimensional plane generated by $v(\theta)$ and $e(\theta)$. Then it is not difficult to prove that the map ψ_θ is an isometry if and only if

$$K_{\sigma(p, v, e, \theta)} = 0 \quad \text{for all } 0 \leq \theta \leq \theta_0. \tag{1.1}$$

LEMMA 1.1. *If the holonomy of νS is trivial then, for all v and θ, all ψ_θ are isometries and* (1.1) *holds for all p, v, e, s.*

Proof. For any given $\theta < \theta_0$, choose t such that $\psi_\theta \subset C_t$. Then the map $\phi_t \circ \psi_\theta \colon S \to S$ is the short one and is homotopic to the identity map. Hence we can conclude that ψ_θ is an isometry and that (1.1) holds for all $\theta < \theta_0$. But in this case the claim of the Berger theorem (stating that ψ_θ is a short map) is also true for $\theta_0 < \theta < \theta_1$, and we see (repeating the above arguments) that (1.1) holds for all $\theta < \theta_1$. Therefore the set of all θ such that (1.1) is true is an open set. Obviously, this set is also closed, and therefore (1.1) holds for all θ. □

LEMMA 1.2. *All submanifolds $S_\theta = \psi(S)$ are totally geodesic.*

Proof. The proof is obvious. If we have two points $p' = \psi_\theta(p)$ and $q' = \psi_\theta(q)$ on S_θ such that p and q are from S, then for $\gamma(t)$, which is the minimal geodesic from p to q lying on S, the upper edge of the Synge film $\pi(s,t) = \psi_s(\gamma(t))$ is the geodesic $\gamma_\theta(t) = \psi_\theta(\gamma(t))$ lying on S_θ. Therefore S_θ is totally geodesic. Lemma 1.2 is proved. □

Consider the family of the submanifolds

$$W_p = \exp(\nu_p S),$$

which we will call *fibers*; W_p is the fiber over p. Let us arbitrarily choose two vectors, e from $T_p S$ and v from $\nu_p S$, and let $\rho > 0$. Construct the geodesic

$$\gamma(t) = \exp_p(te) \subset S$$

and the curve

$$q(t) = \psi_\rho(\gamma(t)).$$

Then $q(t) \in W_{\gamma(t)}$, and it is not difficult to verify that the vector $\dot{q}(t)$ is normal to the submanifold $W_{\gamma(t)}$ at point $q(t)$. Denote by $A_\rho(t)$ the second fundamental form of $W_{\gamma(t)}$ corresponding to the normal $\dot{q}(t)$, and by $G_\rho(t)$ its trace:

$$G_\rho(t) = \sum_{i=1}^{d-1} (A_\rho(t)\bar{e}_i(t), \bar{e}_i(t)),$$

where $\bar{e}_i(t)$ is an orthonormal basis of $T_{q(t)}W_{\gamma(t)}$ consisting of the eigenvectors of the form $A_\rho(t)$,

$$A_\rho(t)\bar{e}_i(t) = \frac{D}{\partial e_i(t)}\dot{q}(t) = \lambda_i(t)\bar{e}_i(t)$$

($\bar{e}_i(t)$ is not necessarily continuous in t). We see that $D\dot{q}(t)/\partial\rho \equiv 0$, so we may assume that $\bar{e}_1(t)$ equals the vector $\partial/\partial\rho \equiv -\overline{q(t)\gamma(t)}$ and that $\lambda_1(t) \equiv 0$, where $\overline{pq}$ denotes the unit vector (in the direction of the minimal geodesic pq at point p) connecting points p and q. To compute $\partial G_\rho(t)/\partial t$, let us introduce the special system of the Fermi coordinates in the following way: Let $\operatorname{codim} S = d-1$, let the axis of the coordinate system be the geodesic $\gamma(t)$ on S that is simultaneously the dth coordinate line, and let the geodesic $pq(0)$ be the first coordinate line. In this coordinate system all points $q(t)$ have the following coordinates: $q^i(t) = \rho\delta_{1i} + t\delta_{id}$. By $e_i(\rho, t)$ we denote the coordinate vectors of this system at the point $q(t)$ and choose $e_i(0,0)$ so that $e_i(\rho, 0)$ coincides with $\bar{e}_i(0)$ for $i < d$, while $e_j(0,0)$ for $j \geq d$ generate $T_p S$. It is not difficult to prove that $e_i(0,0)$, $i < d$, generate $\nu_p S$. By definition $e_i(0,t) = I_{p\gamma(t)}(e_i(0,0))$. Therefore, $e_i(0,t)$, $i < d$, generate $\nu_{\gamma(t)}S$, and $e_j(0,t)$, $j \geq d$, generate $T_{\gamma(t)}S$. From the metric tensor estimates we see also that $e_j(\rho, t)$ generate $T_{q(t)}S_\rho$, while $e_i(\rho,t)$, $i < d$, generate $T_{q(t)}W_{\gamma(t)}$, and

$$(e_i(\rho,t), e_j(\rho,t)) \equiv 0, \quad i < d \leq j. \tag{1.2}$$

All S_ρ are totally geodesic submanifolds, so we have

$$\left(\frac{D}{\partial e_d(\rho,t)} e_i(\rho,t), e_j(\rho,t)\right) \equiv 0; \tag{1.3}$$

$e_i(\rho,t)$ are coordinate vectors, so they commute. Therefore, from (1.3) it follows that

$$\begin{aligned} \frac{D}{\partial e_d(\rho,t)} e_i(\rho,t)\Big|_{t=0} &= \frac{D}{\partial e_i(\rho,t)} e_d(\rho,t)\Big|_{t=0} \\ &= \frac{D}{\partial e_i(\rho,0)} \dot{q}(0) = A_\rho(0) e_i(0) = \lambda_i(0)\bar{e}_i(0). \end{aligned} \tag{1.4}$$

LEMMA 1.3.

$$\frac{\partial}{\partial t} G_\rho(t) = -\sum_{i=2}^{d-1} (R[e_d(\rho,t), e_i(\rho,t)] + \lambda_i^2(t)), \tag{1.5}$$

where $R[v,w]$ is the sectional curvature of the plane spanned by v and w.

Proof. Without loss of generality we may assume that $t=0$. By definition,

$$G_\rho(t) = \sum_{i=1}^{d-i} (A_\rho(t)\bar{e}_i(t), \bar{e}_i(t))$$

and, if $t=0$, then $\bar{e}_i(0) = e_i(\rho,0)$, $i<d$, and $e_i(\rho,0)$ is the orthonormal basis. Since $A_\rho(t)$ is symmetric, we have

$$\begin{aligned} \frac{\partial}{\partial t} G_\rho(t) &= \frac{\partial}{\partial t} \sum_{i=1}^{d-1} (A_\rho(t)\bar{e}_i(t), \bar{e}_i(t)) \\ &= \sum_{i=1}^{d-1} \left(\frac{D}{\partial t} A_\rho(t)\Big|_{t=0} \bar{e}_i(0), \bar{e}_i(0)\right) = \sum_{i=1}^{d-1} \left(\frac{D}{\partial t} A_\rho(t)\Big|_{t=0} e_i(0), e_i(0)\right) \\ &= \frac{\partial}{\partial t} \sum_{i=1}^{d-1} (A_\rho(t) e_i(\rho,t), e_i(\rho,t))\Big|_{t=0} \\ &\quad -2\sum_{i=1}^{d-1} \left(A_\rho(t) e_i(\rho,t), \frac{D}{\partial t} e_i(\rho,t)\right)\Big|_{t=0}. \end{aligned}$$

Therefore, from (1.4) we see that

$$\frac{\partial}{\partial t} G_\rho(t) = \frac{\partial}{\partial t} \sum_{i=1}^{d-1} (A_\rho(t) e_i(\rho,t), e_i(\rho,t))\Big|_{t=0} - 2\sum_{i<1} \lambda_i^2.$$

But

$$\begin{aligned} &\frac{\partial}{\partial t} \sum_{i=1}^{d-1} (A_\rho(t) e_i(\rho,t), e_i(\rho,t))\Big|_{t=0} \\ &\quad = \sum_{i=1}^{d-1} \left(\frac{D^2}{\partial t^2} e_i(\rho,t), e_i(\rho,t)\right)\Big|_{t=0} + \left(\frac{D}{\partial t} e_i(\rho,t), \frac{D}{\partial t} e_i(\rho,t)\right)\Big|_{t=0}. \end{aligned} \tag{1.6}$$

The vector fields $e_i(\rho,t)$ along $q(t)$ are the variation fields of d-coordinate lines:

$$e_i(\rho,t) = \frac{\partial}{\partial \theta} \psi_{v(\theta),\rho}(\gamma(t)),$$

where $v(\theta) = v + \theta e_i(0, 0)$. But all lines $\psi_{v(\theta),\rho}(\gamma(t))$ are geodesics. Therefore the vector fields $e_i(\rho, t)$ are Jacobi fields along $q(t)$ and

$$\frac{D^2}{\partial t^2} e_i(\rho, t) = -R(e_i(\rho, t), e_d(\rho, t))e_d(\rho, t),$$

where R is the curvature tensor of V^n. Inserting (1.4) and the last equality into (1.6), we obtain the claim of the lemma. □

LEMMA 1.4. *All submanifolds W_p are totally geodesic.*

Proof. All values in (1.5) do not depend on the particular choice of the basis $\bar{e}_i(t)$, but only on the point $\gamma(t)$ and the vector $\dot{\gamma}(t)$ (if v from $\nu_p S$ and $\rho > 0$ are given). Let us assign

$$\sum_{i=2}^{d-1} (R[e_d(\rho, t), \bar{e}_i(t)] = K(\gamma(t), \dot{\gamma}(t))$$

and

$$\sum_{i=2}^{d-1} \lambda_i^2(t) = \Lambda(\gamma(t), \dot{\gamma}(t)).$$

Using the compactness of S, we see that the function $G_\rho(t)$ is bounded: There exists some constant K such that

$$|G_\rho(t)| \le K \quad \text{for all } -\infty < t < \infty.$$

Therefore, for arbitrary $T > 0$, from Lemma 1.3 we have

$$\left| \frac{1}{2T} \int_{-T}^{T} K(\gamma(t), \dot{\gamma}(t))\, dt \right| \le \frac{K}{T}, \tag{1.7}$$

$$\left| \frac{1}{2T} \int_{-T}^{T} \Lambda(\gamma(t), \dot{\gamma}(t))\, dt \right| \le \frac{K}{T}. \tag{1.8}$$

A geodesic flow—that is, the map sending (p, e) to $(\gamma(t), \dot{\gamma}(t))$—preserves the volume form of the bundle T^1S of the unit vectors tangent to S. So, from the Birkhoff-Khintchine theorem, we see that the left-hand sides of (1.7) and (1.8) under the constraint $T \to \infty$ tend to the mean values of the functions K and Λ on T^1S, which equal zero according to (1.7) and (1.8). Hence the non-negativity of K and Λ implies $K \equiv 0$ and $\Lambda \equiv 0$. Since ρ and v were chosen arbitrarily, W_p is totally geodesic for every p. Lemma 1.4 is proved. □

Now we can complete the proof of Theorem 1.

Let q and q' be arbitrary points in the r_{in}-neighborhood of S. Find p and p' so that $q \in W_p$ and $q' \in W_{p'}$. Connect p and p' by some minimal geodesic $\gamma(t)$, $0 \le t \le t_0$, on S, and define the map $\omega(t)\colon W_p \to W_{\gamma(t)}$ by

$$\omega(t)(\exp_p v) = \exp_{\gamma(t)}(I_{p\gamma(\tau)}(v)).$$

From (1.1) it follows that $\omega(t)(r)$, for fixed r, is the geodesic, and that $\omega(t)$ is an isometry if all $W_{\gamma(t)}$ are totally geodesic. Therefore, from Lemma 1.4 we see that all maps $\omega(t)$ are isometries. Let us connect q' with $\omega(t)(q)$ by some minimal geodesic $l(\xi)$. The fiber $W_{q'}$ is totally geodesic, so $l(\xi) \subset W_{q'}$. Consider the film

$$\pi(\xi, t) = \omega(t)(\omega^{-1}(t_0) l(\xi)).$$

If ξ is fixed then $\pi(\xi, t)$ is geodesic, and from $K=0$ we easily see that the vector field $\partial\pi(\xi, t)/\partial\xi$ is parallel along this geodesic. Therefore $\pi(\xi, t)$ is locally isometric to the Euclidean plane and, in fact, is totally geodesic, because $l(\xi)$ is geodesic. So

$$\rho(q, q') = \sqrt{\rho^2(p, p') + \rho^2(q', \omega(t_0))}$$

and $i\colon S \times W_p \to V^n$, where $i(p', \exp_p v) = \exp_{p'}(I_{pp'}(v))$ is an isometry in the considered r_{in}-neighborhood of S. We can extend the given consideration to a larger neighborhood of S, replacing S by some $S_{v,\rho}$ with $\rho < r_{\text{in}}$. Thus we can prove that the domain where V^n is a direct product is open. Obviously this region is closed. Therefore, from standard arguments we easily obtain the claim of the theorem: V^n is isometric to the direct product $S \times W_p$. □

2. Proofs of Theorems 2, 3, and 4

According to Theorem 1, the claims of Theorems 2, 3, and 4 will follow from the vanishing of the holonomy, that is, $I_\omega \equiv \text{id}$ for every contractible curve $\omega \subset S$ (or $\omega \subset D$ in Theorem 3). If $\omega = \partial\Omega$ for some surface Ω in S (or in D in Theorem 3) then, according to the Ambrose–Singer theorem, to prove this it is sufficient to check that

$$R(e_1, e_2)v \equiv 0 \tag{2.1}$$

at all points p on Ω and all e_1 and e_2 of $T_p\Omega \subset T_pS$ and v of ν_pS.

To simplify the notation we choose a Fermi coordinate system in some neighborhood of an arbitrarily chosen p such that e_1 and e_2 are the first coordinate vectors at p, e_1 is the direction of the axis, and v is the third vector. Because S is totally geodesic for every e tangent to S at p, we have $(R(e_1, e_2)v, e) \equiv 0$. So, to prove (2.1) it is enough to check that, for every w normal to S,

$$(R(e_1, e_2)v, w) \equiv 0. \tag{2.2}$$

Choose such a w and denote it by e_4 of our coordinate system.

Proof of Theorem 2 (due to G. Perelman). As explained above, to prove Theorem 2 it is sufficient to verify (2.2) under the conditions of the theorem. According to these conditions $R_{34,34} = 0$, and nonnegativity of the curvature leads to $R_{s4,34} = 0$ and $R_{3s,34} = 0$ for all s. By direct computation one gets $(R(e_1+e_3, e_4)(e_1+e_3), e_4) = 0$ which, again because of the nonnegativity of the curvature, leads to

$$(R(e_1+e_3, e_2)(e_1+e_3), e_4) = 0 \tag{2.3}$$

or, if we take into account that by the same reasoning $R_{12,14}=R_{32,34}=0$, to $R_{12,34}=R_{23,14}$. In the same way, considering $(R(e_1+e_4,e_3)(e_1+e_4),e_3)=0$ we have $R_{12,34}=R_{13,24}$, and from the first Bianchi identity

$$R_{12,34}+R_{14,23}+R_{13,42}=0$$

we obtain the claim of the theorem:

$$R_{12,34}=0.$$

Theorem 2 is proved. □

Proof of Theorem 3. In the same way as above, it is easy to see that $R_{12,12}=0$ also leads to the vanishing of (2.3), which yields $R_{12,34}=0$ and the claim of the theorem. □

Proof of Theorem 4. Note that, since e_3 and e_4 are normal to S, we have

$$(R(e_1,e_3)e_s,e_3)=0,\qquad (R(e_1,e_4)e_s,e_4)=0,$$

and

$$(R(e_1,e_3+e_4)e_s,e_3+e_4)=0,$$

or

$$R_{14,s3}=R_{13,4s}. \tag{2.4}$$

Therefore, from the first Bianchi identity

$$R_{12,34}+R_{14,23}+R_{13,42}=0$$

it follows that

$$R_{12,34}=-2R_{14,23}=-2R_{13,42}. \tag{2.5}$$

Obviously the same is true for every s instead of 3, if the vector e_s is normal to S,

$$R_{12,s4}=-2R_{14,2s}=-2R_{1s,42},$$

and for every s instead of 2 if the vector e_s is tangent to S,

$$R_{1s,34}=-2R_{14,s3}=-2R_{13,4s}. \tag{2.6}$$

According to the conditions of Theorem 3, we have

$$(R_{13,13})''_{44}=0.$$

These conditions also imply that

$$(R_{13,34})''_{14}=(R_{13,41})''_{34}=0,$$

because from the nonnegativity of the sectional curvature of V^n we have

$$|R_{13,34}|\le\sqrt{|R_{13,31}||R_{43,34}|}=o(\rho)$$

and

$$|R_{13,41}|\le\sqrt{|R_{13,31}||R_{41,14}|}=o(\rho^2),$$

correspondingly. Taking the derivative of the second Bianchi identity,

$$(R_{13,13})'_4+(R_{13,34})'_1+(R_{13,41})'_3$$
$$=\Gamma^s_{14}R_{s3,13}+\Gamma^s_{11}R_{s3,34}+\Gamma^s_{13}R_{s3,41}+\Gamma^s_{34}R_{1s,13}+\Gamma^s_{31}R_{1s,34}+\Gamma^s_{33}R_{1s,41}$$

along the fourth coordinate we obtain at the point p (where all Christoffel symbols are zero) the following equality:

$$0=(\Gamma^s_{11})'_4R_{s3,34}+(\Gamma^s_{31})'_4(R_{1s,34}+R_{s3,41}). \tag{2.7}$$

In the Fermi coordinate system with axis of direction e_1 we have

$$D^2/\partial e_1^2(e_4)=0,$$

because all coordinate vectors are parallel along the axis. Therefore, from $R_{14,1s}=0$ we conclude that

$$(\Gamma^s_{11})'_4=0.$$

As above, for e_s normal to S, from

$$(R(e_1,e_4+e_s)e_4+e_s,e_1)=0$$

we have

$$(R(e_1,e_4+e_s)e_4+e_s,e_3)=0$$

and $R_{1s,34}+R_{s3,41}=0$. For e_s tangent to S, from (2.5) we have

$$R_{1s,34}+R_{s3,41}=3R_{s3,41}.$$

Therefore (2.7) gives

$$(\Gamma^s_{31})'_4R_{s3,41}=0,$$

where summation is over all s such that e_s is tangent to S. Interchanging 3 and 4, we have

$$(\Gamma^s_{41})'_3R_{s4,31}=0,$$

which according to (2.4) and (2.5) gives

$$((\Gamma^s_{31})'_4+(\Gamma^s_{41})'_3)R_{1s,34}=\sum_s(R_{1s,34})^2=0,$$

where summation is over all s such that e_s is tangent to S. In particular, $R_{12,34}=0$. This completes the proof. □

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