Interpolating Varieties and Counting Functions in $\mathbf{C}^n$

CARLOS A. BERENSTEIN, DER-CHEN CHANG, & BAO QIN LI

1. Introduction

In this paper, we shall study the extension problem of an analytic function satisfying growth conditions on an analytic variety of codimension 1 to an entire function on $\mathbf{C}^n$ satisfying the same kind of growth conditions, where

$$V = f^{-1}(0) := \{\zeta \in \mathbf{C}^n : f(\zeta) = 0\}$$

defined as the zero set of an entire function f in a given weighted space. If each analytic function on V with growth conditions has such a global extension, V is then called an *interpolating variety* (defined below).

The above interpolation problem has been studied by many authors (see e.g. [Be, De, Oh, Sk]) and is related to harmonic analysis (cf. [BT3]). An open problem [BT2] is to find geometric interpolation conditions that depend only on the geometry of varieties. When $n=1$, it has been shown that V is an interpolating variety for A_p (resp. A_p^0) if and only if

$$N(|\zeta|, \zeta, V) \leq Ap(\zeta) + B, \quad \zeta \in V, \tag{1.1}$$

for some $A, B > 0$ (resp. $N(|\zeta|, \zeta, V) = o\{p(\zeta)\}$, $\zeta \in V$, $\zeta \to \infty$) (see [BL, BLV, Sq]), where $N(|\zeta|, \zeta, V)$ is the counting function of V and p is the given weight (defined below). The advantage of this condition is that one can determine whether V is an interpolating variety by estimating the geometric quantity $N(|\zeta|, \zeta, V)$, which depends only on the "value distribution" of V. Note that the counting function is one of the most important quantities in studying value distribution of holomorphic mappings in one and several complex variables (see e.g. [Gr]). It is a natural goal to consider whether it could give geometric interpolation conditions in the higher-dimensional case. The paper is concerned with this problem. We shall prove that (1.1) can still be used to give a sufficient interpolation condition. However, a counterexample shows that (1.1) is no longer necessary for interpolation in the higher-dimensional case.

Received January 6, 1994. Revision received January 17, 1995.
This research is supported in part by NSF grants DMS-92-25043, DMS-92-06185, and CDR 88-03012.
Michigan Math. J. 42 (1995).

We arrange the paper as follows. In Section 2, we give some preliminaries and state our main results. Theorem 2.4 uses the counting function of V to give a sufficient interpolation condition, and Theorem 2.7 uses the counting function of slices of V by complex lines to give a sufficient interpolation condition. We note that, given V, these conditions can be verified by a direct calculation, since the counting functions depend only on the geometry of V. We shall see from the proofs that the hypotheses of Theorem 2.4 imply the well-known analytic sufficient condition on the lower bound on the gradient of the defining function, and thus V is interpolating. The hypotheses of Theorem 2.7 do not imply this gradient condition but instead have V interpolating uniformly along each slice and thus interpolating by passing from slices to the variety. Theorem 2.9 and 2.10 give the corresponding results for the space A_p^0. The proofs of the above results will be given in Section 3. Finally, in Section 4 we shall present a class of interpolating varieties defined by exponential polynomials.

The authors would like to thank the referee for valuable suggestions.

2. Interpolating Varieties

First of all, let us fix the notation to be used throughout this paper: $\zeta = (\zeta_1, \dots, \zeta_n)$ is a point in $\mathbf{C}^n$; $|\zeta|^2 = \sum_{j=1}^n |\zeta_j|^2$ is the square of its modulus; and

$$\phi = dd^c|\zeta|^2 = \frac{i}{2\pi} \sum_{j=1}^{n} d\zeta_j \wedge d\bar{\zeta}_j$$

is the standard Kähler form on $\mathbf{C}^n$, where $d = \partial + \bar{\partial}$ and $d^c = (\partial - \bar{\partial})/4\pi i$ are defined in the usual way. Denote $V = f^{-1}(0)$, where f is an entire function. Using the form ϕ, one defines the function

$$n(r, V) = \frac{1}{r^{2(n-1)}} \int_{V \cap B_r} \phi^{n-1}$$

(counted with the multiplicity of f) and the counting function

$$N(r, V) = \int_0^r \frac{n(t, V) - n(0, V)}{t}\, dt + n(0, V) \log r,$$

where

$$B_r := \{\zeta \in \mathbf{C}^n : |\zeta| \le r\}.$$

In the same way, one can define the functions $n(r, \zeta, V)$ and $N(r, \zeta, V)$ with respect to $B(\zeta, r)$, the ball with the center at ζ and radius r. Note that $n(0, \zeta, V)$ is the Lelong number of V at ζ. Clearly, when $n = 1$, $n(r, \zeta, V)$ is the number of points of V in $B(\zeta, r)$.

For any ζ, $\mathcal{L}_\zeta$ denotes the line through ζ and the origin and $N(r, \zeta, V \cap \mathcal{L}_\zeta)$ is the counting function corresponding to the function $n(r, \zeta, V \cap \mathcal{L}_\zeta)$, which denotes the number of points of $V \cap \mathcal{L}_\zeta$ in the ball $B(\zeta, r)$ (counted with multiplicities).

DEFINITION 2.1. A plurisubharmonic function $p: \mathbf{C}^n \to [0,\infty)$ is called a weight (function) if it satisfies the following three conditions:

$$\log(1+|\zeta|^2) = O(p(\zeta)); \tag{2.1}$$

$$p(|\zeta|) = p(\zeta); \tag{2.2}$$

$$p(2\zeta) = O\{p(\zeta)\}. \tag{2.3}$$

Here (2.2) means that p is radial.

DEFINITION 2.2. Let $A(\mathcal{V})$ be the ring of all analytic functions on $\mathcal{V}$, where $\mathcal{V}$ is a subset of $\mathbf{C}^n$. Then

$$A_p(\mathcal{V}) = \{f \in A(\mathcal{V}): |f(\zeta)| \le A\exp(Bp(\zeta)) \text{ for some } A, B > 0\}$$

endowed with the inductive limit topology and

$$A_p^0(\mathcal{V}) = \{f \in A(\mathcal{V}): \forall \epsilon > 0, \sup_{\zeta\in \mathbf{C}^n}|f(\zeta)|e^{-\epsilon p(\zeta)} < \infty\}$$

endowed with the topology of the projective limit. In the case when $\mathcal{V} = \mathbf{C}^n$, we simply write $A_p = A_p(\mathbf{C}^n)$ and $A_p^0 = A_p^0(\mathbf{C}^n)$.

A basic example of such weight function is $p(\zeta) = |\zeta|$, corresponding to the space A_p of all entire functions of order ≤ 1 and finite type as well as the space A_p^0 of all entire functions of infraexponential type.

DEFINITION 2.3. Let V be as before. Define the restriction map $\rho: A_p \to A_p(V)$ by sending $g \in A_p$ to $g|_V$, the restriction of g to V. Then V is said to be an interpolating variety for A_p if the restriction map ρ is onto from A_p to $A_p(V)$. Similarly, we say that V is an interpolating variety for A_p^0 if the restriction map $\rho_0: A_p^0 \to A_p^0(V)$ sending $f \in A_p^0$ to $f|_V$ is onto from A_p^0 to $A_p^0(V)$.

Clearly, that V is an interpolating variety for A_p (resp., A_p^0) means that any analytic function in $A_p(V)$ (resp., $A_p^0(V)$) has an extension to an entire function in A_p (resp., A_p^0).

Now let us state our results, whose proofs will be given in the next section. We denote $\nabla f = (\partial f/\partial\zeta_1, \ldots, \partial f/\partial\zeta_n)$, the gradient of f, and assume that p is a weight.

THEOREM 2.4. *Let* $V = f^{-1}(0)$ *for* $f \in A_p(\mathbf{C}^n)$ *with* $\nabla f \ne 0$ *on* V. *If*

$$N(|\zeta|, \zeta, V) \le Ap(\zeta) + B, \quad \zeta \in V, \tag{2.4}$$

for some constants $A, B > 0$, *then* $|\nabla f| \ge \epsilon\exp(-Cp)$ *on* V *for some* $\epsilon, C > 0$ *and thus* V *is an interpolating variety for* $A_p(\mathbf{C}^n)$.

REMARK 2.5. We note that condition (2.4) alone, without assuming $\nabla f \ne 0$ on V, does not imply the lower bound of the gradient of f in the conclusion of the theorem. An easy example may be given by considering the function

$f(\zeta)=f(\zeta_1,\zeta_2)=g(\zeta_1)$, where g is an entire function in $A_p(\mathbf{C})$ whose zero set V satisfies the condition (1.1) and some of whose zeros are of order >1. For example, let $g(z)=(\cos z)^2$. Then, for $f(\zeta_1,\zeta_2)=g(\zeta_1)=(\cos\zeta_1)^2$, it is easy to check that (2.4) holds with $p(\zeta)=|\zeta|$; however, $\nabla f=0$ on $f^{-1}(0)$.

REMARK 2.6. As pointed out earlier, condition (2.4) is not necessary for the variety V to be interpolating in cases of dimension >1. Let us look at the following counterexample. Set

$$p(\zeta)=|\zeta|\colon \mathbf{C}^2\to[0,\infty) \quad \text{and} \quad V=f^{-1}(0),$$

where

$$f(\zeta_1,\zeta_2)=\zeta_2^2-\frac{\sin\zeta_1\sin\lambda\zeta_1}{\zeta_1}$$

and λ is a Liouville number (i.e., $\lambda=\sum(10)^{-n_k}$, where $\{n_k\}$ is a sequence increasing to infinity sufficiently fast). Then clearly $\nabla f(\zeta)\neq 0$ for $\zeta\in V$, and it is known ([BD] or [BT2]) that V is an interpolating variety for A_p. However, one can verify that (2.4) does not hold.

THEOREM 2.7. *Let $V=f^{-1}(0)$ for $f\in A_p(\mathbf{C}^n)$. If*

$$N(|\zeta|,\zeta,V\cap\mathcal{L}_\zeta)\le Ap(\zeta)+B,\quad \zeta\in V, \tag{2.5}$$

for some constants $A,B>0$, then V is an interpolating variety for $A_p(\mathbf{C}^n)$.

REMARK 2.8. In Theorem 2.7, we did not assume that $\nabla f\neq 0$ on V. We note that (2.5) does not imply a lower bound of the gradient of f, and that the gradient of f may vanish on V; this can be shown by using the same example in Remark 2.5.

For the corresponding results in the space A_p^0, we have the following results.

THEOREM 2.9. *Let $V=f^{-1}(0)$ for $f\in A_p^0(\mathbf{C}^n)$ with $\nabla f\neq 0$ on V. If*

$$N(|\zeta|,\zeta,V)=o\{p(\zeta)\},\quad \zeta\in V,\ \zeta\to\infty, \tag{2.6}$$

then V is an interpolating variety for $A_p^0(\mathbf{C}^n)$.

THEOREM 2.10. *Let $V=f^{-1}(0)$ for $f\in A_p^0(\mathbf{C}^n)$. If*

$$N(|\zeta|,\zeta,V\cap\mathcal{L}_\zeta)=o\{p(\zeta)\},\quad \zeta\in V,\ \zeta\to\infty,$$

then V is an interpolating variety for $A_p^0(\mathbf{C}^n)$.

3. Proofs of the Results

In the sequel, we shall use A and B to denote positive constants that depend only on the dimension n and may vary in value from one occurrence to the next.

Proof of Theorem 2.4. First of all, for the sake of convenience, we can assume that $f(0)=1$ (otherwise we need only make an obvious modification in the following proof).

For any $\zeta_0=(\zeta_1^0,\dots,\zeta_n^0)\in V$, since $\nabla f(\zeta_0)\neq 0$ the set

$$P_{\zeta_0}=\{\zeta\in\mathbf{C}^n:\langle\nabla f(\zeta_0),\zeta-\zeta_0\rangle=0\}$$

is an $(n-1)$-dimensional complex hyperplane through the point ζ_0 with normal vector $\nabla f(\zeta_0)$, where

$$\langle\nabla f(\zeta_0),\zeta-\zeta_0\rangle=\frac{\partial f(\zeta_0)}{\partial\zeta_1}(\zeta_0)(\zeta_1-\zeta_1^0)+\cdots+\frac{\partial f(\zeta_0)}{\partial\zeta_n}(\zeta_0)(\zeta_n-\zeta_n^0).$$

It is easy to see that there exists a point $w_0\in B(0,|\zeta_0|/2)$ with $|w_0|=\frac{1}{4}|\zeta_0|$ such that

$$B(w_0,|\zeta_0|/8)\cap P_{\zeta_0}=\emptyset.$$

(In fact, the complex line through the origin and perpendicular to P_{ζ_0} will intersect the sphere $\partial B(0,\frac{1}{4}|\zeta_0|)$ at two points. At least one of them can serve as w_0.) Now let C_{ζ_0} be the cone formed by all lines through ζ_0 and all points $w\in B(w_0,|\zeta_0|/8)$. Then $C_{\zeta_0}\cap P_{\zeta_0}=\{\zeta_0\}$. This implies that, for $\zeta\in C_{\zeta_0}-\{\zeta_0\}$,

$$\langle\nabla f(\zeta_0),\zeta-\zeta_0\rangle\neq 0. \tag{3.1}$$

It is readily seen that, for any $w\in B(w_0,|\zeta_0|/16)$, we have

$$B(w,|\zeta_0|/16)\subset B(w_0,|\zeta_0|/8).$$

Therefore, for any such w and $\zeta\in B(w,|\zeta_0|/16)$, it holds from (3.1) that

$$\langle\nabla f(\zeta_0),\zeta-\zeta_0\rangle\neq 0.$$

Let $\mathcal{L}_{\zeta\zeta_0}$ denote the line through the ζ and ζ_0. Then, by the Jensen formula,

$$\begin{aligned}N(|\zeta-\zeta_0|,\zeta_0,V\cap\mathcal{L}_{\zeta\zeta_0})=&\frac{1}{2\pi}\int_0^{2\pi}\log|f(\zeta_0+(\zeta-\zeta_0)e^{i\theta})|\,d\theta\\&-\log\left|\left\langle\nabla f(\zeta_0),\frac{\zeta-\zeta_0}{|\zeta-\zeta_0|}\right\rangle\right|.\end{aligned}$$

Clearly, the right-hand side of the above equality is a plurisubharmonic function for $\zeta\in B(w,|\zeta_0|/16)$. By the mean value inequality for plurisubharmonic functions we thus obtain that, for $w\in B(w_0,|\zeta_0|/16)$,

$$\begin{aligned}&N(|w-\zeta_0|,\zeta_0,V\cap\mathcal{L}_{w\zeta_0})\\&\quad\leq\frac{1}{\mathrm{vol}(B(w,|\zeta_0|/16))}\int_{B(w,|\zeta_0|/16)}N(|\zeta-\zeta_0|,\zeta_0,V\cap\mathcal{L}_{\zeta\zeta_0})\phi^n(\zeta)\\&\quad=\frac{1}{(|\zeta_0|/16)^{2n}}\int_{B(w,|\zeta_0|/16)}N(|\zeta-\zeta_0|,\zeta_0,V\cap\mathcal{L}_{\zeta\zeta_0})\phi^n(\zeta).\end{aligned}$$

Note that for any $w\in B(w_0,|\zeta_0|/16)$,

$$|w-\zeta_0|\geq|\zeta_0|-\left(|w_0|+\frac{|\zeta_0|}{16}\right)=\frac{11}{16}|\zeta_0|$$

and, for $\zeta\in B(w,|\zeta_0|/16)$,

$$|\zeta-\zeta_0|\le|\zeta_0|+\frac{|\zeta_0|}{16}+\frac{|\zeta_0|}{16}+|w_0|=\frac{11}{8}|\zeta_0|.$$

We deduce that

$$N\left(\frac{11}{16}|\zeta_0|,\zeta_0,V\cap\mathcal{L}_{w\zeta_0}\right)\le\left(\frac{16}{|\zeta_0|}\right)^{2n}\int_{B(\zeta_0,11|\zeta_0|/8)}N\left(\frac{11}{8}|\zeta_0|,\zeta_0,V\cap\mathcal{L}_{\zeta\zeta_0}\right)\phi^n(\zeta).$$

Let $\omega=dd^c\log|\zeta|^2$ be the pull-back to $\mathbf{C}^n-\{0\}$ of the form ϕ on the projective space $\mathbf{P}^{n-1}$ of lines through the origin in $\mathbf{C}^n$. It is then easy to check that

$$\phi^n(\zeta-\zeta_0)=n|\zeta-\zeta_0|^{2(n-2)}\omega^{n-1}(\zeta-\zeta_0)\wedge d|\zeta-\zeta_0|^2\wedge d^c|\zeta-\zeta_0|^2.$$

We next apply Fubini's theorem to the above integral by first integrating over the ball $B(\zeta_0,11|\zeta_0|/8)$ and then over the set of all the lines $\mathcal{L}$ through ζ_0, i.e., the space $\mathbf{P}^{n-1}$. In view of the fact that on the line

$$d|\zeta-\zeta_0|^2\wedge d^c|\zeta-\zeta_0|^2=\frac{i}{2\pi}|\zeta-\zeta_0|^2d(\zeta-\zeta_0)\wedge d(\bar\zeta-\bar\zeta_0),$$

we deduce that

$$\begin{aligned}N&\left(\frac{11}{16}|\zeta_0|,\zeta_0,V\cap\mathcal{L}_{w\zeta_0}\right)\\
&\le n\left(\frac{16}{|\zeta_0|}\right)^{2n}\int_{\mathbf{P}^{n-1}}N\left(\frac{11}{8}|\zeta_0|,\zeta_0,V\cap\mathcal{L}_{\zeta\zeta_0}\right)\omega^{n-1}\\
&\quad\times\int_{B(\zeta_0,11|\zeta_0|/8)}\frac{i}{2\pi}|\zeta-\zeta_0|^{2n-2}d(\zeta-\zeta_0)\wedge d(\bar\zeta-\bar\zeta_0)\\
&=n\left(\frac{16}{|\zeta_0|}\right)^{2n}\left(\frac{11}{8}|\zeta_0|\right)^{2n}\int_{\mathbf{P}^{n-1}}N\left(\frac{11}{8}|\zeta_0|,\zeta_0,V\cap\mathcal{L}_{\zeta\zeta_0}\right)\omega^{n-1}\\
&=n(22)^{2n}N\left(\frac{11}{8}|\zeta_0|,\zeta_0,V\right).\end{aligned}$$

Here we have used the known equality that

$$\int_{\mathbf{P}^{n-1}}N(r,V\cap\mathcal{L})\omega^{n-1}=N(r,V)$$

for $r>0$, where the integral is taken over all the lines through the origin (see e.g. [Gr]).

It follows from the hypothesis (2.4) that for any $w\in B(w_0,|\zeta_0|/16)$,

$$N\left(\frac{11}{16}|\zeta_0|,\zeta_0,V\cap\mathcal{L}_{w\zeta_0}\right)\le Ap(\zeta_0)+B. \tag{3.2}$$

Now consider the restriction $g(z)=f(z(w_0/|w_0|))$ of the function f to the line

$$\mathcal{L}_{w_0}=\left\{\zeta=\frac{w_0}{|w_0|}z:z\in\mathbf{C}\right\}.$$

Then $g(0)=f(0)=1$ and $\log|g(z)|\le Ap(\zeta_0)+B$ for $|z|\le 2e|\zeta_0|$. Recall the following minimum modulus theorem (see [Le]): For a function F holomorphic in $|z|\le 2eR$ with $F(0)=1$,

$$\log|F(z)|>-C_\eta \log \max_{|z|=2eR}\{|F(z)|\}$$

inside $|z|\le R$ but outside a union of circles the sum of whose radii $\le 4\eta R$ (η is a constant with $0<\eta<3e/2$.) Applying this theorem to the function g, we obtain that, in $|z|\le|\zeta_0|$ but outside a union of circles the sum of whose radii $\le|\zeta_0|/32$,

$$\log|g(z)|\ge -Ap(\zeta_0)-B. \tag{3.3}$$

Thus, there exists a ϱ with $|\zeta_0|/4<\varrho<5|\zeta_0|/16$ such that for $|z|=\varrho$, (3.3) holds. Take $z_0=\varrho$ and set $w^*=z_0(w_0/|w_0|)$. Then

$$\log|f(w^*)|=\log|g(z_0)|>-Ap(\zeta_0)-B. \tag{3.4}$$

Note that

$$|w^*-w_0|=z_0-|w_0|=z_0-\frac{|\zeta_0|}{4}<\left(\frac{5}{16}-\frac{1}{4}\right)|\zeta_0|=\frac{|\zeta_0|}{16}.$$

This implies that $w^*\in B(w_0,|\zeta_0|/16)$ and thus that, from (3.2),

$$N\left(\frac{11}{16}|\zeta_0|,\zeta_0,V\cap\mathfrak{L}_{w^*\zeta_0}\right)\le Ap(\zeta_0)+B. \tag{3.5}$$

Next, let

$$G(z)=\frac{f\left(\zeta_0+z\dfrac{w^*-\zeta_0}{|w^*-\zeta_0|}\right)}{f(w^*)}.$$

Then

$$G(|w^*-\zeta_0|)=\frac{f(w^*)}{f(w^*)}=1.$$

Applying again the above minimum modulus theorem to $G(z)$ in

$$\big|z-|w^*-\zeta_0|\big|\le 4e(|w^*-\zeta_0|),$$

we deduce that, inside $\big|z-|w^*-\zeta_0|\big|\le 2|w^*-\zeta_0|$ but outside a union of circles the sum of whose radii $\le|w^*-\zeta_0|/84$,

$$\log|G(z)|>-Ap(\zeta_0)-B.$$

Therefore there exists a ϱ^* with $|w^*-\zeta_0|/42<\varrho^*<|w^*-\zeta_0|/21$ such that

$$\min_{|z|=\varrho^*}\{\log|G(z)|\}>-Ap(\zeta_0)-B. \tag{3.6}$$

Now by the Jensen formula we have

$$N(\varrho^*,\zeta_0,V\cap\mathfrak{L}_{w^*\zeta_0})=\frac{1}{2\pi}\int_0^{2\pi}\log|G(\varrho^*e^{i\theta})|\,d\theta-\log|G'(0)|.$$

It is immediate to check that

$$\varrho^* < \frac{|w^*-\zeta_0|}{21} \le \frac{1}{21}(|w^*|+|\zeta_0|) \le \frac{1}{21}\left(\frac{5}{16}+1\right)|\zeta_0| = \frac{|\zeta_0|}{16}$$

and

$$G'(0) = \frac{\left\langle \nabla f(\zeta_0), \dfrac{w^*-\zeta_0}{|w^*-\zeta_0|}\right\rangle}{f(w^*)}.$$

It thus follows, in virtue of (3.4), (3.5), and (3.6), that

$$\log\left|\left\langle \nabla f(\zeta_0), \frac{w^*-\zeta_0}{|w^*-\zeta_0|}\right\rangle\right| \ge -Ap(\zeta_0)-B.$$

However,

$$\left|\left\langle \nabla f(\zeta_0), \frac{w^*-\zeta_0}{|w^*-\zeta_0|}\right\rangle\right| \le |\nabla f(\zeta_0)|\left|\frac{w^*-\zeta_0}{|w^*-\zeta_0|}\right| = |\nabla f(\zeta_0)|.$$

Hence we have proved that, for any $\zeta_0 \in V$,

$$\log|\nabla f(\zeta_0)| \ge -Ap(\zeta_0)-B. \tag{3.7}$$

The proof is then concluded by the result in [BT2] that V is an interpolating variety if (3.7) holds on V. ☐

From the proof of Theorem 2.4, we see that if (2.4) holds only for $\zeta \in V$ with sufficiently large modulus, then (3.7) holds for $\zeta \in V$ with sufficiently large modulus and thus holds for all $\zeta \in V$ because, for any $R > 0$, in the intersection $V \cap B(0, R)$ (which is a compact subset of $\mathbf{C}^n$) the function $\log|\nabla f(\zeta)|$ is continuous and thus attains its minimum modulus. This shows that Theorem 2.4 is still valid in this case.

Proof of Theorem 2.7. There is no loss of generality in assuming that $f(0) = 1$. For any fixed $\zeta \in V$, let

$$\zeta^* = \zeta/|\zeta| \quad \text{and} \quad g(z) = f(\zeta + z\zeta^*), \tag{3.8}$$

where $z \in \mathbf{C}$. Then by the assumption there exists a $k \in \mathbf{N}$ such that $g^{(j)}_{(0)} = 0$ for $j < k$ and $g^{(k)}(0) \ne 0$. Also, $g(-|\zeta|) = f(0) = 1$. Using the minimum modulus theorem for g in

$$|z+|\zeta|| \le 4e|\zeta|,$$

we deduce that inside $|z+|\zeta|| \le 2|\zeta|$ but outside a union of circles the sum of whose radii $\le \frac{1}{2}|\zeta|$,

$$\log|g(z)| > -Ap(\zeta)-B.$$

Therefore there exists a η with $0 < \eta < |\zeta|$ such that

$$\min_{|z|=\eta}\{|g(z)|\} > -Ap(\zeta)-B. \tag{3.9}$$

Now, by the Jensen formula,

$$N(\eta, \zeta, V \cap \mathcal{L}_\zeta) = \frac{1}{2\pi}\int_0^{2\pi} \log|g(\eta e^{i\theta})|\,d\theta - \log\left|\frac{g^{(k)}(0)}{k!}\right|,$$

where $\mathcal{L}_\zeta$ denotes the line through the origin and the point ζ. It follows from (2.5) and (3.9) that

$$\log\left|\frac{g^{(k)}(0)}{k!}\right| > -Ap(\zeta)-B, \tag{3.10}$$

where $A, B > 0$ are two constants independent of ζ. Now let

$$V_\zeta = V \cap \mathcal{L}_\zeta;$$
$$d_\zeta = \min\{1, \operatorname{dist}(\zeta, V_\zeta \backslash \{\zeta\})\};$$
$$G(z) = \frac{g(z)}{z^k}.$$

Then on $|z|=1$ and thus $|z|\le 1$, by the maximum modulus theorem we have

$$|G(z)| < A\exp(Bp(\zeta)). \tag{3.11}$$

Therefore, the function

$$G_1(z) := \frac{G(z)-G(0)}{2A\exp(Bp(\zeta))}$$

satisfies that $G_1(0)=0$ and $|G_1(z)|<1$ in $|z|<1$. By the Schwarz lemma (see e.g. [BG1]), $|G_1(z)|\le |z|$ in $|z|<1$. In particular, letting $a\ne 0$ be any zero of g in $|z|<1$, by (3.10) we then have

$$\begin{aligned}|a| \ge |G_1(a)| &= \left|\frac{G(0)}{2A\exp(Bp(\zeta))}\right| \\ &= \left|\frac{g^{(k)}(0)/k!}{2A\exp(Bp(\zeta))}\right| \ge \epsilon_1 \exp(-c_1 p(\zeta))\end{aligned}$$

for some positive constants ϵ_1, c_1 ($\epsilon_1 < 1$) independent of ζ. This shows that

$$d_\zeta \ge \epsilon_1 \exp(-c_1 p(\zeta)) := d_\zeta^*. \tag{3.12}$$

Applying the Caratheodory inequality to $G(z)$ (see e.g. [Le, p. 19]), we have that in $|z|\le d_\zeta/2$,

$$\log\left|\frac{G(z)}{G(0)}\right| \ge -\frac{2\times d_\zeta/2}{d_\zeta - d_\zeta/2}\log\left(\max_{|z|=d_\zeta}\left\{\left|\frac{G(z)}{G(0)}\right|\right\}\right)$$

and so, by (3.10), (3.11), and the fact that $G(0)=g^{(k)}(0)/k!$,

$$|G(z)| \ge \epsilon_2 \exp(-c_2 p(\zeta)),$$

where ϵ_2, c_2 are positive numbers independent of ζ. For $|z| = d_\zeta^*/2$, in view of (3.8) and (3.12), we thus have that

$$|f(\zeta + z\zeta^*| = |g(z)| = |z^k G(z)| \ge \epsilon \exp(-cp(\zeta)) \tag{3.13}$$

for two constants $\epsilon, c > 0$.

Now let

$$U_\zeta := \{\xi \in \mathcal{L}_\zeta : |\xi - \zeta| \le d_\zeta^*/2\},$$
$$S(f, \mathcal{L}_\zeta; \epsilon, c) = \{\xi \in \mathcal{L}_\zeta : |f(\xi)| < \epsilon \exp(-cp(\xi))\},$$

and let W_ζ be the component of $S(f, \mathcal{L}_\zeta; \epsilon, c)$ containing ζ. Then obviously (3.13) holds for $\xi \in \partial U_\zeta$ and thus

$$W_\zeta \subset U_\zeta. \tag{3.14}$$

To show that V is interpolating, we take any analytic function λ on V (we need to prove that it has a global extension to the space A_p). Then there exist two constants C and D such that

$$\lambda(\xi) \leq C \exp(Dp(\xi)) \tag{3.15}$$

for all $\xi \in \mathbf{C}^n$. For the line $L := \mathcal{L}_\zeta$ through the origin and ζ, we define

$$\lambda_L(\xi) = \begin{cases} \lambda(\xi) & \text{if } \xi \in W_\zeta,\ \zeta \in V \cap L; \\ 0 & \text{if } \xi \in S(f, L; \epsilon, c) \setminus \bigcup_{\zeta \in V \cap L} W_\zeta. \end{cases}$$

By (3.14), the function λ_L is well-defined and analytic on $S(f, L; \epsilon, c)$. Clearly, λ_L satisfies (3.15) on $S(f, L; \epsilon, c)$. This implies that there exists an analytic function $\tilde{\lambda}_L$ on L such that $\tilde{\lambda}_L = \lambda$ on the intersection $V \cap L$ and

$$|\tilde{\lambda}_L(\xi)| \leq A \exp(Bp(\xi)), \quad \xi \in L,$$

for constants $A, B > 0$ independent of the choices of the line L by the semi-local-to-global extension theorem and its proof in [BT1].

Next, let $\mathcal{L}$ be the family of all the lines through the origin. Then $\mathcal{L}$ is almost analytic parallel and f is slowly decreasing in the sense of Berenstein and Taylor ([BT3, Thm. 7.1]) that there exist constants $\epsilon_1, C_1, K_1, K_2 > 0$ such that, for each $L \in \mathcal{L}$: the set

$$\mathcal{O} = \mathcal{O}(L, \epsilon_1, C_1) = \{\xi \in L : |f(\xi)| < \epsilon_1 \exp(-C_1 p(\xi))\}$$

has relatively compact components; and if ξ, w are in the same component of $\mathcal{O}$ then $p(\xi) \leq K_1 p(w) + K_2$. We can now conclude our proof by recalling the following theorem in [BT3, Thm. 5.6]: Let F be slowly decreasing with respect to an analytic almost parallel family of lines $\mathcal{L}$, and let ω be an analytic function on $V = Z(F)$. Assume that for some constants $A, B > 0$ and for every line $L \in \mathcal{L}$, there exists a function $\tilde{\omega}_L$, analytic on L, such that

$$|\tilde{\omega}_L(\xi)| \leq A \exp(Bp(\xi)), \quad \xi \in L,$$

and the restriction of $\tilde{\omega}_L$ to $V \cap L$ is equal to the restriction of ω to $V \cap L$. Then there exists a $\tilde{\lambda} \in A_p$ such that $\tilde{\omega}$ restricted to V coincides with ω. Applying this result to our case with $F = f$ and $\omega = \lambda$, we see that λ has a global extension to the space A_p and thus V is an interpolating variety.

The proof is thus complete. □

Proof of Theorem 2.9. By Riesz's convexity theorem ([BG1, 4.4.27]), (2.2) and the subharmonicity of $p(z)$ imply that $p(e^r)$ is convex and $p(r)$ is increasing.

Also, by [BMT, 1.7 & 1.8], for any continuous and increasing function $\beta(r)$: if $\beta(r)$ satisfies (2.1) and (2.3) and $\beta(e^r)$ is convex then, for any function g: $[0, \infty) \to [0, \infty)$ satisfying $g(r) = o(\beta(r))$ as $r \to \infty$, there exists an increasing function $q(r)\colon [0,\infty)\to[0,\infty)$ such that $q(r)$ also satisfies (2.1) and (2.3) and $q(e^r)$ is convex, and moreover $g(r)=o(q(r)), q(r)=o(\beta(r))$ as $r\to\infty$. By (2.6), there exists a sequence $\{C_m\}$ ($m\in\mathbf{N}$) of positive numbers such that $\zeta\in V$ and $\zeta\to\infty$,

$$N(|\zeta|,\zeta,V)\le \min_{m\in\mathbf{N}}\left\{\frac{1}{m}p(\zeta)+C_m\right\}=o\{p(\zeta)\}.$$

Therefore it follows from the above result with $\beta=p$ that

$$N(|\zeta|,\zeta,V)=o(q(|\zeta|)), \tag{3.16}$$

where $q(r)\colon[0,\infty)\to[0,\infty)$ is a function that satisfies (2.1) and (2.3). Moreover, $q(e^r)$ is convex and $q(r)=o(p(r))$ as $r\to\infty$. By the fact that $h\circ u$ is subharmonic if h is convex increasing and u is subharmonic [BG1, 4.4.18], we deduce that $q(|\zeta|)=q(e^{\ln|\zeta|})$ is subharmonic. Therefore, it turns out that q is also a weight.

Now for any function ψ, analytic on V and satisfying that for any $\epsilon>0$ there exists an $A_\epsilon>0$ such that

$$|\psi(\zeta)|\le A_\epsilon\exp(\epsilon p(\zeta)),$$

there exists a sequence $\{D_m\}$ of positive numbers such that

$$|\psi(\zeta)|\le\exp\left(\frac{1}{m}p(\zeta)+D_m\right).$$

Therefore, by the same argument as before, we can find a weight q_1 with $q_1(r)=o\{p(r)\}$ as $r\to\infty$ such that

$$|\psi(\zeta)|\le\exp(q_1(\zeta))\le\exp(q_1(\zeta)+q(\zeta)).$$

Since (3.16) implies that V is an interpolating variety for A_{q+q_1} (by Theorem 2.4), there exists a function $\Psi\in A_{q+q_1}\subset A_p^0$ such that $\Psi|_V=\psi$. This shows that V is an interpolating variety for A_p^0 by Definition 2.3. The proof is thus complete. □

Proof of Theorem 2.10. The proof is similar to the one for Theorem 2.9. □

4. Interpolating Varieties Defined by Exponential Polynomials

In this section we present a class of interpolating varieties defined by exponential polynomials. An *exponential polynomial* is an entire function f in $\mathbf{C}^n$ of the form

$$f(\zeta)=\sum_{j=1}^{N}p_j(\zeta)e^{\alpha_j\cdot\zeta}, \tag{4.1}$$

where p_j are polynomials in $\mathbf{C}^n$, called the *coefficients* of f; $\alpha_j\in\mathbf{C}^n$ are called the *frequencies* of f; and $\alpha_j\cdot\zeta$ is the bilinear product of α_j and ζ. We assume nonzero p_j and that the α_j are distinct (cf. [BY]).

In the 1-dimensional case, it is known (see [BG2] and [BT1]) that $V=f^{-1}(0)=\{z_k\}$ $(k\in\mathbf{N})$ is an interpolating variety for A_p, $p(z)=|z|$, if the zeros of f are well-separated in the sense that

$$|z_j-z_k|\geq\epsilon_1\exp(-C_1p(z_k)),\quad j\neq k,$$

for two constants $\epsilon_1>0$ and $C_1>0$. This is equivalent to saying that there exist $\epsilon>0$ and $C>0$ such that, for $\epsilon_k=\epsilon\exp(-Cp(z_k))$,

$$N(\epsilon_k,z_k,f^{-1}(0))\leq Ap(z_k)+B$$

for some constants $A,B>0$. We will see that this result is also true in the higher-dimensional case. That is, we have the following.

PROPOSITION 4.1. *Let $V=f^{-1}(0)$ with f as given in* (4.1) *and $\nabla f(\zeta)\neq 0$ on V. Then V is an interpolating variety for A_p, where $p(\zeta)=|\zeta|$, if*

$$N(\epsilon(\zeta),\zeta,V)\leq A|\zeta|+B,\quad \zeta\in V, \tag{4.2}$$

for some $A,B>0$, where $\epsilon(\zeta)=\epsilon\exp(-Cp(\zeta))$ and ϵ, C are two fixed positive constants.

Proof. Let ζ_0 be a point in V and let

$$\mathcal{L}=\{\zeta=\zeta_0+az\colon z\in\mathbf{C}\}$$

be a line through ζ_0, where $a\in\mathbf{C}^n$ is a unit vector. Then, for almost all such lines $\mathcal{L}$ except a lower-dimension set (i.e., a subset of the sphere $\mathbf{S}^{2n-1}$ of lower dimension), we have that

$$\mathcal{L}\cap V\cap B(\zeta_0,r)$$

for $r>0$ has finitely many points. Consider

$$f|_{\mathcal{L}}=\sum_{j=1}^{N}p_j(\zeta_0+az)e^{\alpha_j\cdot(\zeta_0+az)},$$

the restriction of f to the line $\mathcal{L}$, and write

$$\alpha_j=(\alpha_1^j,\dots,\alpha_n^j);$$
$$a=(a_1,\dots,a_n);$$
$$\zeta_0=(\zeta_1^0,\dots,\zeta_n^0).$$

Then

$$\alpha_j\cdot(\zeta_0+az)=(\alpha_1^ja_1+\cdots+\alpha_n^ja_n)z+(\alpha_1^j\zeta_1^0+\cdots+\alpha_n^j\zeta_n^0). \tag{4.3}$$

Recall [BG2] that for any exponential polynomial

$$F(z)=\sum_{j=1}^{N}q_j(z)e^{\beta_jz}$$

in $\mathbf{C}$ and $z_0\in\mathbf{C}$,

$$n(r,z_0,Z(F))\leq 2d^0F+\frac{4\Omega_F}{\pi}r,$$

where $d^0F = m_1 + \cdots + m_N + N - 1$ with m_j being the degree of q_j and

$$\Omega_F = \max_{1 \le j \le N}\{|\beta_j| : \beta_j \text{ the frequencies of } F\}.$$

Applying this result to the function $f|_{\mathcal{L}}$, we obtain

$$n(r, \zeta_0, V \cap \mathcal{L}) \le 2d^0 f|_{\mathcal{L}} + \frac{4\Omega_{f|_{\mathcal{L}}}}{\pi} r,$$

where

$$d^0 f|_{\mathcal{L}} \le n_1 + \cdots + n_N + N - 1 := d^0 f$$

with n_j being the degree of $p_j(\zeta)$, and that, by (4.3),

$$\begin{aligned}\Omega_{f|_{\mathcal{L}}} &\le \max_{1 \le j \le N}\{|\alpha_1^j a_1 + \cdots + \alpha_n^j a_n|\} \\ &\le \max_{1 \le j \le N}\{|a_1^j| + \cdots + |a_n^j|\} := \Omega_f.\end{aligned}$$

Therefore, for $r \ge 0$,

$$n(r, \zeta_0, V \cap \mathcal{L}) \le 2d^0 f + \frac{4\Omega_f}{\pi} r.$$

Notice that

$$n(r, \zeta_0, V) - n(0, \zeta_0, V) = \int_{\mathbf{P}^{n-1}} n(r, \zeta_0, V \cap \mathcal{L})\omega^{n-1},$$

where the form $\omega = dd^c \log|\zeta|$, and that $\int_{\mathbf{P}^{n-1}} \omega^{n-1} = 1$ (see e.g. [Gr]). We therefore have

$$n(r, \zeta_0, V) - n(0, \zeta_0, V) \le 2d^0 f + \frac{4\Omega_f}{\pi} r,$$

and so, by (4.2), that

$$\begin{aligned} &N(|\zeta_0|, \zeta_0, V) \\ &\quad = N(\epsilon(\zeta_0), \zeta_0, V) + \int_{\epsilon(\zeta_0)}^{|\zeta_0|} \frac{n(r, \zeta_0, V) - n(0, \zeta_0, V)}{r}\, dr + n(0, \zeta_0, V)\log|\zeta_0| \\ &\quad \le A|\zeta_0| + B + \frac{4\Omega_f}{\pi}(|\zeta_0| - \epsilon(\zeta_0)) + 2d^0 f \log\frac{|\zeta_0|}{\epsilon(\zeta_0)} + \log|\zeta_0| \\ &\quad \le A|\zeta_0| + B. \end{aligned}$$

It follows from Theorem 2.4 that V is an interpolating variety for A_p, $p(\zeta) = |\zeta|$. □

An easy example of a variety defined by an exponential function satisfying condition (4.2) is a union of parallel planes $\mathcal{P}_j$ given by $V = f^{-1}(0) = \bigcup_j \mathcal{P}_j$ with

$$\operatorname{dist}(\mathcal{P}_j, \mathcal{P}_k) \ge \epsilon \exp(-Cp(\zeta)), \quad j \ne k,$$

for some $\epsilon, C > 0$. Of course, this example can also be reduced to the single-variable case. The corresponding result for arbitrary functions of exponential type is false [BT1].

To conclude this section we note that, by using the same method, it can be shown that Proposition 4.1 also holds for the so-called pseudo-polynomials

$$f=\sum_{j=1}^{N} f_j(\zeta_1,\dots,\zeta_{n-1})\zeta_n^j, \tag{4.4}$$

with

$$f_j=f_j(\zeta')=\sum_{k=1}^{N_j} p_{j,k}(\zeta')e^{\alpha_{j,k}\cdot\zeta'}$$

exponential polynomials, $\zeta'=(\zeta_1,\dots,\zeta_{n-1})$. That is, we have our next result.

PROPOSITION 4.2. *Let* $V=f^{-1}(0)$, *with* f *given as in* (4.4) *and* $\nabla f(\zeta)\neq 0$ *on* V. *Then* V *is an interpolating variety for* A_p, *where* $p(\zeta)=|\zeta|$, *if*

$$N(\epsilon(\zeta),\zeta,V)\leq A|\zeta|+B,\quad \zeta\in V,$$

for some $A,B>0$, *where* $\epsilon(\zeta)=\epsilon\exp(-Cp(\zeta))$ *with* ϵ, C *two positive constants.*

Proof. Just as in the proof of Proposition 4.1, let ζ_0 be a point in V and

$$\mathcal{L}=\{\zeta=\zeta_0+az: z\in\mathbf{C}\}$$

be a line through ζ_0, where $a=(a',a_n)\in\mathbf{C}^n$ is a unit vector. Consider the restriction $f|_{\mathcal{L}}$ of the function

$$f=\sum_{j=1}^{N}\sum_{k=1}^{N_j}(p_{j,k}(\zeta')e^{\alpha_{j,k}\cdot\zeta'})\zeta_n^j \tag{4.5}$$

to the line $\mathcal{L}$, which is clearly an exponential polynomial in one variable, and write

$$\alpha_{j,k}=(\alpha_1^{j,k},\dots,\alpha_{n-1}^{j,k});$$
$$a=(a_1,\dots,a_n);$$
$$\zeta_0=(\zeta_1^0,\dots,\zeta_n^0).$$

Then

$$\alpha_{j,k}\cdot(\zeta_0+az)'=(\alpha_1^{j,k}a_1+\dots+\alpha_{n-1}^{j,k}a_{n-1})z+(\alpha_1^{j,k}\zeta_1^0+\dots+\alpha_{n-1}^{j,k}\zeta_{n-1}^0), \tag{4.6}$$

where $(\zeta_0+az)'$ denotes the first $n-1$ coordinates of (ζ_0+az). We then have that

$$n(r,\zeta_0,V\cap\mathcal{L})\leq 2d^0f|_{\mathcal{L}}+\frac{4}{\pi}\Omega_{f|_{\mathcal{L}}}r,$$

where $d^0f|_{\mathcal{L}}$ and $\Omega_{f|_{\mathcal{L}}}$ are defined similarly as in the proof of Proposition 4.1. By (4.5), it is easy to check that

$$d^0f|_{\mathcal{L}}\leq\sum_{j=1}^{N}\sum_{k=1}^{N_j}(n_{j,k}+j)+\sum_{1}^{N}N_j-1:=d^0f$$

with $n_{j,k}$ being the degree of $p_{j,k}$, and that by (4.6)

$$\Omega_{f|_{\mathcal{L}}} \le \max_{1\le j\le N}\max_{1\le k\le N_j}\{|\alpha_1^{j,k}a_1+\cdots+\alpha_{n-1}^{j,k}a_{n-1}|\}$$
$$\le \max_{1\le j\le N}\max_{1\le k\le N_j}\{|\alpha_1^{j,k}|+\cdots+|\alpha_{n-1}^{j,k}|\} := \Omega_f.$$

Therefore, for $r\ge 0$,

$$n(r,\zeta_0,V\cap\mathcal{L}) \le 2d^0 f+\frac{4\Omega_f}{\pi}r.$$

Next, by the exact same reasoning as in the proof of Proposition 4.1, we deduce that

$$N(|\zeta_0|,\zeta_0,V)\le A|\zeta_0|+B.$$

It then follows from Theorem 2.4 that V is an interpolating variety for A_p, $p(\zeta)=|\zeta|$. □

We note that similar results for exponential polynomials and pseudo-polynomials corresponding to Theorems 2.7, 2.9, and 2.10 also hold. We omit the details here.

References

[BD] C. A. Berenstein and M. Dostal, *Analytically uniform spaces and their applications to convolution equations,* Lecture Notes in Math., 256, Springer, Berlin, 1972.

[BG1] C. A. Berenstein and R. Gay, *Complex variables, an introduction,* Springer, New York, 1991.

[BG2] ———, *Complex variables, an introduction,* vol. II, Springer, New York, 1995.

[BL] C. A. Berenstein and B. Q. Li, *Interpolating varieties for spaces of meromorphic functions,* J. Geom. Anal. 5 (1995), 1–48.

[BLV] C. A. Berenstein, B. Q. Li, and A. Vidras, *Geometric characterization of interpolating varieties for the (FN)-space A_p^0 of entire functions,* Canad. J. Math. 47 (1995), 28–43.

[BT1] C. A. Berenstein and B. A. Taylor, *A new look at interpolating theory for entire functions of one variable,* Adv. in Math. 33 (1979), 109–143.

[BT2] ———, *On the geometry of interpolating varieties,* Sem. Lelong-Skoda, Lecture Notes in Math., 919, pp. 1–25, Springer, Berlin, 1982.

[BT3] ———, *Interpolation problems in $\mathbf{C}^n$ with application to harmonic analysis,* J. Analyse Math. 38 (1980), 188–254.

[BY] C. A. Berenstein and A. Yger, *Ideals generated by exponential polynomials,* Adv. in Math. 60 (1986), 1–80.

[Be] B. Berndtsson, *A formula for interpolation and division in $\mathbf{C}^n$*, Math. Ann. 263 (1983), 399–418.

[BMT] R. W. Braun, R. Meise, and B. A. Taylor, *Ultradifferentiable functions and Fourier analysis,* Resultate Math. 17 (1990), 206–237.

[De] J. P. Demailly, *Scindage holomorphe d'un morphisme de fibrés vectoriels semi-positifs avec estimation L^2*, Sem. Lelong-Skoda, Lecture Notes in Math., 919, pp. 77–107, Springer, Berlin, 1982.

[Gr] P. Griffiths, *Entire holomorphic mappings in one and several complex variables,* Ann. of Math. Stud., 85, Princeton Univ. Press, Princeton, NJ, 1976.
[Le] B. J. Levin, *Distribution of zeros of entire functions,* Amer. Math. Soc., Providence, RI, 1964.
[Oh] T. Ohsawa, *On the extension of L^2 holomorphic functions II,* Publ. Res. Inst. Math. Sci. 24 (1988), 265–275.
[Sk] H. Skoda, *Application des techniques L^2 à la théorie des idéaux d'une algèbre de fonctions holomorphes avec poids,* Ann. Sci. École Norm. Sup. (4) 5 (1972), 545–579.
[Sq] W. A. Squires, *Geometric condition for universal interpolation in $\hat{\mathcal{E}}'$,* Trans. Amer. Math. Soc. 280 (1983), 401–413.

C. A. Berenstein & D.-C. Cheng
Department of Mathematics
University of Maryland
College Park, MD 20742

B. Q. Li
Department of Mathematics
University of Michigan
Ann Arbor, MI 48109