Boundary Limits of the Bergman Kernel and Metric

HAROLD P. BOAS, EMIL J. STRAUBE, & JIYE YU

1. Introduction and Results

Bergman computed, for certain special domains, the precise boundary behavior of the Bergman kernel function $K(z)$ on the diagonal and of the Bergman metric (see [3; 4]). Subsequently, Hörmander [25] showed for any bounded strongly pseudoconvex domain in $\mathbf{C}^n$ that the limit of $K(z)d(z)^{n+1}$ at the boundary (where $d(z)$ is the Euclidean distance from z to the boundary) equals the determinant of the Levi form times $n!/4\pi^n$. Diederich [10; 11] computed the boundary limit of the Bergman metric for strongly pseudoconvex domains. Later, Klembeck [29], using Fefferman's asymptotic expansion for the Bergman kernel function [16], found the boundary limit of the holomorphic sectional curvature of the Bergman metric in strongly pseudoconvex domains. (See [19] and [27] for more on the curvature of the Bergman metric.)

Although there has been considerable progress in estimating the size of the Bergman kernel and metric on weakly pseudoconvex domains of finite type (see e.g. [6; 13; 14; 21; 23; 24; 32; 34; 35; 36]), boundary limits in the sense of Hörmander's result are not well understood. Examples of Herbort [20; 22] show that in general the growth of the Bergman kernel function is not an algebraic function of the distance to the boundary. In this paper we show that the kernel function, weighted by a suitable power of the distance to the boundary, does have a nontangential limit for a large class of weakly pseudoconvex domains of finite type in $\mathbf{C}^n$. We also show the existence of limits for the Bergman metric and its holomorphic sectional curvature. Moreover, we evaluate these limits in terms of the corresponding Bergman invariants of an unbounded local model of the finite type domain; in favorable cases, these admit explicit computation.

Our method is based on Bergman's original approach of minimum integrals. The idea, following [43; 28], is first to localize the minimum integrals as in [25]; then to blow up the domain via dilations, in the spirit of the scaling method [2; 39]; and finally to observe that the minimum integrals of the dilates approach the minimum integrals of the local model. In the last step,

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an ad hoc technical stability result for the minimum integrals on *unbounded* domains is required.

The domains to which our technique applies (defined below) are known variously as h-extendible [42; 43; 44] and semiregular [14]. This class includes, for example, all bounded convex domains of finite type in $\mathbb{C}^n$ and all bounded pseudoconvex domains of finite type in dimension 2.

Note added February 1995. (1) Diederich and Herbort have kindly informed us of an alternative approach [15] that does not use our stability lemma; instead, localization is done differently (cf. [10; 12; 38]).

(2) Recently Krantz and Yu [31] derived representations for some other invariants, such as the Ricci curvature, in terms of (not necessarily monotone) domain functionals; their boundary limits were determined by methods similar to ours.

2. Definitions and Statements of Results

Throughout the paper Ω denotes a pseudoconvex domain, not necessarily bounded, in $\mathbb{C}^{n+1}$. The objects we shall study are: first, the Bergman kernel function $K_\Omega(z)$ on the diagonal, which is the sum $\sum_j |\varphi_j(z)|^2$ over an orthonormal basis $\{\varphi_j\}$ for the square-integrable holomorphic functions on Ω; second, the Bergman metric $B_\Omega(z,X)$ given by $(\sum_{j,k} g_{j\bar{k}} X_j \bar{X}_k)^{1/2}$, where

$$g_{j\bar{k}} = \frac{\partial^2 \log K_\Omega(z)}{\partial z_j \partial \bar{z}_k};$$

and third, the holomorphic sectional curvature $R_\Omega(z,X)$ of the Bergman metric, given by $B_\Omega(z,X)^{-4} \sum_{h,j,k,l} R_{\bar{h}jk\bar{l}} \bar{X}_h X_j X_k \bar{X}_l$, where

$$R_{\bar{h}jk\bar{l}} = -\frac{\partial^2 g_{j\bar{h}}}{\partial z_k \partial \bar{z}_l} + \sum_{\mu,\nu} g^{\nu\bar{\mu}} \frac{\partial g_{j\bar{\mu}}}{\partial z_k} \frac{\partial g_{\nu\bar{h}}}{\partial \bar{z}_l},$$

$g^{\nu\bar{\mu}}$ being (as usual) the inverse matrix to $g_{j\bar{k}}$. (With the most frequently used normalization, the holomorphic sectional curvature is actually $2R_\Omega(z,X)$.)

According to Catlin's theory [5] of *multitype,* there is associated to a smooth, finite-type boundary point p of Ω a certain biholomorphically invariant nondecreasing sequence of numbers $(m_0, m_1, \dots, m_n)$, with $m_0 = 1$, such that $m_{n-q+1} \le \Delta_q$ for $1 \le q \le n$, where Δ_q is the q-type of p in the sense of D'Angelo [8; 9] (roughly speaking, the maximal order of contact of q-dimensional varieties with the boundary of Ω at p). In suitable local coordinates $(z_0, z') = (z_0, z_1, \dots, z_n)$ in which $p = 0$, there exists a real-valued, plurisubharmonic, weighted homogeneous polynomial P with no pluriharmonic terms such that Ω is defined locally near p by

$$\operatorname{Re} z_0 + P(z') + o\left(\sum_{j=0}^{n} |z_j|^{m_j}\right) < 0.$$

By *weighted homogeneous* we mean that $P(\pi_t(z')) = tP(z')$, where π_t is the anisotropic dilation acting (on $\mathbb{C}^{n+1}$, or on the last n coordinates by restriction) via $\pi_t(z)=(tz_0, t^{1/m_1}z_1, \ldots, t^{1/m_n}z_n)$. The unbounded domain $D=\{(z_0,z'): \operatorname{Re} z_0+P(z')<0\}$ is a local *model* for Ω at p.

The domains we shall consider are the ones for which equality holds at p in the inequality between the components of the multitype and the D'Angelo type: $m_{n-q+1}=\Delta_q<\infty$ for $1\le q\le n$. When this condition holds, we say that Ω is *h-extendible* at p. It was proved independently in [14] (where the terminology "semiregular" is used instead of "h-extendible") and in [44] that Ω is h-extendible at p if and only if the local model D admits a *bumping function* $a(z')$ with the following properties:

(i) on $\mathbb{C}^n\setminus\{0\}$, the function a is C^∞ smooth and positive;
(ii) a is weighted homogeneous (with the same weights as for P); and
(iii) $P(z')-\epsilon a(z')$ is strictly plurisubharmonic on $\mathbb{C}^n\setminus\{0\}$ when $0<\epsilon\le 1$.

These conditions state that the model domain for Ω at p can be approximated from outside by the pseudoconvex domains $\{(z_0,z'): \operatorname{Re} z_0+P(z')-\epsilon a(z')<0\}$ having the same homogeneity as D. In other words, the local model is "homogeneously extendible" to a larger pseudoconvex domain—hence the terminology "h-extendible."

The class of h-extendible domains is quite large. It trivially contains the strictly pseudoconvex domains (since P serves as its own bumping function in this case). More generally, if the Levi form at the finite-type pseudoconvex point p has at most one zero eigenvalue, then p is an h-extendible point: the arguments of [2, pp. 168–169 and proof of Lemma 1, p. 171] or [13, proof of Lemma 2.1] reduce P to the form $\sum_{j=1}^{n-1}|z_j|^2+\Psi(z_n,\bar z_n)$, and Ψ can then be bumped by [37, Prop. 4.1]. In particular, every pseudoconvex domain of finite type in $\mathbb{C}^2$ is h-extendible—for instance, the famous Kohn–Nirenberg example [30] of a real-analytic pseudoconvex domain lacking a holomorphic support function is nonetheless h-extendible. For similar reasons, the so-called decoupled domains [33] of finite type in $\mathbb{C}^n$ are h-extendible. It was shown in [41] that *convex* domains of finite type in $\mathbb{C}^n$ are h-extendible. (It turns out that, in convex domains, it is enough to consider orders of contact with complex linear subspaces.)

Later we will use that h-extendible domains admit peaking functions, an important property recently shown in [14; 44]. In particular, h-extendible domains are complete in the Carathéodory and Kobayashi metrics.

By [43], we may choose local holomorphic coordinates near the h-extendible point p (which we always assume has been translated to the origin) such that near p the domain Ω is defined by the equation $r<0$ with

$$r(z_0,z')=\operatorname{Re} z_0+P(z')+O(\sigma(z')^{1+\alpha})+O((\operatorname{Im} z_0)^2),$$

where the polynomial P is as described above, α is some positive constant (which we may assume is less than 1), and $\sigma(z')=\sum_{j=1}^n|z_j|^{m_j}$. Rescaling the coordinates, if necessary, we may assume that at p both the Jacobian determinant of the transformation to local coordinates and the derivative of $\operatorname{Re} z_0$ with respect to the unit normal to the boundary of Ω are equal to 1. These normalized coordinates and the corresponding local model D are fixed from now on.

THEOREM 1. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^{n+1}$ that is h-extendible at the boundary point p with multitype $(m_0, m_1, \ldots, m_n)$ and local model D. If Γ is a nontangential cone in Ω with vertex at p, then*

$$\lim_{\substack{z\to p\\ z\in\Gamma}} K_\Omega(z)\, d(z)^{\sum_{j=0}^{n} 2/m_j} = K_D(\varpi),$$

where $d(z)$ is the distance from z to the boundary of Ω and $\varpi = (-1, 0, \ldots, 0)$.

For the Bergman metric and curvature, we must weight different directions appropriately. When X is a nonzero (tangent) vector, we set $\pi_{1/d(z)}(X) = (d(z)^{-1}X_0, d(z)^{-1/m_1}X_1, \ldots, d(z)^{-1/m_n}X_n)$ in the local coordinates described above, and we define the unit vector $\hat{X} = \lim_{z\to p} \pi_{1/d(z)}(X)/|\pi_{1/d(z)}(X)|$.

THEOREM 2. *Under the hypotheses of Theorem 1,*

$$\lim_{\substack{z\to p\\ z\in\Gamma}} \frac{B_\Omega(z,X)}{|\pi_{1/d(z)}(X)|} = B_D(\varpi,\hat{X}) \quad \textit{and} \quad \lim_{\substack{z\to p\\ z\in\Gamma}} R_\Omega(z,X) = R_D(\varpi,\hat{X}).$$

It is not a priori obvious that the unbounded model domains have nontrivial Bergman spaces, but we shall see in Section 3 that $K_D(\varpi)$ and $B_D(\varpi, \hat{X})$ are in fact positive. The theorems therefore imply, in particular, sharp bounds for the Bergman kernel and metric, so that we recover the growth exponents recently determined by Diederich and Herbort [14].

We emphasize that although the theorems require Ω to be globally pseudoconvex, it is only in a neighborhood of p that we need the boundary of Ω to be smooth and of finite type.

The hypothesis of nontangential approach cannot be removed in general. It is essential, for example, for the Bergman metric on the unit ball [18] and for the holomorphic sectional curvature on the domain $\{(z_0, z_1) \in \mathbb{C}^2 : |z_0|^2 + |z_1|^4 < 1\}$ [1].

The theorems evaluate the boundary limit as the value at an interior point of a model domain. This value can be computed explicitly if the model is simple. For example, if p is a strongly pseudoconvex point, then a biholomorphic image of the ball serves as a local model at p. Since the curvature is biholomorphically invariant, it follows that $\lim_{z\to p} 2R_\Omega(z, X) = -4/(n+2)$ for every direction X, since this is the curvature of the ball in $\mathbb{C}^{n+1}$. This result for strongly pseudoconvex points is due to Kim and Yu [28]; as mentioned above, it was proved earlier by Klembeck [29] under the assumption that the boundary is globally strongly pseudoconvex and globally C^∞ smooth. Another case in which the limits in the theorems admit explicit evaluation is the case of models with circular symmetry; see Section 5.

3. Minimum Integrals

We first recall how the Bergman invariants are related to certain minimum integrals; these facts can be found in [3; 4; 17; 26]. Then we will prove a stability lemma for the minimum integrals on unbounded model domains.

For the moment, Ω can be any domain in $\mathbb{C}^{n+1}$ with a nontrivial Bergman space $A^2(\Omega)$ of square-integrable holomorphic functions; we write $\|\cdot\|_\Omega$ for the norm in $L^2(\Omega)$. Let X be a (nonzero) vector in $\mathbb{C}^{n+1}$, and let ζ be a fixed point in Ω. We define the minimum integrals

$$I_\Omega^0(\zeta) = \inf\{\|f\|_\Omega^2 : f \in A^2(\Omega), f(\zeta) = 1\},$$

$$I_\Omega^1(\zeta, X) = \inf\left\{\|f\|_\Omega^2 : f \in A^2(\Omega), f(\zeta) = 0, \sum_{j=0}^{n} X_j \frac{\partial f}{\partial z_j}(\zeta) = 1\right\},$$

$$I_\Omega^2(\zeta, X) = \inf\left\{\|f\|_\Omega^2 : f \in A^2(\Omega), f(\zeta) = \frac{\partial f}{\partial z_0}(\zeta) = \cdots = \frac{\partial f}{\partial z_n}(\zeta) = 0, \right.$$
$$\left. \sum_{j,k=0}^{n} X_j X_k \frac{\partial^2 f}{\partial z_j \partial z_k}(\zeta) = 1\right\}.$$

These quantities vary smoothly with respect to the point ζ and the nonzero vector X, and

$$K_\Omega(\zeta) = \frac{1}{I_\Omega^0(\zeta)}, \qquad B_\Omega(\zeta, X) = \left(\frac{I_\Omega^0(\zeta)}{I_\Omega^1(\zeta, X)}\right)^{1/2},$$
$$R_\Omega(\zeta, X) = 2 - \frac{[I_\Omega^1(\zeta, X)]^2}{I_\Omega^0(\zeta) I_\Omega^2(\zeta, X)}.$$

It is easy to see from the definitions that the minimum integrals increase when the domain does, and under a biholomorphic mapping $F: \Omega_1 \to \Omega_2$ they transform according to

$$I_{\Omega_1}^0(\zeta)|\det J_{\mathbb{C}} F(\zeta)|^2 = I_{\Omega_2}^0(F(\zeta)),$$
$$I_{\Omega_1}^j(\zeta, X)|\det J_{\mathbb{C}} F(\zeta)|^2 = I_{\Omega_2}^j(F(\zeta), F_{*\zeta}(X)), \quad j = 1, 2,$$

where $J_{\mathbb{C}} F(\zeta)$ denotes the complex Jacobian matrix of F at ζ.

We shall need some stability results for the minimum integrals under perturbations of the domain. Small modifications of Ramadanov's argument [40] show that if domains Ω_j are all contained in a limit domain Ω, and if the Ω_j converge to Ω in the sense that every compact subset of Ω is eventually contained in Ω_j, then the minimum integrals for Ω_j converge to the integrals for Ω. Moreover, the convergence is uniform for the point ζ in a compact subset of Ω and for the vector X on the unit sphere.

If the Ω_j are not contained in Ω, then there is no corresponding theorem without some further hypotheses. We prove here an ad hoc result for unbounded model domains that suffices for our purposes. We also verify that h-extendible models support large numbers of square-integrable holomorphic functions, justifying the claim in Section 2 that K_D and B_D are nonzero. Indeed, on an h-extendible model we can find a square-integrable holomorphic function with prescribed Taylor polynomial at a given point.

LEMMA. *Let P be a real-valued, class C^2, plurisubharmonic, weighted homogeneous function that admits a bumping function a with properties (i)–(iii) of Section 2.*

(a) *As $\delta \to 0$, the minimum integrals for the models*

$$D_\delta := \{(z_0, z') : \operatorname{Re} z_0 + P(z') - \delta a(z') < 0\}$$

converge to the integrals for the model D_0. The convergence is uniform for z in compact subsets of D_0 and for vectors X on the unit sphere.

(b) *Fix a point ζ in D_0, a positive integer m, and a holomorphic polynomial q. There exists a function f in $A^2(D_{1/2})$ such that $f(z) - q(z) = O(|z-\zeta|^m)$ as $z \to \zeta$. Moreover, the norm of f in $A^2(D_{1/2})$ is bounded by a constant (depending on ζ and m) multiplied by the sum of the moduli of the coefficients of q.*

Proof. The proof is based on Hörmander's L^2 theory for solving the $\bar\partial$ equation with plurisubharmonic weights. We will use two related plurisubharmonic functions, ψ_a and ψ_b, in proving parts (a) and (b) of the lemma.

By hypothesis, the function $P(z') - a(z')$ is strictly plurisubharmonic away from the origin in $\mathbb{C}^n$, so $P(z') - a(z') + \log(1+|z'|^2)$ is strictly plurisubharmonic for all z'. Consequently, the function

$$\psi_a(z) := \exp(\operatorname{Re} z_0 + P(z') - a(z') + \log(1+|z'|^2))$$

is strictly plurisubharmonic in $\mathbb{C}^{n+1}$. The homogeneous function $a(z')$ grows faster than some power of $|z'|$, so $-\frac{1}{2}a(z') + \log(1+|z'|^2)$ is negative outside a compact set in $\mathbb{C}^n$. Since $\operatorname{Re} z_0 + P(z') - \frac{1}{2}a(z') < 0$ on $D_{1/2}$, the strictly plurisubharmonic function ψ_a is bounded on $D_{1/2}$, and hence is uniformly bounded on D_δ for $0 \le \delta \le 1/2$. By the same reasoning, the strictly plurisubharmonic function

$$\psi_b(z) := \psi_a(z) + \operatorname{Re} z_0 + P(z') - a(z') + (2n+2m)\log|z-\zeta|$$

is bounded above on $D_{1/2}$ (the bound depending on ζ and m).

We prove part (b) first. Fix a smooth cutoff function χ supported in D_0 that is identically equal to 1 in a neighborhood of ζ. By [25, Thm. 2.2.1′], there is a solution u of the equation $\bar\partial u = \bar\partial(\chi q) = (\bar\partial\chi)q$ on $D_{1/2}$ satisfying the estimate

$$\int_{D_{1/2}} |u|^2 e^{-\psi_b} \le \int_{D_{1/2}} |\bar\partial\chi|^2 |q|^2 \frac{e^{-\psi_b}}{c},$$

where c is a positive lower bound for the eigenvalues of the complex Hessian of ψ_b on the support of $\bar\partial\chi$. Since ψ_b is bounded above on $D_{1/2}$, the left-hand side of this inequality dominates the square of the $L^2(D_{1/2})$ norm of u. Since $\bar\partial\chi$ vanishes in a neighborhood of ζ, the function ψ_b is bounded below on the support of $\bar\partial\chi$, so the right-hand side is bounded above by a constant multiplied by the square of the sum of the moduli of the coefficients of the polynomial q. Consequently, the function $f = \chi q - u$ belongs to $A^2(D_{1/2})$ and satisfies the required norm estimate. That $u = O(|z-\zeta|^m)$ as $z \to \zeta$ follows because the left-hand side of the above inequality is finite, u

is analytic near ζ, and $e^{-\psi_b}$ is bounded below near ζ by a positive constant times $|z-\zeta|^{-2n-2m}$. This completes the proof of part (b) of the lemma.

To prove part (a), let I denote any of the quantities I^0, I^1, and I^2 for a fixed point ζ in D_0. Since $I_{D_\delta} \geq I_{D_0}$ (because $D_0 \subset D_\delta$), we need only show, for an arbitrary positive ϵ, that $I_{D_\delta} \leq I_{D_0} + \epsilon$ for sufficiently small positive δ. It will suffice to prove the following approximation property: Given a holomorphic function f that is square-integrable on D_0, there exists, for every sufficiently small positive δ, a holomorphic function f_δ on D_δ such that f_δ agrees with f at ζ to second order and $\|f_\delta\|_{D_\delta} = \|f\|_{D_0} + O(\epsilon)$. The constants involved in O may depend on f and ζ, but not on δ. Our strategy to produce f_δ is to cut f off at infinity and then correct it on D_δ by solving a $\bar\partial$ problem.

Fix a smooth, compactly supported function χ that is identically equal to 1 in a neighborhood V of the origin in $\mathbb{C}^{n+1}$. Choose a positive t so small that the L^2 norm of f on $D_0 \backslash \pi_{1/t}(V)$ is less than ϵ. When $\lambda \to 0$ through positive numbers, the functions $f(z_0 - \lambda, z')$ are holomorphic in a neighborhood of the closure of D_0 and converge to $f(z_0, z')$ in $L^2(D_0)$. Fix a value of λ so small that the function $\tilde f$ defined by $\tilde f(z_0, z') = f(z_0 - \lambda, z')$ differs from f in $L^2(D_0)$ by less than ϵ, and the second-order Taylor polynomial at ζ of the difference $\tilde f - f$ has coefficients of modulus less than ϵ.

When δ is a sufficiently small positive number, the function $\tilde f$ will be defined on $D_\delta \cap \operatorname{supp}(\chi \circ \pi_t)$ and will have L^2 norm on this set no more than $\epsilon + \|f\|_{D_0}$. We apply Hörmander's theorem with the plurisubharmonic weight function $\varphi = \psi_a \circ \pi_t$ to obtain a solution u of the equation $\bar\partial u = \bar\partial((\chi \circ \pi_t)\tilde f)$ on D_δ satisfying the estimate

$$\int_{D_\delta} |u|^2 e^{-\varphi} \leq \int_{D_\delta} \sum_{j=0}^{n} \left| t^{-1/m_j} \frac{\partial(\chi \circ \pi_t)}{\partial \bar z_j} \right|^2 |\tilde f|^2 \frac{e^{-\varphi}}{c},$$

where c is a positive lower bound for the eigenvalues of the Hessian of ψ_a on the support of χ. Here we have used that the Hessian of $\psi_a \circ \pi_t$ evaluated at z and applied to a vector w is the same as the Hessian of ψ_a evaluated at $\pi_t(z)$ and applied to the vector $\pi_t(w)$, and we have used an *anisotropic* version of Hörmander's theorem that is implicit in [25, Thms. 1.1.4, 2.1.4, & 2.2.1′]. This estimate reduces, by the chain rule, to

$$\int_{D_\delta} |u|^2 e^{-\varphi} \leq \int_{D_\delta} (|\bar\partial \chi|^2 \circ \pi_t) |\tilde f|^2 \frac{e^{-\varphi}}{c}.$$

Since ψ_a is bounded on $D_{1/2}$, so is φ, and therefore the left-hand side dominates the square of the $L^2(D_\delta)$ norm of u. Because f has small norm in $L^2(D_0 \backslash \pi_{1/t}(V))$ while $(\bar\partial\chi) \circ \pi_t$ is zero on $\pi_{1/t}(V)$, and because $\tilde f$ has small norm on $(D_\delta \backslash D_0) \cap \operatorname{supp}(\chi \circ \pi_t)$, the square root of the right-hand side is $O(\epsilon)$. Consequently, the function $(\chi \circ \pi_t)\tilde f - u$ is in $A^2(D_\delta)$, and its norm is $\|f\|_{D_0} + O(\epsilon)$.

The function just constructed may not have the correct second-order Taylor polynomial at the fixed point ζ. However, since the constructed function is close to f in $L^2(D_0)$, it is also close to f in C^2 norm on a compact neighborhood of ζ. By part (b) of the lemma, we can make the necessary small correction in the second-order Taylor polynomial by adding a holomorphic function of small $L^2(D_\delta)$ norm.

The uniformity statement in (a) is immediate from Dini's theorem. This completes the proof of the lemma. □

4. Proofs of the Theorems

The two theorems are proved by the same method. We will prove Theorem 1 in detail and then briefly sketch what changes are needed to prove Theorem 2.

The idea of the proof is to localize, dilate, and pass to the limit. To make this work, we need a barrier for the blow-ups of Ω. Our first step, therefore, is to make a preliminary local change of coordinates to ensure that a bumped model contains Ω locally near p.

Recall from Section 2 that near p the domain Ω is defined by the equation $r<0$ with

$$r(z_0,z') = \operatorname{Re} z_0 + P(z') + O(\sigma(z')^{1+\alpha}) + O((\operatorname{Im} z_0)^2).$$

Let a be a bumping function for P, and suppose $0<\delta<1$. Put

$$\rho_\delta = \operatorname{Re}(z_0+kz_0^2)+P(z')-\delta a(z').$$

We claim that there is a value of k, independent of δ, such that for each δ there is a neighborhood U_δ of the origin in $\mathbf{C}^{n+1}$ for which $\{z\in U_\delta: \rho_\delta(z)<0\}$ contains $\Omega\cap U_\delta$.

Indeed, $\rho_\delta = r-\delta a+O(\sigma^{1+\alpha})+O((\operatorname{Im} z_0)^2)+k(\operatorname{Re} z_0)^2-k(\operatorname{Im} z_0)^2$ and $(\operatorname{Re} z_0)^2\le 4r^2+4P^2+O(\sigma^{2+2\alpha})+O((\operatorname{Im} z_0)^4)$, so if $|\operatorname{Im} z_0|$ is small and k is large then $\rho_\delta\le r+4kr^2-\delta a+kO(\sigma^{1+\alpha})$. Fixing k, we see that the first two terms on the right-hand side sum to no more than $r/2$ when r is negative and small, and the last two terms have a negative sum in a sufficiently small (depending on δ) neighborhood of the origin. Consequently, there is a neigborhood U_δ of the origin in which $\rho_\delta\le r/2$ when r is negative.

Thus, after the local change of variables $(z_0,z')\mapsto(z_0+kz_0^2,z')$, we may assume that we have the following situation: Ω has local model $D=\{z: \operatorname{Re} z_0+P(z')<0\}$ at 0, and for each δ between 0 and 1, there is a neighborhood U_δ of the origin such that the bumped model

$$D_\delta := \{z: \operatorname{Re} z_0+P(z')-\delta a(z')<0\}$$

contains $\Omega\cap U_\delta$.

Our coordinate changes preserve nontangential approach to 0. Moreover, when $z\to 0$ in a nontangential approach region, the ratio of any two of the quantities $d(z)$, $|\operatorname{Re} z_0|$, and $|r(z)|$ has limit 1. To prove Theorem 1, we may compute the limit of $K_\Omega(z)|r(z)|^\beta$, where we write β as an abbreviation for $\sum_{j=0}^n 2/m_j$.

For the time being, we fix δ and the neighborhood U_δ. Since there exist local peak functions at h-extendible points [14; 44], the localization lemma

[25, Lemma 3.5.2] shows that $\lim_{z\to 0} K_\Omega(z)/K_{\Omega\cap U_\delta}(z) = 1$. We have therefore reduced our problem to finding the limit of $K_{\Omega\cap U_\delta}(z)|r(z)|^\beta$ as $z\to 0$. Because of the normalization at p of the transformation to local coordinates, we need not distinguish subsequently between the original coordinates and the local coordinates chosen above.

The factor $1/|r(z)|^\beta$ is precisely the square of the modulus of the complex Jacobian determinant of the anisotropic dilation $\pi_{1/|r(z)|}$. The transformation rule for the Bergman kernel function under biholomorphic mappings implies that the quantity we want to compute is equal to the limit as $z\to 0$ of $K_{\pi_{1/|r(z)|}(\Omega\cap U_\delta)}(\pi_{1/|r(z)|}(z))$. We will bound the lim sup and the lim inf by separate arguments.

We will write Ω_z^δ for $\pi_{1/|r(z)|}(\Omega\cap U_\delta)$ and $\zeta(z)$ for $\pi_{1/|r(z)|}(z)$. The Bergman kernel function increases when the domain decreases, so

$$\limsup_{\substack{z\to 0\\ z\in\Gamma}} K_{\Omega_z^\delta}(\zeta(z)) \le \limsup_{\substack{z\to 0\\ z\in\Gamma}} K_{\Omega_z^\delta\cap D}(\zeta(z)),$$

where D is the local model for Ω. Since the $\Omega_z^\delta\cap D$ converge to D from inside as $z\to 0$, Ramadanov's convergence theorem for the Bergman kernel (see Section 3) applies and shows that $K_{\Omega_z^\delta\cap D}(\xi)$ converges to $K_D(\xi)$ uniformly for ξ in a compact subset of D. When $z\to 0$ in a cone, the point $\zeta(z)$ approaches a compact portion of the line $\{\operatorname{Re} z_0 = -1, z' = 0\}$; but K_D is independent of the $\operatorname{Im} z_0$ variable since D is. Consequently

$$\limsup_{\substack{z\to 0\\ z\in\Gamma}} K_{\Omega_z^\delta}(\zeta(z)) \le K_D(\varpi),$$

where $\varpi = (-1, 0, \ldots, 0)$, or, equivalently,

$$\limsup_{\substack{z\to 0\\ z\in\Gamma}} K_\Omega(z)|r(z)|^\beta \le K_D(\varpi).$$

On the other hand, $K_{\Omega_z^\delta}(\zeta(z)) \ge K_{D_\delta}(\zeta(z))$ because $\Omega_z^\delta \subset D_\delta$. Since D_δ too is independent of the $\operatorname{Im} z_0$ variable, it follows that

$$\liminf_{\substack{z\to 0\\ z\in\Gamma}} K_{\Omega_z^\delta}(\zeta(z)) \ge K_{D_\delta}(\varpi)$$

simply because K_{D_δ} is a continuous function on the interior of D_δ. Equivalently,

$$\liminf_{\substack{z\to 0\\ z\in\Gamma}} K_\Omega(z)|r(z)|^\beta \ge K_{D_\delta}(\varpi).$$

Now we let $\delta\to 0$ and invoke the lemma to obtain

$$\liminf_{\substack{z\to 0\\ z\in\Gamma}} K_\Omega(z)|r(z)|^\beta \ge K_D(\varpi).$$

This completes the proof of Theorem 1.

Although we have written the proof in terms of the Bergman kernel function, we could equally well have phrased it in terms of the minimum integral I^0. Theorem 2 is proved by a completely analogous argument for the minimum integrals I^1 and I^2. (Localization of the minimum integrals I^1 and I^2 is explicit in [22; 28].) The metric is homogeneous (in the Euclidean sense) of degree 1 in X, so $B_\Omega(z,X)/|\pi_{1/d(z)}(X)| = B_\Omega(z, X/|\pi_{1/d(z)}(X)|)$. When we apply the anisotropic dilation $\pi_{1/|r|}$, the vector $X/|\pi_{1/d(z)}(X)|$ is transformed to $\pi_{1/|r|}(X)/|\pi_{1/d}(X)|$, which has limit $\hat{X}$. For the curvature, we use the same argument together with the observation that

$$R_\Omega(z,X) = R_\Omega(z, X/|\pi_{1/d(z)}(X)|),$$

since the curvature is homogeneous of degree 0 with respect to X.

REMARK. Theorems 1 and 2 do not generalize to unbounded domains Ω without some further hypothesis, for the localization of the minimum integrals involves constants (from Hörmander's L^2 estimates) that blow up with the diameter of the domain. However, the proof carries over if Ω is an unbounded pseudoconvex domain that is h-extendible at p and is contained in some barrier domain that supports a bounded strictly plurisubharmonic function whose complex Hessian has eigenvalues bounded away from zero under approach to the point p. This condition is satisfied, for example, if Ω is globally contained in some h-extendible model domain (not necessarily of the same homogeneity as the local model for Ω).

5. Explicit Formulas

If the model domain has circular symmetries, we can compute the limits in Theorems 1 and 2 explicitly. Some related results for a special class of models were obtained by Chalmers [7].

COROLLARY. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^{n+1}$. Suppose that Ω has the local model $D := \{z \in \mathbb{C}^{n+1} : \operatorname{Re} z_0 + P(z') < 0\}$ at the boundary point p, where P has circular symmetry in each variable. The model D is biholomorphically equivalent to the bounded domain $G := \{z \in \mathbb{C}^{n+1}: |z_0|^2 + P(|z_1|, \dots, |z_n|) < 1\}$ and, with the notation of Theorem 2,*

$$\lim_{\substack{z\to p\\ z\in\Gamma}} K_\Omega(z)\, d(z)^{\sum_{j=0}^{n} 2/m_j} = \frac{1}{4\operatorname{Vol} G},$$

$$\lim_{\substack{z\to p\\ z\in\Gamma}} \frac{B_\Omega(z,X)}{|\pi_{1/d(z)}(X)|} = \frac{\sqrt{\operatorname{Vol} G}}{2}\left(a_0|\hat{X}_0|^2 + \sum_{j=1}^{n} 4a_j|\hat{X}_j|^2\right)^{1/2},$$

and

$$\lim_{\substack{z\to p\\ z\in\Gamma}} R_\Omega(z,X) = 2 - \frac{4}{\operatorname{Vol} G}\left(a_0|\hat{X}_0|^2 + \sum_{j=1}^{n} 4a_j|\hat{X}_j|^2\right)^{-2} \times$$

$$\times\left(b_{00}|\hat{X}_0|^4+\sum_{0<k\le n}4b_{0k}|\hat{X}_0|^2|\hat{X}_k|^2+\sum_{0<j\le k\le n}16b_{jk}|\hat{X}_j|^2|\hat{X}_k|^2\right),$$

where $a_j=1/\|z_j\|^2_{L^2(G)}$ *and* $b_{jk}=1/\|z_jz_k\|^2_{L^2(G)}$. *In particular, the limiting value of the holomorphic sectional curvature* $2R_\Omega(z,X)$ *in the direction normal to the boundary is* $4-8b_{00}/a_0^2\,\mathrm{Vol}\,G$.

Sketch of the proof. Since the weighted homogeneous polynomial P is plurisubharmonic and has circular symmetry, it must be positive except at the origin. Therefore the model D is h-extendible, since P serves as its own bumping function. Consequently, Theorems 1 and 2 are applicable.

The mapping $w_0=(z_0-1)/(z_0+1)$, $w'=\pi_{1/(z_0+1)^2}(z')$, takes G biholomorphically onto the model D with the origin corresponding to the special point ϖ. The complex Jacobian matrix of this mapping at the origin is diagonal, with a 2 in the upper left-hand corner and 1s down the rest of the diagonal. Therefore the transformation rule from Section 3 for the minimum integrals implies that $K_D(\varpi)=K_G(0)/4$, $B_D(\varpi,\hat{X})=B_G(0,Y)$, and $R_D(\varpi,\hat{X})=R_G(0,Y)$, where the components of the vector Y are given by $Y_0=\frac{1}{2}\hat{X}_0$ and $Y_j=\hat{X}_j$ for $1\le j\le n$.

We have reduced the problem to a computation on the bounded Reinhardt domain G. There the monomials z^α form a complete orthogonal system in the Bergman space, and we can write the Bergman kernel function explicitly as a power series $\sum_\alpha c_a|z^\alpha|^2$, where $c_\alpha=\|z^\alpha\|^{-2}$. To compute the kernel, metric, and curvature at the origin, we may truncate the series at the fourth-order terms, and a routine calculation completes the proof. □

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Department of Mathematics
Texas A & M University
College Station, TX 77843