Convolutions and Fourier–Feynman Transforms of Functionals Involving Multiple Integrals

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1. Introduction

Let $C_0[0,T]$ denote one-parameter Wiener space, that is, the space of $\mathbb{R}$-valued continuous functions $x(t)$ on $[0,T]$ with $x(0)=0$. The concept of an L_1-analytic Fourier–Feynman transform for functionals on Wiener space was introduced by Brue in [1]. In [3], Cameron and Storvick introduced an L_2-analytic Fourier–Feynman transform. In [11], Johnson and Skoug developed an L_p-analytic Fourier–Feynman transform for $1 \le p \le 2$ that extended the results in [1; 3]. In [9], Huffman, Park, and Skoug defined a convolution product for functionals on Wiener space and, for a class of functionals of the type

$$F(x)=f\left(\int_0^T \alpha_1(t)\,dx(t),\dots,\int_0^T \alpha_n(t)\,dx(t)\right),$$

showed that the Fourier–Feynman transform of the convolution product was a product of Fourier–Feynman transforms. In [10], they obtain similar results for functionals of the form

$$G(x)=\exp\left\{\int_0^T g(t,x(t))\,dt\right\},$$

which play an important role in quantum mechanics.

In this paper we consider functionals, on Wiener space, of the form

$$F(x)=\exp\left\{\int_0^T\int_0^T f(s,t,x(s),x(t))\,ds\,dt\right\} \tag{1.1}$$

for appropriate $f\colon [0,T]^2\times\mathbb{R}^2\to\mathbb{C}$. Such functionals were discussed in the book by Feynman and Hibbs [8, Secs. 3-10] on path integrals, and in Feynman's original paper [7, Sec. 13]. Feynman obtained such functionals by formally integrating out the oscillator coordinates in a system involving a harmonic oscillator interacting with a particle moving in a potential. The double dependence on time occurs because, as Feynman and Hibbs [8, p. 71] explain, "The separation of past and future can no longer be made. This

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happens because the variable x at some previous time affects the oscillator which, at some later time, reacts back to affect x."

Moreover, functionals like (1.1), but involving multiple integrals of more time dimensions than two, arise when more particles are involved. We consider a class of such functionals in Section 3 where $f\colon [0,T]^n \times \mathbb{R}^n \to \mathbb{C}$ is quadratic in the space variables.

In Section 3 we establish several results involving Fourier–Feynman transforms and convolutions for functionals in a Banach algebra $\mathcal{S}$ introduced by Cameron and Storvick in [4]. These results can then be immediately applied to many functionals of the form (1.1) which are known to belong to $\mathcal{S}$ [6; 14]. In addition, we establish a Parseval's identity for functionals F and G in $\mathcal{S}$.

In Section 4 we consider functionals of the form (1.1) with $f \in L_{1\infty}([0,T]^2 \times \mathbb{R}^2)$. These functionals in general do not belong to $\mathcal{S}$ (all of whose elements are bounded), and the resulting theory is considerably more complicated than the theory in Section 3.

2. Definitions and Preliminaries

Let $\mathfrak{M}$ denote the class of all Wiener measurable subsets of $C_0[0,T]$ and let m denote Wiener measure. $(C_0[0,T], \mathfrak{M}, m)$ is a complete measure space, and we denote the Wiener integral of a functional F by $\int_{C_0[0,T]} F(x)m(dx)$.

A subset E of $C_0[0,T]$ is said to be scale-invariant measurable [5; 13] provided $\rho E \in \mathfrak{M}$ for all $\rho > 0$, and a scale-invariant measurable set N is said to be scale-invariant null provided $m(\rho N) = 0$ for each $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). If two functionals F and G are equal s-a.e., we write $F \approx G$.

Let $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$ and $\tilde{\mathbb{C}}_+ = \{\lambda \in \mathbb{C} : \lambda \neq 0 \text{ and } \operatorname{Re}\lambda \geq 0\}$. Let F be a $\mathbb{C}$-valued scale-invariant measurable functional on $C_0[0,T]$ such that

$$J(\lambda) = \int_{C_0[0,T]} F(\lambda^{-1/2}x)m(dx)$$

exists for all $\lambda > 0$. If there exists a function $J^*(\lambda)$ analytic in $\mathbb{C}_+$ such that $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$, then $J^*(\lambda)$ is defined to be the analytic Wiener integral of F over $C_0[0,T]$ with parameter λ, and for $\lambda \in \mathbb{C}_+$ we write

$$\int_{C_0[0,T]}^{\mathrm{anw}_\lambda} F(x)m(dx) = J^*(\lambda).$$

Let $q \neq 0$ be a real number and F a functional such that $\int_{C_0[0,T]}^{\mathrm{anw}_\lambda} F(x)m(dx)$ exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists, we call it the analytic Feynman integral of F with parameter q and we write

$$\int_{C_0[0,T]}^{\mathrm{anf}_q} F(x)m(dx) = \lim_{\lambda \to -iq} \int_{C_0[0,T]}^{\mathrm{anw}_\lambda} F(x)m(dx),$$

where $\lambda \to -iq$ through $\mathbb{C}_+$.

NOTATION.

(i) For $\lambda \in \mathbb{C}_+$ and $y \in C_0[0,T]$, let

$$(T_\lambda(F))(y) = \int_{C_0[0,T]}^{\mathrm{anw}_\lambda} F(x+y)m(dx). \tag{2.1}$$

(ii) Given a number p with $1 \le p \le +\infty$, p and p' will always be related by $1/p + 1/p' = 1$.

(iii) Let $1 < p \le 2$ and let $\{H_n\}$ and H be scale-invariant measurable functionals such that, for each $\rho > 0$,

$$\lim_{n\to\infty} \int_{C_0[0,T]} |H_n(\rho y) - H(\rho y)|^{p'} m(dy) = 0. \tag{2.2}$$

Then we write

$$\operatorname*{l.i.m.}_{n\to\infty}(w_s^{p'})(H_n) \approx H \tag{2.3}$$

and call H the scale-invariant limit in the mean of order p'. A similar definition is understood when n is replaced by the continuously varying parameter λ.

We are finally ready to state the definition of the L_p-analytic Fourier–Feynman transform [12] and our definition of the convolution product [9].

DEFINITION. Let $q \ne 0$ be a real number. For $1 < p \le 2$ we defined the L_p-analytic Fourier–Feynman transform $T_q^{(p)}(F)$ of F by the formula ($\lambda \in \mathbb{C}_+$)

$$(T_q^{(p)}(F))(y) = \operatorname*{l.i.m.}_{\lambda\to -iq}(w_s^{p'})(T_\lambda(F))(y) \tag{2.4}$$

whenever this limit exists. We define the L_1-analytic Fourier–Feynman transform $T_q^{(1)}(F)$ of F by ($\lambda \in \mathbb{C}_+$)

$$(T_q^{(1)}(F))(y) = \lim_{\lambda\to -iq}(T_\lambda(F))(y) \tag{2.5}$$

for s-a.e. y. We note that, for $1 \le p \le 2$, $T_q^{(p)}(F)$ is defined only s-a.e. We also note that if $T_q^{(p)}(F_1)$ exists and if $F_1 \approx F_2$, then $T_q^{(p)}(F_2)$ exists and $T_q^{(p)}(F_2) \approx T_q^{(p)}(F_1)$.

DEFINITION. Let F and G be functionals on $C_0[0,T]$. For $\lambda \in \tilde{\mathbb{C}}_+$ we define their convolution product (if it exists) by

$$(F*G)_\lambda(y) = \begin{cases} \displaystyle\int_{C_0[0,T]}^{\mathrm{anw}_\lambda} F\left(\frac{y+x}{\sqrt{2}}\right) G\left(\frac{y-x}{\sqrt{2}}\right) m(dx), & \lambda \in \mathbb{C}_+; \\ \displaystyle\int_{C_0[0,T]}^{\mathrm{anf}_q} F\left(\frac{y+x}{\sqrt{2}}\right) G\left(\frac{y-x}{\sqrt{2}}\right) m(dx), & \lambda = -iq,\ q \in \mathbb{R},\ q \ne 0. \end{cases} \tag{2.6}$$

REMARK. When $\lambda = -iq$, we will denote $(F*G)_\lambda$ by $(F*G)_q$.

3. Transforms and Convolutions of Functionals in $\mathcal{S}$

The Banach algebra $\mathcal{S}$ of functionals on $C_0[0,T]$, each of which is a type of stochastic Fourier transform of a bounded $\mathbb{C}$-valued Borel measure, was introduced in [4] by Cameron and Storvick. The Banach algebra $\mathcal{S}$ consists of functionals expressible in the form

$$F(x)=\int_{L_2[0,T]}\exp\left\{i\int_0^T v(s)\,dx(s)\right\}df(v) \tag{3.1}$$

for s-a.e. x in $C_0[0,T]$ (i.e., except on a scale-invariant null set), where f is an element of $M(L_2[0,T])$, the space of $\mathbb{C}$-valued countably additive Borel measures on $L_2[0,T]$.

THEOREM 3.1. *Let $F\in\mathcal{S}$ be given by* (3.1). *Then, for all $p\in[1,2]$, the Fourier-Feynman transform $T_q^{(p)}(F)$ exists for all $q\in\mathbb{R}-\{0\}$ and is given by the formula*

$$(T_q^{(p)}(F))(y)=\int_{L_2[0,T]}\exp\left\{i\int_0^T v(t)\,dy(t)-\frac{i}{2q}\int_0^T v^2(t)\,dt\right\}df(v). \tag{3.2}$$

Proof. First of all, using the Fubini theorem and the well-known Wiener integration formula

$$\int_{C_0[0,T]}\exp\left\{i\int_0^T h(t)\,dx(t)\right\}m(dx)=\exp\left\{-\frac{1}{2}\int_0^T h^2(t)\,dt\right\}, \tag{3.3}$$

we obtain, for all $\lambda>0$ and s-a.e. $y\in C_0[0,T]$, the formula

$$\begin{aligned}(T_\lambda(F))(y)&=\int_{C_0[0,T]}F(\lambda^{-1/2}x+y)m(dx)\\&=\int_{C_0[0,T]}\int_{L_2[0,T]}\exp\left\{i\int_0^T v(t)\,d[\lambda^{-1/2}x(t)+y(t)]\right\}df(v)m(dx)\\&=\int_{L_2[0,T]}\exp\left\{i\int_0^T v(t)\,dy(t)\right\}\\&\quad\cdot\int_{C_0[0,T]}\exp\left\{\frac{i}{\sqrt{\lambda}}\int_0^T v(t)\,dx(t)\right\}m(dx)\,df(v)\\&=\int_{L_2[0,T]}\exp\left\{i\int_0^T v(t)\,dy(t)-\frac{1}{2\lambda}\int_0^T v^2(t)\,dt\right\}df(v).\end{aligned} \tag{3.4}$$

But the last expression is an analytic function of λ throughout $\mathbb{C}_+$, and is a bounded continuous function of λ on $\tilde{\mathbb{C}}_+$ for all y in $C_0[0,T]$ since f is a finite Borel measure. Hence $T_q^{(p)}(F)$ exists and is given by (3.2) for all desired values of p and q. □

THEOREM 3.2. *Let F and G be elements of $\mathcal{S}$ with corresponding finite Borel measures f and g in $M(L_2[0,T])$. Then their convolution product $(F*G)_q$ exists for all $q\in\mathbb{R}-\{0\}$ and is given by the formula*

$$(F*G)_q(y) = \int_{L_2^2[0,T]} \exp\left\{\frac{i}{\sqrt{2}} \int_0^T [v(t)+w(t)]\,dy(t)\right\}$$
$$\cdot \exp\left\{-\frac{i}{4q} \int_0^T [v(t)-w(t)]^2\,dt\right\} df(v)\,dg(w). \tag{3.5}$$

Proof. Proceeding as in the proof of Theorem 3.1, for all $\lambda > 0$ and s-a.e. y in $C_0[0,T]$ we obtain

$$\begin{aligned}
(F*G)_\lambda(y) &= \int_{C_0[0,T]} F\left(\frac{y+\lambda^{-1/2}x}{\sqrt{2}}\right) G\left(\frac{y-\lambda^{-1/2}x}{\sqrt{2}}\right) m(dx) \\
&= \int_{C_0[0,T]} \int_{L_2[0,T]} \exp\left\{\frac{i}{\sqrt{2}} \int_0^T v(t)\,d[y(t)+\lambda^{-1/2}x(t)]\right\} df(v) \\
&\quad \cdot \int_{L_2[0,T]} \exp\left\{\frac{i}{\sqrt{2}} \int_0^T w(t)\,d[y(t)-\lambda^{-1/2}x(t)]\right\} dg(w)\,m(dx) \\
&= \int_{L_2^2[0,T]} \exp\left\{\frac{i}{\sqrt{2}} \int_0^T [v(t)+w(t)]\,dy(t)\right\} \\
&\quad \cdot \int_{C_0[0,T]} \exp\left\{\frac{i}{\sqrt{2\lambda}} \int_0^T [v(t)-w(t)]\,dx(t)\right\} m(dx)\,df(v)\,dg(w) \\
&= \int_{L_2^2[0,T]} \exp\left\{\frac{i}{\sqrt{2}} \int_0^T [v(t)+w(t)]\,dy(t)\right\} \\
&\quad \cdot \exp\left\{-\frac{1}{4\lambda} \int_0^T [v(t)-w(t)]^2\,dt\right\} df(v)\,dg(w).
\end{aligned} \tag{3.6}$$

But the last expression is an analytic function of λ throughout $\mathbb{C}_+$, and is a bounded continuous function of λ on $\tilde{\mathbb{C}}_+$ for all y in $C_0[0,T]$ since f and g are finite Borel measures. Hence $(F*G)_q$ exists and is given by (3.5) for all $q \in \mathbb{R}-\{0\}$. □

Our next theorem shows that the Fourier–Feynman transform of the convolution product is a product of transforms.

THEOREM 3.3. *Let F, G, f, and g be as in Theorem 3.2. Then, for all $q \in \mathbb{R}-\{0\}$,*

$$(T_q^{(p)}(F*G)_q)(z) = (T_q^{(p)}(F))(z/\sqrt{2})(T_q^{(p)}(G))(z/\sqrt{2}) \tag{3.7}$$

for $1 \le p \le 2$.

Proof. We first show that

$$(T_\lambda(F*G)_\lambda)(z) = (T_\lambda(F))(z/\sqrt{2})(T_\lambda(G))(z/\sqrt{2}) \tag{3.8}$$

holds for $\lambda > 0$. For $\lambda > 0$, using (2.1) and (2.6) we see that

$$\begin{aligned}
(T_\lambda(F*G)_\lambda)(z) &= \int_{C_0[0,T]}^{\mathrm{anw}_\lambda} (F*G)_\lambda(z+y)\,m(dy) \\
&= \int_{C_0^2[0,T]}^{\mathrm{anw}_\lambda} F\left(\frac{z+y+x}{\sqrt{2}}\right) G\left(\frac{z+y-x}{\sqrt{2}}\right) m(dx)\,m(dy).
\end{aligned}$$

But $w_1=(y+x)/\sqrt{2}$ and $w_2=(y-x)/\sqrt{2}$ are independent standard Wiener processes and hence, for all $\lambda>0$,

$$\begin{aligned}(T_\lambda(F*G)_\lambda)(z) &= \int_{C_0^2[0,T]}^{\mathrm{anw}_\lambda} F(w_1+z/\sqrt{2})G(w_2+z/\sqrt{2})m(dw_1)m(dw_2)\\ &= \int_{C_0[0,T]}^{\mathrm{anw}_\lambda} F(w_1+z/\sqrt{2})m(dw_1)\int_{C_0[0,T]}^{\mathrm{anw}_\lambda} G(w_2+z/\sqrt{2})m(dw_2)\\ &= (T_\lambda(F))(z/\sqrt{2})(T_\lambda(G))(z/\sqrt{2}).\end{aligned}$$

But both expressions on the right-hand side of equation (3.8) are analytic functions of λ throughout $\mathbb{C}_+$, and are bounded continuous functions of λ on $\tilde{\mathbb{C}}_+$ for all $z\in C_0[0,T]$. Hence $T_q^{(p)}(F*G)_q$ exists and is given by equation (3.7) for all desired values of p and q. □

In our next theorem we establish an interesting Parseval's identity for functionals F and G in the Banach algebra $\mathcal{S}$.

THEOREM 3.4. *Let F and G be as in Theorem 3.2. Then, for all $q\in\mathbb{R}-\{0\}$, the Parseval's identity*

$$\begin{aligned}&\int_{C_0[0,T]}^{\mathrm{anf}_{-q}} (T_q^{(p)}(F*G)_q)(z)m(dz)\\ &\quad\equiv \int_{C_0[0,T]}^{\mathrm{anf}_{-q}} (T_q^{(p)}(F))(z/\sqrt{2})(T_q^{(p)}(G))(z/\sqrt{2})m(dz)\\ &\quad= \int_{C_0[0,T]}^{\mathrm{anf}_{q}} F(z/\sqrt{2})G(-z/\sqrt{2})m(dz) \qquad (3.9)\end{aligned}$$

holds for $1\le p\le 2$.

Proof. Fix p and q. Then, for $\lambda>0$, using (3.7) and (3.2) we obtain

$$\begin{aligned}&\int_{C_0[0,T]} (T_q^{(p)}(F*G)_q)(z/\sqrt{\lambda})m(dz)\\ &\quad= \int_{C_0[0,T]}\int_{L_2[0,T]} \exp\left\{\frac{i}{\sqrt{2\lambda}}\int_0^T v(t)\,dz(t)-\frac{i}{2q}\int_0^T v^2(t)\,dt\right\}df(v)\\ &\qquad\cdot\int_{L_2[0,T]} \exp\left\{\frac{i}{\sqrt{2\lambda}}\int_0^T w(t)\,dz(t)-\frac{i}{2q}\int_0^T w^2(t)\,dt\right\}dg(w)m(dz)\\ &\quad= \int_{L_2^2[0,T]} \exp\left\{-\frac{i}{2q}\int_0^T [v^2(t)+w^2(t)]\,dt\right.\\ &\qquad\qquad \left.-\frac{1}{4\lambda}\int_0^T [v(t)+w(t)]^2\,dt\right\}df(v)\,dg(w).\end{aligned}$$

But the last expression is a continuous function of λ on $\tilde{\mathbb{C}}_+$, and so letting $\lambda=-(-q)i=qi$ we obtain

$$\int_{C_0[0,T]}^{\text{anf}_{-q}} (T_q^{(p)}(F*G)_q)(z)m(dz)$$
$$= \int_{L_2^2[0,T]} \exp\left\{-\frac{i}{4q}\int_0^T [v(t)-w(t)]^2\,dt\right\} df(v)\,dg(w).$$

On the other hand, for $\lambda > 0$,

$$\int_{C_0[0,T]} F(z/\sqrt{2\lambda})G(-z/\sqrt{2\lambda})m(dz)$$
$$= \int_{C_0[0,T]}\int_{L_2^2[0,T]} \exp\left\{\frac{i}{\sqrt{2\lambda}}\int_0^T [v(t)-w(t)]\,dz(t)\right\} df(v)\,dg(w)m(dz)$$
$$= \int_{L_2^2[0,T]} \exp\left\{-\frac{1}{4\lambda}\int_0^T [v(t)-w(t)]^2\,dt\right\} df(v)\,dg(w).$$

But the last expression is a continuous function of λ on $\mathbb{C}_+^\sim$, and so letting $\lambda = -qi$ we obtain that

$$\int_{C_0[0,T]}^{\text{anf}_q} F(z/\sqrt{2})G(-z/\sqrt{2})m(dz)$$
$$= \int_{L_2^2[0,T]} \exp\left\{-\frac{i}{4q}\int_0^T [v(t)-w(t)]^2\,dt\right\} df(v)\,dg(w)$$
$$= \int_{C_0[0,T]}^{\text{anf}_{-q}} (T_q^{(p)}(F*G)_q)(z)m(dz). \qquad \square$$

The following corollary follows immediately from equations (3.9) and (3.7) by choosing $G \equiv F$ for (i) and $G \equiv 1$ for (ii) below.

COROLLARY 3.1. *For $F \in \mathcal{S}$,*

(i) $$\int_{C_0[0,T]}^{\text{anf}_{-q}} [(T_q^{(p}(F))(z/\sqrt{2})]^2 m(dz) = \int_{C_0[0,T]}^{\text{anf}_q} F(z/\sqrt{2})F(-z/\sqrt{2})m(dz)$$

and

(ii) $$\int_{C_0[0,T]}^{\text{anf}_{-q}} (T_q^{(p)}(F))(z/\sqrt{2})m(dz) = \int_{C_0[0,T]}^{\text{anf}_q} F(z/\sqrt{2})m(dz).$$

REMARK. An interesting alternative form of Parseval's identity is

$$\int_{C_0[0,T]}^{\text{anf}_{-q}} (T_{q/2}^{(p)}(F))(z)(T_{q/2}^{(p)}(G))(z)m(dz) = \int_{C_0[0,T]}^{\text{anf}_q} F(z)G(-z)m(dz). \quad (3.9\text{a})$$

However, it is not true that

$$\int_{C_0[0,T]}^{\text{anf}_{-q}} (T_q^{(p)}(F))(z)(T_q^{(p)}(G))(z)m(dz) = \int_{C_0[0,T]}^{\text{anf}_q} F(z)G(-z)m(dz).$$

Next, for $F \in \mathcal{S}$, we obtain an inverse transform theorem.

THEOREM 3.5. *Let $F \in \mathcal{S}$ be given by* (3.1). *Then $T_{-q}^{(p)}(T_q^{(p)}(F)) \approx F$ for all $q \in \mathbb{R} - \{0\}$.*

Proof. First, proceeding as in the proof of Theorem 3.4, for $\lambda > 0$ calculate $T_\lambda(T_q^{(p)}(F))$. Then extend analytically in λ to $\mathbb{C}_+$ and finally let $\lambda \to -(-qi) = qi$ through values in $\mathbb{C}_+$ to obtain $T_{-q}^{(p)}(T_q^{(p)}(F)) \approx F$. □

Next we exhibit some classes of functionals on $C_0[0,T]$ of the form (1.1), or the n-dimensional version of (1.1). Every functional in each of these classes is known to belong to the Banach algebra $\mathcal{S}$ by Corollaries 3.2, 3.3, and 3.4 of [14].

Class A_1: Let A_1 denote the class of all functionals of the form

$$F(x) = \exp\left\{-\int_0^T \int_0^T \langle A(t_1,t_2)(x(t_1),x(t_2)),(x(t_1),x(t_2))\rangle \, dt_1\, dt_2\right\} \tag{3.10}$$

for s-a.e. x in $C_0[0,T]$, where $\{A(t_1,t_2) = (a_{ij}(t_1,t_2)) : (t_1,t_2) \in [0,T]^2\}$ is a commutative family of 2×2 real, symmetric, nonnegative definite matrices with the functions $a_{ij}(t_1,t_2)$ all in $L_1([0,T]^2)$. Note that A_1 contains the functionals of the form (3.10) with $A(t_1,t_2) = g(t_1,t_2)A$, where $g(t_1,t_2) \geq 0$ is in $L_1([0,T]^2)$ and A is a real, constant, symmetric, nonnegative definite matrix.

Class A_2: Let $\{A(t_1,t_2)\}$ be as in class A_1. Let η be a Borel measure on $[0,T]^2$. Let $\theta : [0,T]^2 \times \mathbb{R}^2 \to \mathbb{C}$ be such that, for all $(t_1,t_2) \in [0,T]^2$,

$$\theta(t_1,t_2,u_1,u_2) = \int_{\mathbb{R}^2} \exp\{iu_1v_1 + iu_2v_2\}\, d\sigma_{(t_1,t_2)}(v_1,v_2)$$

where (i) $\sigma_{(t_1,t_2)} \in M(\mathbb{R}^2)$; (ii) for all $E \in \mathcal{B}(\mathbb{R}^2)$, $\sigma_{(t_1,t_2)}(E)$ is a Borel measurable function of (t_1,t_2); and (iii) $\|\sigma_{(t_1,t_2)}\| \in L_1([0,T]^2, \mathcal{B}([0,T]^2), \eta)$. Then A_2 is the class of all functionals of the form

$$\begin{aligned} F(x) = \exp&\left\{-\int_0^T \int_0^T \langle A(t_1,t_2)(x(t_1),x(t_2)),(x(t_1),x(t_2))\rangle \, dt_1\, dt_2\right\} \\ &\cdot \exp\left\{\int_0^T \int_0^T \theta(t_1,t_2,x(t_1),x(t_2))\, d\eta(t_1,t_2)\right\}. \end{aligned} \tag{3.11}$$

Class A_3: Let $n \geq 2$ be a fixed positive integer. A_3 is the class of all functionals of the form

$$F(x) = \exp\left\{-\int_{[0,T]^n} \langle A(t_1,\dots,t_n)(x(t_1),\dots,x(t_n)),(x(t_1),\dots,x(t_n))\rangle \, d\vec{t}\right\} \tag{3.12}$$

for s-a.e. x in $C_0[0,T]$, where

$$\{A(t_1,\dots,t_n) = (a_{ij}(t_1,\dots,t_n)) : i,j = 1,2,\dots,n,\ \vec{t} = (t_1,\dots,t_n) \in [0,T]^n\}$$

is a commutative family of $n \times n$ real, symmetric nonnegative definite matrices with the functions $a_{ij}(t_1,\dots,t_n)$ all in $L_1([0,T]^n)$.

Class A_4: Let η be a finite Borel measure on $[0,T]^n$. Let A_4 be the class of all functionals of the form

$$F(x)=\exp\left\{-\int_{[0,T]^n}\langle A(t_1,\ldots,t_n)(x(t_1),\ldots,x(t_n)),(x(t_1),\ldots,x(t_n))\rangle\,d\vec{t}\right\}$$
$$\cdot\exp\left\{\int_{[0,T]^n}\theta(t_1,\ldots,t_n,x(t_1),\ldots,x(t_n))\,d\eta(t_1,\ldots,t_n)\right\} \tag{3.13}$$

for s-a.e. x in $C_0[0,T]$, where $\{A(t_1,\ldots,t_n)\}$ is as in class A_3 and

$$\theta(t_1,\ldots,t_n,u_1,\ldots,u_n)=\int_{\mathbb{R}^n}\exp\left\{i\sum_{j=1}^n u_jv_j\right\}d\sigma_{(t_1,\ldots,t_n)}(v_1,\ldots,v_n),$$

where (i) $\sigma_{(t_1,\ldots,t_n)}\in M(\mathbb{R}^n)$; (ii) $\sigma_{(t_1,\ldots,t_n)}(E)$ is a Borel measurable function of $(t_1,\ldots,t_n)$ for all $E\in\mathcal{B}(\mathbb{R}^n)$; and (iii) $\|\sigma_{(t_1,\ldots,t_n)}\|\in L_1([0,T]^n,\mathcal{B}([0,T]^n),\eta)$.

THEOREM 3.6. *Let $F,G\in\bigcup_{j=1}^4 A_j$. Then, for $q\in\mathbb{R}-\{0\}$,*

$$(T_q^{(p)}(F*G)_q)(z)=(T_q^{(p)}(F))(z/\sqrt{2})(T_q^{(p)}(G))(z/\sqrt{2}) \tag{3.14}$$

and

$$\int_{C_0[0,T]}^{\mathrm{anf}_{-q}}(T_q^{(p)}(F*G)_q)(z)m(dz)=\int_{C_0[0,T]}^{\mathrm{anf}_q}F(z/\sqrt{2})G(-z/\sqrt{2})m(dz) \tag{3.15}$$

for $1\le p\le 2$.

Proof. We simply note that $F,G\in\bigcup_{j=1}^4 A_j$ implies that F and G belong to $\mathcal{S}$; then we can use Theorems 3.3 and 3.4. □

4. Transforms and Convolution of Functionals in $\mathcal{A}$

First we describe the class $\mathcal{A}$ of functionals that we will be considering in this section. Let $L_{1\infty}([0,T]^2\times\mathbb{R}^2)$ be the space of all $\mathbb{C}$-valued Lebesgue measurable functions f on $[0,T]^2\times\mathbb{R}^2$ such that $f(s,t,\cdot,\cdot)$ is in $L_1(\mathbb{R}^2)$ for almost all $(s,t)\in[0,T]^2$, and as a function of s and t, $\|f(s,t,\cdot,\cdot)\|_1$ is in $L_\infty([0,T]^2)$. We define $\mathcal{A}$ to be the class of all functionals F such that, for some $f\in L_{1\infty}([0,T]^2\times\mathbb{R}^2)$,

$$F(x)=\exp\left\{\int_0^T\int_0^T f(s,t,x(s),x(t))\,ds\,dt\right\} \tag{4.1}$$

for s-a.e. x in $C_0[0,T]$. Functionals F in the class $\mathcal{A}$ were studied by Cameron and Storvick in [2] and by Johnson and Skoug in [11].

In our first theorem, for $F\in\mathcal{A}$ we obtain a series expansion for $T_\lambda(F)$ for $\lambda\in\mathbb{C}_+$.

THEOREM 4.1. *Let $F\in\mathcal{A}$ be given by* (4.1) *with $\|f(s,t,\cdot,\cdot)\|_1\le B$ for almost all $(s,t)\in[0,T]^2$. Then, for all $\lambda\in\mathbb{C}_+$,*

$$(T_\lambda(F))(y)$$
$$= 1+\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{P_n}\int_{\Delta_{2n}(T)}\int_{\mathbb{R}^{2n}}\left[\prod_{j=1}^{2n}\left(\frac{\lambda}{2\pi(r_j-r_{j-1})}\right)^{1/2}\exp\left\{-\frac{\lambda(u_j-u_{j-1})^2}{2(r_j-r_{j-1})}\right\}\right]$$
$$\cdot\left[\prod_{j=1}^{n} f(r_{m_j}, r_{k_j}, u_{m_j}+y(r_{m_j}), u_{k_j}+y(r_{k_j}))\right] d\vec{u}\, d\vec{r}, \tag{4.2}$$

where

$$\Delta_{2n}(T) = \{\vec{r} = (r_1, \ldots, r_{2n}) \in [0,T]^{2n}: 0 < r_1 < \cdots < r_{2n} \le T\},$$

P_n *is the set of all* $(2n)!$ *permutations of the set of integers* $\{1, 2, \ldots, 2n\}$, $r_0 = 0$, *and* $u_0 = 0$.

Proof. We first note that, for $F \in \mathfrak{A}$ of the form (4.1), we have the series expansion

$$F(x) = 1+\sum_{n=1}^{\infty}\frac{1}{n!}\left[\int_0^T\int_0^T f(s,t,x(s),x(t))\,ds\,dt\right]^n$$
$$= 1+\sum_{n=1}^{\infty}\frac{1}{n!}\int_{[0,T]^{2n}}\left[\prod_{j=1}^{n} f(s_j, t_j, x(s_j), w(t_j))\right] d\vec{s}\,d\vec{t}. \tag{4.3}$$

Hence, for $\lambda > 0$ we see that

$$(T_\lambda(F))(y) = \int_{C_0[0,T]} F(\lambda^{-1/2}x+y)\,m(dx)$$
$$= 1+\sum_{n=1}^{\infty}\int_{[0,T]^{2n}}\int_{C_0[0,T]}$$
$$\cdot\left[\prod_{j=1}^{n} f(s_j, t_j, \lambda^{-1/2}x(s_j)+y(s_j), \lambda^{-1/2}x(t_j)+y(t_j))\right] m(dx)\,d\vec{s}\,d\vec{t}.$$

Now let $r_1, r_2, \ldots, r_{2n}$ be the set of numbers $s_1, \ldots, s_n, t_1, \ldots, t_n$ in some rearrangement and let P_n be the set of all permutations of $\{1, 2, \ldots, 2n\}$. Now if we set $s_j = r_{m_j}$ and $t_j = r_{k_j}$ and evaluate the foregoing Wiener integrals, we obtain the series expansion on the right-hand side of (4.2) and thus equation (4.2) holds for $\lambda > 0$. But $\lambda \in \mathbb{C}_+$ implies that $\operatorname{Re}\lambda > 0$ and so the right-hand side of (4.2) is an analytic function of λ throughout $\mathbb{C}_+$ for each $y \in C_0[0,T]$. □

We will use our next lemma to help obtain a bound on the series in (4.2).

LEMMA 4.1. *Let*

$$\Delta_{2n}(T) = \{\vec{r} = (r_1, \ldots, r_{2n}) \in [0,T]^{2n}: 0 < r_1 < \cdots < r_{2n} \le T\}.$$

Then

$$\int_{\Delta_{2n}(T)}\prod_{j=1}^{2n}[r_j - r_{j-1}]^{-1/2}\,d\vec{r} = \frac{(\pi T)^n}{n!}. \tag{4.4}$$

Proof. We first note that

$$\int_{\Delta_{2n}(T)} \prod_{j=1}^{2n}[r_j-r_{j-1}]^{-1/2}\,d\vec{r}$$
$$=\int_0^T\int_0^{r_{2n}}\int_0^{r_{2n-1}}\cdots\int_0^{r_4}\int_0^{r_3}\int_0^{r_2}$$
$$\cdot[r_1(r_2-r_1)(r_3-r_2)\cdots(r_{2n}-r_{2n-1})]^{-1/2}\,dr_1\,dr_2\cdots dr_{2n}.$$

Making the substitution $v=r_1/r_2$, we see that

$$\int_0^{r_2}[r_1(r_2-r_1)]^{-1/2}\,dr_1=\int_0^1[v(1-v)]^{-1/2}\,dv=\beta\Big(\frac{1}{2},\frac{1}{2}\Big),$$

where β denotes the beta function. At the next state we make the substitution $v=r_2/r_3$ and obtain

$$\int_0^{r_3}\int_0^{r_2}[r_1(r_2-r_1)(r_3-r_2)]^{-1/2}\,dr_1\,dr_2=\beta\Big(\frac{1}{2},\frac{1}{2}\Big)\int_0^1 r_3^{1/2}(1-v)^{-1/2}\,dv$$
$$=\beta\Big(\frac{1}{2},\frac{1}{2}\Big)\beta\Big(1,\frac{1}{2}\Big)(r_3)^{1/2}.$$

At the next stage we let $v=r_3/r_4$, which yields

$$\int_0^{r_4}\int_0^{r_3}\int_0^{r_2}[r_1(r_2-r_1)(r_3-r_2)(r_4-r_3)]^{-1/2}\,dr_1\,dr_2\,dr_3$$
$$=\beta\Big(\frac{1}{2},\frac{1}{2}\Big)\beta\Big(1,\frac{1}{2}\Big)\beta\Big(\frac{3}{2},\frac{1}{2}\Big)r_4.$$

Continuing on for $2n-1$ stages, we obtain

$$\int_{\Delta_{2n}(T)}\prod_{j=1}^{2n}[r_j-r_{j-1}]^{-1/2}\,d\vec{r}=\prod_{j=1}^{2n-1}\beta\Big(\frac{j}{2},\frac{1}{2}\Big)\int_0^T r_{2n}^{n-1}\,dr_{2n}$$
$$=\frac{T^n}{n}\prod_{j=1}^{2n-1}\beta\Big(\frac{j}{2},\frac{1}{2}\Big)=\frac{(\pi T)^n}{n!},$$

since $\Gamma(\frac{1}{2})=\sqrt{\pi}$, $\beta(v,w)=[\Gamma(v)\Gamma(w)]/\Gamma(v+w)$, and $\Gamma(n)=(n-1)!$. □

REMARK. Similar calculations were carried out in [2; 11].

THEOREM 4.2. *Let $F\in\mathfrak{A}$ be as in Theorem 4.1. Then, for $1\le p\le 2$, $T_q^{(p)}(F)$ exists for all $q\in\mathbb{R}-\{0\}$ such that $|q|<1/2BT$ and is given by*

$$(T_q^{(p)}(F))(y)$$
$$=1+\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{P_n}\int_{\Delta_{2n}(T)}\int_{\mathbb{R}^{2n}}\Bigg[\prod_{j=1}^{2n}\Big(\frac{-qi}{2\pi(r_j-r_{j-1})}\Big)^{1/2}\exp\Big\{-\frac{qi(u_j-u_{j-1})^2}{2(r_j-r_{j-1})}\Big\}\Bigg]$$
$$\cdot\Bigg[\prod_{j=1}^{n}f(r_{m_j},r_{k_j},u_{m_j}+y(r_{m_j}),u_{k_j}+y(r_{k_j}))\Bigg]\,d\vec{u}\,d\vec{r}. \quad (4.5)$$

Proof. For all $(y,\lambda)\in C_0[0,T]\times\{\lambda\in\mathbb{C}_+^{\sim}:|\lambda|<1/2BT\}$, the series on the right-hand side of (4.2) is dominated by the series

$$1+\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{P_n}\int_{\Delta_{2n}(T)}\int_{\mathbb{R}^{2n}}\left[\prod_{j=1}^{2n}\left(\frac{|\lambda|}{2\pi(r_j-r_{j-1})}\right)^{1/2}\right]$$
$$\cdot\left[\prod_{j=1}^{n}|f(r_{m_j},r_{k_j},u_{m_j}+y(r_{m_j}),u_{k_j}+y(r_{k_j}))|\right]d\vec{u}\,d\vec{r}$$
$$=1+\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{P_n}\int_{\Delta_{2n}(T)}\int_{\mathbb{R}^{2n}}\left[\prod_{j=1}^{2n}\left(\frac{|\lambda|}{2\pi(r_j-r_{j-1})}\right)^{1/2}\right]$$
$$\cdot\left[\prod_{j=1}^{n}|f(r_{m_j},r_{k_j},u_{m_j},u_{k_j})|\right]d\vec{u}\,d\vec{r}$$
$$\le 1+\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{P_n}\int_{\Delta_{2n}(T)}\left[\prod_{j=1}^{2n}\left(\frac{|\lambda|}{2\pi(r_j-r_{j-1})}\right)^{1/2}\right]\left[\prod_{j=1}^{n}\|f(r_{m_j},r_{k_j},\cdot,\cdot)\|_1\right]d\vec{r}$$
$$\le 1+\sum_{n=1}^{\infty}\frac{B^n}{n!}\sum_{P_n}\int_{\Delta_{2n}(T)}\left[\prod_{j=1}^{2n}\left(\frac{|\lambda|}{2\pi(r_j-r_{j-1})}\right)^{1/2}\right]d\vec{r}$$
$$=1+\sum_{n=1}^{\infty}\frac{B^n(2n)!}{n!}\int_{\Delta_{2n}(T)}\left[\prod_{j=1}^{2n}\left(\frac{|\lambda|}{2\pi(r_j-r_{j-1})}\right)^{1/2}\right]d\vec{r}$$
$$\le 1+\sum_{n=1}^{\infty}\frac{B^n(2n)!\,|\lambda|^n}{n!\,(2\pi)^n}\frac{(\pi T)^n}{n!}$$
$$=1+\sum_{n=1}^{\infty}\frac{(2BT|\lambda|)^n(2n)!}{n!\,n!\,2^{2n}},$$

where we used Lemma 4.1 in the next-to-last step. Next we use Stirling's formula

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n < n! < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left[1+\frac{1}{12n-1}\right]$$

to obtain

$$\frac{(2n!)}{n!\,n!\,2^{2n}} < \frac{\sqrt{4\pi n}(2n/e)^{2n}}{2\pi n(n/e)^{2n}2^{2n}}\left[1+\frac{1}{24n-1}\right] < \frac{2}{\sqrt{\pi n}}.$$

Thus, the above series is dominated by the series

$$1+2\sum_{n=1}^{\infty}\frac{(2BT|\lambda|)^n}{\sqrt{\pi n}},$$

which converges by the ratio test for $|\lambda|<1/2BT$. Hence the series on the right-hand side of (4.2) converges absolutely and uniformly for all $(y,\lambda)\in C_0[0,T]\times\{\lambda\in\mathbb{C}_+^{\sim}:|\lambda|<1/2BT\}$, and so $T_q^{(p)}(F)$ exists and is given by equation (4.5). □

THEOREM 4.3. *Let F be as in Theorem 4.1, and let $G\in\mathfrak{A}$ be given by*

$$G(x)=\exp\left\{\int_0^T\int_0^T g(s,t,x(s),x(t))\,ds\,dt\right\}, \tag{4.6}$$

with $\|g(s,t,\cdot,\cdot)\|_1\le B$ for a.e. $(s,t)\in[0,T]^2$. Then their convolution product $(F*G)_\lambda$ exists for all $\lambda\in\mathbb{C}_+$ and is given by

$$
\begin{aligned}
&(F*G)_\lambda(y)\\
&=1+\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{P_n}\int_{\Delta_{2n}(T)}\int_{\mathbb{R}^{2n}}\left[\prod_{j=1}^{2n}\left(\frac{\lambda}{2\pi(r_j-r_{j-1})}\right)^{1/2}\exp\left\{\frac{-\lambda(u_j-u_{j-1})^2}{2(r_j-r_{j-1})}\right\}\right]\\
&\quad\cdot\prod_{j=1}^{n}\left[f\left(r_{m_j},r_{k_j},\frac{y(r_{m_j})+u_{m_j}}{\sqrt{2}},\frac{y(r_{k_j})+u_{k_j}}{\sqrt{2}}\right)\right.\\
&\qquad\left.+g\left(r_{m_j},r_{k_j},\frac{y(r_{m_j})-u_{m_j}}{\sqrt{2}},\frac{y(r_{k_j})-u_{k_j}}{\sqrt{2}}\right)\right]d\vec{u}\,d\vec{r}. \qquad (4.7)
\end{aligned}
$$

Proof. By definition of $(F*G)_\lambda$, for $\lambda>0$ we have

$$
\begin{aligned}
&(F*G)_\lambda(y)\\
&=\int_{C_0[0,T]}F\left(\frac{y+\lambda^{-1/2}x}{\sqrt{2}}\right)G\left(\frac{y-\lambda^{-1/2}x}{\sqrt{2}}\right)m(dx)\\
&=\int_{C_0[0,T]}\exp\left\{\int_0^T\int_0^T\left[f\left(s,t,\frac{y(s)+\lambda^{-1/2}x(s)}{\sqrt{2}},\frac{y(t)+\lambda^{-1/2}x(t)}{\sqrt{2}}\right)\right.\right.\\
&\qquad\left.\left.+g\left(s,t,\frac{y(s)-\lambda^{-1/2}x(s)}{\sqrt{2}},\frac{y(t)-\lambda^{-1/2}x(t)}{\sqrt{2}}\right)\right]ds\,dt\right\}m(dx)\\
&=1+\sum_{n=1}^{\infty}\frac{1}{n!}\int_{[0,T]^{2n}}\int_{C_0[0,T]}\\
&\quad\cdot\prod_{j=1}^{n}\left[f\left(s_j,t_j,\frac{y(s_j)+\lambda^{-1/2}x(s_j)}{\sqrt{2}},\frac{y(t_j)+\lambda^{-1/2}x(t_j)}{\sqrt{2}}\right)\right.\\
&\qquad\left.+g\left(s_j,t_j,\frac{y(s_j)-\lambda^{-1/2}x(s_j)}{\sqrt{2}},\frac{y(t_j)-\lambda^{-1/2}x(t_j)}{\sqrt{2}}\right)\right]m(dx)\,d\vec{s}\,d\vec{t}.
\end{aligned}
$$

Then, letting $r_1,\dots,r_{2n}$ be the set of numbers $s_1,\dots,s_n,t_1,\dots,t_n$ in some rearrangement and P_n the set of all permutations of $\{1,2,\dots,2n\}$ and evaluating the Wiener integrals as in the proof of Theorem 4.1, we obtain (4.7) for $\lambda>0$. But $\lambda\in\mathbb{C}_+$ implies that $\operatorname{Re}\lambda>0$ and so the right-hand side of (4.7) is an analytic function of λ for each $y\in C_0[0,T]$. □

THEOREM 4.4. *Let F and G be as in Theorem 4.3. Then, for all $\lambda\in\mathbb{C}_+$, $T_\lambda(F*G)_\lambda$ exists and is given by*

$$
\begin{aligned}
(T_\lambda(F*G)_\lambda)(z)&=1+\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{P_n}\int_{\Delta_{2n}(T)}\int_{\mathbb{R}^{4n}}\prod_{j=1}^{2n}\left(\frac{\lambda}{2\pi(r_j-r_{j-1})}\right)\\
&\quad\cdot\exp\left\{-\frac{\lambda}{2}\sum_{j=1}^{2n}\frac{[(v_j-v_{j-1})^2+(l_j-l_{j-1})^2]}{r_j-r_{j-1}}\right\}\\
&\quad\cdot\prod_{j=1}^{n}\left[f\left(r_{m_j},r_{k_j},\frac{z(r_{m_j})}{\sqrt{2}}+v_{m_j},\frac{z(r_{k_j})}{\sqrt{2}}+v_{k_j}\right)\right.\\
&\qquad\left.+g\left(r_{m_j},r_{k_j},\frac{z(r_{m_j})}{\sqrt{2}}+l_{m_j},\frac{z(r_{k_j})}{\sqrt{2}}+l_{k_j}\right)\right]d\vec{v}\,d\vec{l}\,d\vec{r}. \qquad (4.8)
\end{aligned}
$$

Proof. For $\lambda > 0$, using (4.7) we obtain

$$
\begin{aligned}
(T_\lambda(F*G)_\lambda)(z) &= \int_{C_0[0,T]} (F*G)_\lambda(\lambda^{-1/2}x+z)m(dx) \\
&= 1+\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{P_n}\int_{\Delta_{2n}(T)}\int_{\mathbb{R}^{4n}} \\
&\quad \cdot\left[\prod_{j=1}^{2n}\left(\frac{\lambda}{2\pi(r_j-r_{j-1})}\right)\exp\left\{\frac{-\lambda(u_j-u_{j-1})^2-\lambda(w_j-w_{j-1})^2}{2(r_j-r_{j-1})}\right\}\right] \\
&\quad \cdot\prod_{j=1}^{n}\left[f\left(r_{m_j}, r_{k_j}, \frac{z(r_{m_j})+u_{m_j}+w_{m_j}}{\sqrt{2}}, \frac{z(r_{k_j})+u_{k_j}+w_{k_j}}{\sqrt{2}}\right)\right. \\
&\quad \left. +g\left(r_{m_j}, r_{k_j}, \frac{z(r_{m_j})+w_{m_j}-u_{m_j}}{\sqrt{2}}, \frac{z(r_{k_j})+w_{k_j}-u_{k_j}}{\sqrt{2}}\right)\right] d\vec{u}\, d\vec{w}\, d\vec{r}.
\end{aligned}
$$

Next, in the above expression we make the substitutions

$$v_j = \frac{w_j+u_j}{\sqrt{2}} \quad\text{and}\quad l_j = \frac{w_j-u_j}{\sqrt{2}}$$

for $j = 1, 2, \dots, 2n$. The Jacobian of this transformation is unity and for $j = 1, 2, \dots, 2n$ we have that

$$(u_j-u_{j-1})^2+(w_j-w_{j-1})^2 = (v_j-v_{j-1})^2+(l_j-l_{j-1})^2.$$

Hence $(T_\lambda(F*G)_\lambda)(z)$ is given by (4.8) for $\lambda > 0$. Again, by analytic continuation in λ, we obtain that equation (4.8) is valid throughout $\mathbb{C}_+$. □

Our next theorem shows that the Fourier–Feynman transform of the convolution product is the product of their transforms.

THEOREM 4.5. *Let F and G be as in Theorem 4.3. Then, for all $q \in \mathbb{R}-\{0\}$ such that $|q| < 1/2BT$,*

$$(T_q^{(p)}(F*G)_q)(z) = (T_q^{(p)}(F)(z/\sqrt{2})(T_q^{(p)}(G))(z/\sqrt{2}) \tag{4.9}$$

for $1 \le p \le 2$.

Proof. The proof used in establishing equation (3.8) shows that

$$(T_\lambda(F*G)_\lambda)(z) = (T_\lambda(F))(z/\sqrt{2})(T_\lambda(G))(z/\sqrt{2}) \tag{4.10}$$

holds for all $\lambda > 0$. But both sides of (4.10) are analytic functions of λ throughout $\mathbb{C}_+$ and so equation (4.10) is valid throughout $\mathbb{C}_+$. However, by Theorem 4.2, both of the expressions on the right-hand side of (4.10) are bounded continuous functions of λ on $\{\lambda \in \tilde{\mathbb{C}}_+ : |\lambda| < 1/2BT\}$ for all $z \in C_0[0,T]$. Hence $T_q^{(p)}(F*G)_q$ exists and is given by equation (4.9) for all $(p,q) \in [1,2]\times E$, where $E \equiv (-1/2BT, 0)\cup(0, 1/2BT)$. □

REMARK. An alternative (but much longer) method of establishing Theorem 4.5 is to show that the series expansion for $(T_\lambda(F*G)_\lambda)(z)$ is the Cauchy product of the series expansions for $(T_\lambda(F))(z/\sqrt{2})$ and $(T_\lambda(G))(z/\sqrt{2})$.

References

[1] M. D. Brue, *A functional transform for Feynman integrals similar to the Fourier transform,* thesis, Univ. of Minnesota, Minneapolis, 1972.

[2] R. H. Cameron and D. A. Storvick, *An operator valued function space integral applied to multiple integrals of functions of class L_1,* Nagoya Math. J. 51 (1973), 91-122.

[3] ———, *An L_2 analytic Fourier-Feynman transform,* Michigan Math. J. 23 (1976), 1-30.

[4] ———, *Some Banach algebras of analytic Feynman integrable functionals,* Analytic functions (Kozubnik, 1979), Lecture Notes in Math., 798, pp. 18-27, Springer, Berlin, 1980.

[5] K. S. Chang, *Scale-invariant measurability in Yeh-Wiener space,* J. Korean Math. Soc. 19 (1982), 61-67.

[6] K. S. Chang, G. W. Johnson, and D. L. Skoug, *The Feynman integral of quadratic potentials depending on two time variables,* Pacific J. Math. 122 (1986), 11-33.

[7] R. P. Feynman, *Space-time approach to non-relativistic quantum mechanics,* Rev. Mod. Phys. 20 (1948), 367-387.

[8] R. P. Feynman and A. R. Hibbs, *Quantum mechanics and path integrals,* McGraw-Hill, New York, 1965.

[9] T. Huffman, C. Park, and D. Skoug, *Analytic Fourier-Feynman transforms and convolution,* Trans. Amer. Math. Soc. 347 (1995), 661-673.

[10] ———, *Convolution and Fourier-Feynman transforms,* Rocky Mountain Math J. (to appear).

[11] G. W. Johnson and D. L. Skoug, *The Cameron-Storvick function space integral: an $L(L_p, L_{p'})$ theory,* Nagoya Math. J. 60 (1976), 93-137.

[12] ———, *An L_p analytic Fourier-Feynman transform,* Michigan Math. J. 26 (1979), 103-127.

[13] ———, *Scale-invariant measurability in Wiener space,* Pacific J. Math. 83 (1979), 157-176.

[14] C. Park and D. Skoug, *The Feynman integral of quadratic potentials depending on n time variables,* Nagoya Math. J. 110 (1988), 151-162.

[15] J. Yeh, *Convolution in Fourier-Wiener transform,* Pacific J. Math. 15 (1965), 731-738.

[16] I. Yoo, *Convolution in Fourier-Wiener transform on abstract Wiener space,* Rocky Mountain Math. J. 25 (1995), 1577-1587.

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