The Branched Schwarz Lemma: A Classical Result via Circle Packing

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1. Introduction

Our original aim in this work was to establish certain results about circle packings which were suggested by classical complex function theory. We succeeded in doing that, but along the way we found that we could in fact extend the classical results themselves, independently establishing a theorem of Z. Nehari.

A circle packing is a collection of circles in the plane with a prescribed pattern of tangencies satisfying certain combinatoric conditions, which will be described in a moment. Connections between circle packings and analytic functions were introduced by W. Thurston in 1985. The seminal paper in this topic is the proof by Rodin and Sullivan [RS] of Thurston's conjecture on the approximation of conformal mappings via circle packings. Subsequent work by several researchers has refined the approximation results, but has also suggested the possibility of developing a "discrete analytic function" theory based on circle packings which would parallel classical analytic function theory. Here, a thorough mixing of the proofs of certain fundamental classical, discrete, and approximation results suggests that the emerging discrete theory provides far more than mere analogy with its classical counterpart.

It is certainly best for the reader if we state the classical results first. A finite Blaschke product is an n-to-1 proper mapping of the unit disc $\mathbf{D}$ onto itself for some positive integer n and $\operatorname{br}(f)$ denotes the set of branch points of an analytic function f, counting multiplicities.

SCHWARZ'S LEMMA (Branched). *Let $f, b: \mathbf{D} \to \mathbf{D}$ be analytic, with $f(0) = 0 = b(0)$, and assume b is a finite Blaschke product. If $\operatorname{br}(b) \subseteq \operatorname{br}(f)$, counting multiplicities, then $|f'(0)| \leq |b'(0)|$. If $|b'(0)| \neq 0$, then equality holds iff $f \equiv \lambda b$ for some unimodular constant λ.*

DISTORTION LEMMA (Branched). *Let $f: \mathbf{D} \to \mathbf{C}$ be analytic with $f(0) = 0$ and let $r > 0$. Write Ω for the component of $f^{-1}(r\mathbf{D})$ containing 0 and assume that the restriction $f|_{\Omega}: \Omega \to r\mathbf{D}$ is a proper mapping. Let b be a finite*

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Blaschke product satisfying $\text{br}(b) \equiv \text{br}(f) \cap \Omega$, *counting multiplicities, and* $b(0) = 0$. *Then* $|f'(0)| \geq r|b'(0)|$. *When* $|b'(0)| \neq 0$, *equality holds iff* $f \equiv (\lambda r)b$ *for some unimodular constant* λ.

When b is the identity function, $b(z) = z$, these statements reduce to standard versions which are well known. Indeed, the Schwarz lemma is one of the true pillars of complex analysis. The extension here is apparently due to Nehari [Ne, p. 1037], while the companion distortion lemma follows easily from it via subordination.

We do not consider these classical results to be the main goal of this paper—our interest is in circle packing. However, we found a very pleasing and potentially valuable interplay between the two theories as our investigation developed. That interplay is the common thread running through the paper. Briefly, the chain of events was this: The discrete Schwarz lemma (DSL), a circle-packing version of the standard Schwarz lemma, had been shown in [BS2] and proved to be very valuable. Motivated by that, we formulated and proved a discrete distortion lemma (DDL). The DSL and DDL immediately proved their worth by leading to two theorems on circle packing, one concerned with the approximation of derivatives of analytic functions and the other concerned with uniqueness of extremal packings. These were our putative goals. We then observed, however, that the proofs of DSL and DDL could be generalized quite naturally and easily to incorporate discrete branch points, leading to their "branched" versions. At that point things naturally switched around—now the discrete results motivated extensions of the classical lemmas, leading ultimately to the two results just given. Moreover, we proved the classical extensions from the discrete ones via approximation.

Nehari's work was subsequently pointed out to us by C. D. Minda. Nonetheless, we feel that this paper may represent a new chapter in the topic of circle packing, which is itself quite new: It is true that *techniques* arising in circle packing have been used by He and Schramm [HS] in a major advance on Koebe's "Kreisnormierungsproblem". However, here we see the *discrete theory* itself both inspiring and helping to prove classical results; in the future, such "classical" results may turn out to be new.

The paper begins with a review of the definitions and notation associated with circle packing; the reader is assumed to be familiar with the basics of the topic, as presented in [BS1], for example. In Section 3 we formulate the discrete versions of the standard Schwarz and distortion lemmas, along with two theorems that should be valuable in the continuing development of circle packing. One relates to the approximation of derivatives of analytic functions by "ratio functions", and is applied later in this paper; the other establishes the uniqueness of certain extremal hyperbolic circle packings. Both theorems strengthen and considerably simplify results in the circle packing literature. The DDL is proven in Section 4 and the two theorems in Section 5; their proofs are quite similar in spirit to arguments in the classical setting.

In Section 6 we review the notion of branched circle packings and certain material developed in [D1; D2] concerning their use in the approximation of analytic functions. We formulate the extensions of DSL and DDL to incorporate branch points in Section 7; the proofs are essentially the same as the unbranched cases. In Section 8 we motivate the foregoing classical lemmas and then prove them from the discrete versions via approximation.

2. Circle Packing Preliminaries

We recall here the basic definitions, terminology, and notation (for details, the reader is referred to [BS1; BS2]). In manipulating a circle packing, the sizes and locations of the circles change, while the underlying pattern of tangencies is what remains invariant. That pattern is encoded in an abstract simplicial complex that we generally denote by K. A collection of circles P is said to be a *circle packing* for K if there is a one-to-one correspondence between the circles of P and the vertices of K such that two circles are (externally) tangent if the corresponding vertices share an edge of K. In particular, then, the faces of K correspond to mutually tangent triples of circles in P.

Circle packings can be defined in great generality, but those we encounter are of the garden variety: our packings consist of Euclidean circles, and throughout the paper it is assumed that the underlying complex K is (simplicially equivalent to) a triangulation of a topological disc, with the added condition that every boundary vertex has at least one interior vertex as a neighbor. (We use the term *proper complex* if we need to emphasize these conditions.) Note that K triangulates a closed disc iff it is finite, an open disc iff it is infinite and has no boundary vertices, and a disc with partial boundary otherwise. We also assume that K is oriented and that its packings P preserve that orientation; that is, if v_1, v_2, v_3 are the vertices of a face of K taken in positive order, then the corresponding circles C_1, C_2, C_3 of P form a positively oriented triple of circles in the plane. It is important to note that our definition of circle packing does not require that the circles have mutually disjoint interiors. Of course, a pair of circles associated with the endpoints of an edge of K will have disjoint interiors, since they are necessarily tangent. However, the overall pattern may have circles that overlap one another or are accidentally tangent. For purposes of illustration, four circle packings having the same (finite) complex are shown in Figure 1.

It is precisely the fact that a given complex K might give rise to many different circle packings which underlies our topic. Putting aside the question of existence for a moment, there are several pieces of terminology regarding packings that we shall need in the sequel: Each circle packing $P \subset \mathbf{C}$ of K determines a simplicial map $s_P: K \to \mathbf{C}$ by identifying each vertex of K with the center of the corresponding circle of P and extending using barycentric coordinates. The image of K under s_P is therefore a union of Euclidean triangles whose vertices are center points of the circles—that is, it forms a

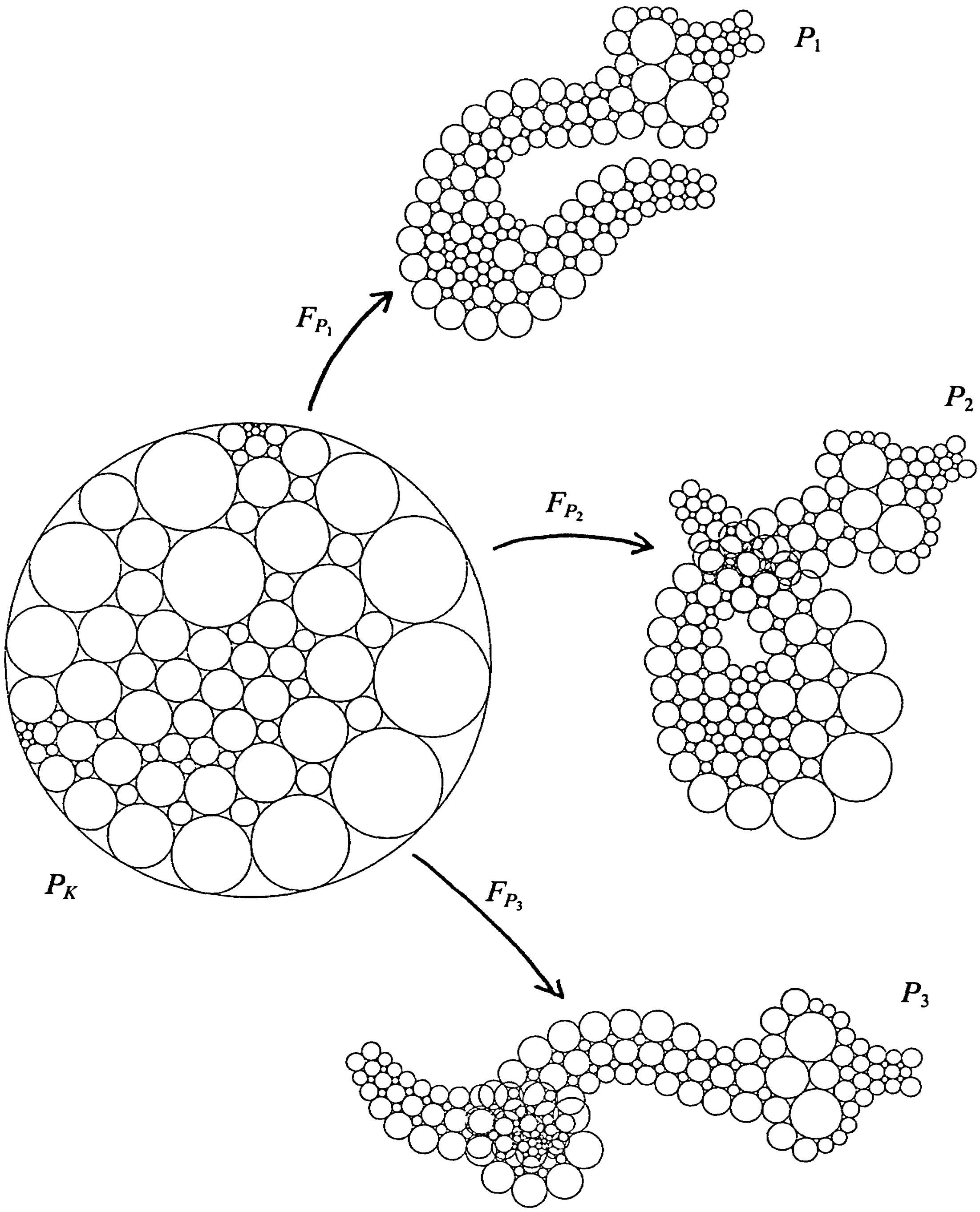

Figure 1 Three packings of the same complex

geometric complex in **C** that is simplicially equivalent to K. We refer to it as the (Euclidean) *carrier* of P, carr(P). A *flower* is a portion of P consisting of a circle associated with a vertex v of K, called the *center* circle, and the tangent circles associated with the neighbors of v, called the *petals*. The number of petals is the *degree* of v. For interior vertices, the petals will necessarily wrap around the center some integral number n times; if $n \geq 2$, then the center of the circle represents a *branch point* of s_P having *multiplicity* n and *order* $n-1$. One easily verifies that s_P is an open, continuous, light-interior, orientation-preserving, possibly branched mapping of K into C; we will say that s_P is an *immersion* of K (whether branched or locally one-to-one). The packing P will be called *univalent* if this immersion is in fact an embedding. A useful sufficient condition for univalence is that the circles of P have mutually disjoint interiors; however, note that this is not necessary, since boundary circles of P might well overlap some others without causing the triangles in the carrier of P to overlap. In Figure 1, P_1 and P_2 are univalent packings.

Regarding existence, it is an important fact from the general theory of circle packing [BS1] that for each complex K there exists an essentially unique extremal circle packing P_K associated with K, termed the *maximal* packing (commonly called the *Andreev* packing when K is finite). Because K, as a proper complex, triangulates a disc, P_K lies in $\mathbf{C}$ and is univalent. Two mutually exclusive possibilities exist: either $\text{carr}(P_K) = \mathbf{C}$ or $\text{carr}(P_K) \subseteq \mathbf{D}$, and these cases are described as *parabolic* or *hyperbolic*, respectively. For instance, the complex K having constant degree 6 is that associated with a regular hexagonal lattice in $\mathbf{C}$; it is parabolic, having as its maximal packing the regular hexagonal packing of $\mathbf{C}$ by circles of constant radius. On the other hand, the complex having constant degree 7 is hyperbolic (see [BS3]). Various criteria have been studied for distinguishing these types, but for purposes of the present paper it is enough to note that, if K has boundary vertices or if there exists a circle packing for K whose carrier is bounded, then K is necessarily hyperbolic. Indeed, the only parabolic complex we will encounter herein is that underlying the regular hexagonal packing. In Figure 1, the circle packing on the left is the maximal packing for the complex underlying all four of the packings, though it may appear to have fewer circles because some of them are too small to be seen.

Suppose now that we have two circle packings P and Q for the same complex K. There is a natural map between these collections, namely, that which identifies each circle of Q with the corresponding circle of P. It would be our preference to treat this as an instance of a "discrete analytic function". Such a formulation is rather more abstract than necessary for this paper, however, so we will work with more explicit point mappings. Proceed as follows: For each vertex $v \in K$, write $z_Q(v)$, $z_P(v)$ and $r_Q(v)$, $r_P(v)$ for the centers and radii of the corresponding circles from Q and P, respectively. (Later in the paper we will rely heavily on hyperbolic data for circles in the unit disc $\mathbf{D}$, but unless stated otherwise, all centers and radii are Euclidean.) The simplicial maps s_Q and s_P from K to Q and P have already been described.

DEFINITION. The *circle packing map* from Q to P is the simplicial map $F_{Q,P}$ defined by $F_{Q,P} = s_P \circ s_Q^{-1}\colon \text{carr}(Q) \to \text{carr}(P)$. The corresponding *ratio map* $F^{\#}_{Q,P}\colon \text{carr}(Q) \to \mathbf{R}$ is defined on the set of centers of Q by $F^{\#}_{Q,P}(z_Q(v)) = r_P(v)/r_Q(v)$, $v \in K$, and is extended affinely to faces.

The mapping $F_{Q,P}$ is piecewise affine, with $F_{Q,P}(z_Q(v)) = z_P(v)$ for all $v \in K$, and it serves as a pointwise version of a discrete analytic function. In this same vein, the ratio function $F^{\#}_{Q,P}$ plays a role parallel to the modulus of the derivative of an analytic function, since at the center of each circle it gives the factor by which that circle is stretched or shrunk by $F_{Q,P}$. Note that $F_{Q,P}$ has been extended to faces purely for the convenience of having a point function on all of carr(Q).

The typical situation we encounter is that in which $Q = P_K$, the maximal packing, and in this case we replace the bulky notation $F_{P_K,P}$ by F_P. It may help the reader to refer to Figure 1 here and to have the parallels with the classical setting laid out explicitly: The packing on the left in Figure 1 is a maximal packing P_K, while on the right are three additional packings, P_1, P_2, P_3. P_K is the common domain for the three circle packing maps F_{P_j}, $j = 1, 2, 3$. Mentally, one should make the following identifications: P_K is the unit disc; F_{P_j} is an analytic function on the unit disc; P_j is the Riemann image surface of F_{P_j}; and $F^{\#}_{P_j}$ is $|F'_{P_j}|$. Note that F_{P_1} is univalent, F_{P_2} is 2-valent but locally univalent, while F_{P_3} is 2-valent and has one branch point.

3. Statements

We will be working primarily with the maps F_P. At almost every stage in our development, it is advantageous to mentally identify F_P as an analytic function on $\mathbf{D}$, as described in the preceding paragraph. Indeed, the original motivation for this paper was to show that there is real substance to the analogy by establishing discrete versions of the following two classical results.

CLASSICAL SCHWARZ LEMMA. *Let $f: \mathbf{D} \to \mathbf{D}$ be analytic with $f(0) = 0$. Then $|f'(0)| \leq 1$, with equality iff $f(z) = \lambda z$ for some unimodular constant λ.*

CLASSICAL DISTORTION LEMMA. *Let f be a univalent analytic function mapping $\mathbf{D}$ onto an open set $\Omega \subset \mathbf{C}$, with $f(0) = 0$. If Ω contains the disc $\{|w| < r\}$ for $r > 0$, then $|f'(0)| \geq r$, with equality iff $f(z) = (\lambda r)z$ for some unimodular constant λ.*

In the discrete setting, we will be given a proper complex K and an associated packing P. The normalization $F_P(0) = 0$ means that in both P and P_K the circle associated with some designated interior vertex $v_0 \in K$ is centered at the origin; this can always be arranged by applying appropriate Möbius transformations to $\mathbf{D}$ and/or $\mathbf{C}$. A discrete version of the Schwarz–Pick lemma was proven for finite complexes in [BS2]; in fact, the proofs go through for infinite complexes as in [BS1, Lemma 5] (even without the bounded degree restriction). We require only the "Schwarz" portion of the statement.

DISCRETE SCHWARZ LEMMA. *Let $P \subset \mathbf{D}$ be a circle packing for K and suppose the associated circle packing map F_P satisfies $F_P(0) = 0$. Then $F^{\#}_P(0) \leq 1$, with equality iff P is a rotated copy of P_K, that is, $P = \lambda P_K$ for some unimodular constant λ.*

Our first goal is to formulate and prove a discrete analog of the distortion lemma. We'll need to introduce one piece of terminology: Suppose P is a circle packing of K in $\mathbf{C}$ and that the circle for vertex v_0 is centered at the origin. Let V be the set of vertices of K whose circles in P intersect $\{|w|=r\}$. We say that P *properly covers* the disc $r\mathbf{D}$ (relative to v_0) if V and the edges of K spanned by V topologically separate v_0 from the boundary of K. That is, no edge path in K starting at v_0 can reach a boundary vertex or approach the ideal boundary without encountering a vertex of V. This is the discrete version of the classical condition that the component of $f^{-1}(r\mathbf{D})$ containing the origin be compact.

DISCRETE DISTORTION LEMMA. *Let P be a univalent circle packing for a hyperbolic complex K and assume that the associated function F_P satisfies $F_P(0)=0$. If P properly covers the disc $\{|w|<r\}$ for $r>0$, then $F_P^\#(0)\geq r$, with equality iff $P=(\lambda r)P_K$ for some unimodular constant λ.*

The proof of the classical case is an easy subordination argument, relying on a composition and an application of the Schwarz lemma. For the discrete result, we use elementary geometric arguments, the discrete Schwarz lemma, and the Perron-type arguments used in [Bo]. A close look will show that these entail a form of discrete "subordination", though the notion of composition is not strictly available in the discrete setting. The proof is carried out in the next section.

Aside from the DDL itself, the main circle-packing results of the paper are the following two theorems, which will be proved in Section 5.

THEOREM 1. *Let $\{K_n\}$ be a sequence of hyperbolic complexes and $\{P_n\}$ a sequence of associated circle packings. Assume that the supremum of the Euclidean radii for circles of the maximal packings P_{K_n} goes to zero uniformly on compacta (of $\mathbf{D}$) and that the circle packing maps F_{P_n} converge uniformly on compacta to an analytic function $f:\mathbf{D}\to\Omega$ as n goes to infinity. Then the ratio functions $F_{P_n}^\#$ converge uniformly on compacta to $|f'|$.*

The conclusion of Theorem 1 was established for packings having hexagonal combinatorics (i.e., in the setting of Thurston's original conjecture on approximation of conformal maps) in work initiated by Rodin [R1] and completed by [He] (see also [Ah]). However, Thurston's conjecture has been shown to hold for circle packings having more general combinatorics (see [St] and [HR]), and more recently approximation has been extended to situations involving branch points [D1; D2]. Theorem 1 applies in all these settings, and, moreover, its proof is entirely in the spirit of complex function theory.

THEOREM 2. *Assume that $P\subset\mathbf{D}$ is a circle packing for a hyperbolic complex K with the property that the hyperbolic metric induced on K by the immersion of P in $\mathbf{D}$ is complete and has constant curvature -1. Then P is Möbius equivalent to the maximal packing P_K.*

The hypotheses of Theorem 2 mean that P "fills" $\mathbf{D}$ in the sense of [BS1]; we will make this more precise as we begin the proof. This theorem completes the picture regarding the uniqueness of maximal packings: The spherical (elliptic) case was covered by the Koebe–Andreev–Thurston theorem and the parabolic case by Sullivan's uniqueness theorem (for bounded degree cases) and more generally by Schramm's rigidity results (see [Sc]). The hyperbolic case for complexes K without boundary was proven by Rodin [R2] for bounded degree, using results of Sullivan and He, and also follows for arbitrary degree from [Sc]. The case of infinite complexes with boundary had remained open. Theorem 2, however, applies to all hyperbolic cases—complexes of arbitrary degree and with or without boundary. In addition, the arguments again parallel the classical ones, and they are easily extended to yield uniqueness of maximal branched packings.

4. Proof of the Discrete Distortion Lemma

The proof of the discrete distortion lemma (DDL) will be carried out in hyperbolic geometry. We review appropriate definitions and notation, following [BS2], and then recall the conclusions of the discrete Schwarz–Pick lemma which are needed here.

A *radius function* for a hyperbolic complex K is a collection R of hyperbolic radii, one associated with each vertex of K. The radii for the three vertices of a face of K can be realized in an essentially unique way as the radii of a triple of mutually tangent circles in the hyperbolic plane, and the hyperbolic metric on the triangle they form can be lifted to give a metric on the face in K. Obtaining these metrics on all faces and pasting them together isometrically along shared edges induces a hyperbolic structure on K (as a topological space), generally with cone-type singularities at interior vertices. The resulting metric space is denoted $K(R)$ and called a *labeled complex*. Each vertex $v \in K$ has an angle sum $\theta_v(R)$ in this structure, defined as the sum of the angles at v in the star of faces containing v. Write the set of vertices of K as the disjoint union $K^{(0)} = K_{\text{int}} \cup K_{\text{bdy}}$ of interior and boundary vertices. We say that $K(R)$ is a *superpacking* if $\theta_v(R) \leq 2\pi$ for all $v \in K_{\text{int}}$, a *subpacking* if $\theta_v(R) \geq 2\pi$ for all $v \in K_{\text{int}}$, and a *packing* if $\theta_v(R) = 2\pi$ for all $v \in K_{\text{int}}$. (We will extend these notions to include branched packings later.)

If P is a circle packing of K whose circles lie in $\mathbf{D}$, then the set R of hyperbolic radii of P defines a radius function such that $K(R)$ is a packing. (Note that horocycles, circles internally tangent to $\partial\mathbf{D}$, are naturally interpreted as circles of infinite hyperbolic radius in $\mathbf{D}$.) Conversely, if $K(R)$ is a packing, then there exists a circle packing P for K having the radii of R, with P being uniquely determined up to orientation and Möbius transformations of $\mathbf{D}$. We study circle packings quantitatively, therefore, by studying their radius functions. The radius function for the maximal packing of K will be denoted R_K.

The following proposition summarizes the important results about circle packings; the reader is referred to [BS2] and [Bo].

PROPOSITION 1. *Let K be a hyperbolic complex, and let R_K be the radius function for its maximal packing.*

(i) *If $K(R)$ is a subpacking then $R \leq R_K$; moreover, equality at any interior vertex implies $R \equiv R_K$.*

(ii) *If L is a subcomplex of K which is itself a proper complex, and if R is the restriction of R_K to the vertices of L, then $R \leq R_L$. Equality for any interior vertex of L implies $L \equiv K$ and $R \equiv R_K$.*

Assuming that K is finite:

(iii) *Given any function g defined on K_{bdy} and taking values in $(0, \infty]$, there exists a unique radius function R that agrees with g at boundary vertices and for which $K(R)$ is a packing.*

(iv) *Assume that $K(R_1)$ is a superpacking, that $K(R_2)$ is a subpacking, and that $R_1 \geq R_2$ on K_{bdy}. Then $R_1 \geq R_2$ on all vertices of K, and equality at any interior vertex implies $R_1 \equiv R_2$.*

Part (i) of the proposition contains our Schwarz lemma and justifies the term "maximal" for P_K. Part (iii) guarantees a huge variety of circle packings for finite complexes. Note in particular that the radius function R_K for the maximal packing is uniquely determined by solving the boundary value problem with infinite boundary radii. The other parts of the proposition may be described as "monotonicity" results, concerned with how radii of packings change as the complex or boundary radii are changed. We will extend and add to these results when we discuss branched packings in Section 6.

Proof of DDL. It suffices to assume that $r = 1$, so that the circle packing P of the statement properly covers $\mathbf{D}$, and to prove that the circle packing map F_P from P_K to P satisfies $F_P^\#(0) \geq 1$. Recall that an interior vertex v_0 has been designated and that the corresponding circles $\tilde{c}_0$ and c_0 of P_K and P, respectively, are centered at the origin. We want to show that the Euclidean radius of c_0 is greater than or equal to that of $\tilde{c}_0$.

Our first task is to restrict attention to a subcomplex L of K associated with the part of P in $\mathbf{D}$. It is clearly sufficient to assume that the circle for v_0 lies inside $\mathbf{D}$. Let V denote the set of vertices of K whose circles in P intersect the unit circle $\partial\mathbf{D}$, let G be the edge-connected component of $K^{(0)} \backslash V$ containing v_0, and let L be the closure in K of the union of all faces having one or more vertices in G. Because P properly covers $\mathbf{D}$, one may easily verify the following properties associated with L:

(a) ∂L is a simple closed chain of edges of K connecting vertices from V. In particular, L is a finite proper complex.

(b) The circles of P associated with boundary vertices of L intersect the unit circle, while the circles associated with interior vertices of L, including c_0, lie in $\mathbf{D}$.

Let Q denote the collection of circles from P corresponding to vertices in L. Then clearly Q is a circle packing for L. Let R be a radius function for Q

prescribed as follows: For each $v \in L_{\text{int}}$, the corresponding circle of Q lies in $\mathbf{D}$, and we set $R(v)$ equal to its hyperbolic radius; for each $w \in L_{\text{bdy}}$, set $R(w) = \infty$.

We claim that $L(R)$ is a superpacking. If v is an interior vertex of L having only interior neighbors, then the corresponding circle of Q is the center of a flower of circles all lying entirely in $\mathbf{D}$; since their hyperbolic radii have been used as the entries in R, the fact that the circles themselves fit together to form a flower implies that $\theta_v(R) = 2\pi$. Therefore, we need only check the angle sum condition for an interior vertex $v \in L$ having one or more boundary vertices as neighbors. Let c be the circle of Q corresponding to such a v. Apply (to the Riemann sphere) a Möbius transformation T that maps $\mathbf{D}$ to itself and moves c to a circle c' centered at the origin. We need to look at the resulting flower centered now on c'. Since the restriction of T to $\mathbf{D}$ is a hyperbolic isometry, c' and any petals that lie in $\mathbf{D}$ will retain their original hyperbolic radii, which were used in the radius function R. On the other hand, any petals that touch or cross $\partial\mathbf{D}$, precisely those associated with boundary vertices of L, will have been assigned infinite radius in R. It is evident that if these latter petals are replaced by horocycles tangent to c', then the angles formed at c' can only decrease (see Figure 2). Consequently, $\theta_v(R) \leq 2\pi$. Indeed, if any petal intersects $\bar{\mathbf{D}}^c$, then this inequality will be strict. We now have

$$R_K(v_0) \leq R_L(v_0) \leq R(v_0), \tag{1}$$

where the first inequality follows from Proposition 1(ii) and the second from (iv). But $R(v_0)$ is simply the hyperbolic radius of c_0 and $R_K(v_0)$ the hyperbolic radius of $\tilde{c}_0$. Since these are both centered at the origin, the inequality $R_K(v_0) \leq R(v_0)$ persists for their Euclidean radii, proving that $F_P^\#(0) \geq 1$. Equality would require two equalities in (1). The first can occur only if $K = L$ by (ii), giving $Q \equiv P$. The second would imply that no boundary circles of P $(= Q)$ extend beyond $\bar{\mathbf{D}}$, so all would be horocycles. Consequently, the radius function R that we defined for L $(= K)$ would in fact be just R_K, and we could conclude that P is the image of P_K under an automorphism of $\mathbf{D}$. Since the circles for v_0 are placed at the origin in each case, this would mean that P is a rotation of P_K. This completes the proof of the DDL for $r = 1$, and the general case follows immediately. □

Observe that our restriction to the subcomplex L is a discrete analog of classical subordination, wherein one studies the restriction of f to some subdomain $\Omega \subset \mathbf{D}$ by considering $f \circ \omega$, where ω is a conformal map $\omega: \mathbf{D} \to \Omega$. Here the map from P_L to the circles of P_K associated with L plays roughly the role of ω.

5. Proofs of Theorems 1 and 2

The proof of Theorem 1 follows very classical lines. We simply couple the meta-theorem that analytic functions map "infinitesimal circles to infinitesimal circles" with a scaled juxtaposition of DSL and DDL, which may be

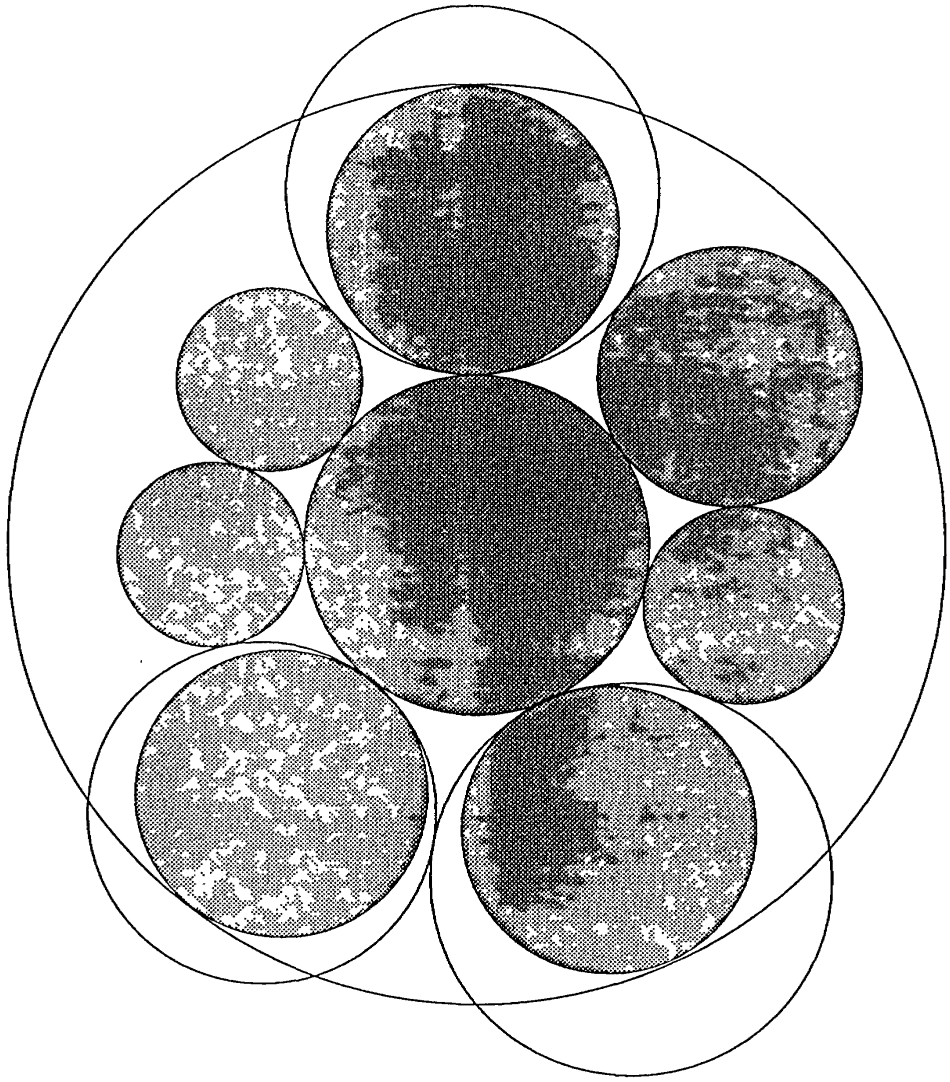

Figure 2 Adjusting a flower

paraphrased as follows: *If a univalent packing Q nearly fills a disc of radius r, then $F_Q^\#(0) \approx r$.*

The argument proceeds informally like this: Suppose $z_0 \in \mathbf{D}$, $w_0 = f(z_0)$, and $f'(z_0) \neq 0$. By the meta-theorem, a disc D_1 with very small radius r centered at z_0 is (approximately) mapped univalently to a disc D_2 of radius $|f'(z_0)|r$ centered at w_0. For large n, a portion of the packing P_{K_n} will almost fill D_1 and, because F_{P_n} approximates f, the corresponding portion of P_n will be univalent and will almost fill D_2. Two applications of the statement paraphrased above will imply that the ratio of radii for the circles at w_0 and z_0 is approximately $|f'(z_0)|r/r = |f'(z_0)|$, as desired.

The details require the following lemma, which is easily obtained by scaling the discrete Schwarz and distortion lemmas.

Lemma 1. *Let P be a univalent packing lying in the disc $r\mathbf{D} = \{|z| < r\}$ and satisfying $(1-\epsilon)r\mathbf{D} \subset \operatorname{carr}(P)$. Assume that the associated circle packing map F_P satisfies $F_P(0) = 0$. Then $(1-\epsilon)r \leq F_P^\#(0) \leq r$.*

Proof of Theorem 1. Recall that the maps F_{P_n}, being simplicial maps between immersions of K_n, are open, continuous, light-interior, possibly branched mappings. Therefore, the notion of winding number, the argument principle, the maximum principle, Hurwitz's theorem, and so forth are applicable to F_{P_n}. In particular, when the limit function f is one-to-one in a neighborhood of a compact set, then the F_{P_n} will be one-to-one on that compact set for sufficiently large n. In the following, write $\Delta(a,t)$ for the disc $\{|z-a|<t\}$.

Given a compact set $E\subset \mathbf{D}$ and $\epsilon>0$, it suffices to prove the existence of $\delta>0$ and N such that if $z_0\in E$ and z is the center of a circle of P_{K_n} for $n>N$ and $|z-z_0|<\delta$, then $|F^{\#}_{P_n}(z)-f'(z_0)|<\epsilon$. We will focus on a fixed $z_0\in \mathbf{D}$ with image $w_0=f(z_0)$; uniformity will follow routinely.

Define the circle packing $\tilde{Q}_n\subset P_{K_n}$ to consist of all circles belonging to flowers of P_{K_n} which lie entirely in $\Delta(z_0,r)$. If L_n denotes the corresponding subcomplex of K_n, then one easily checks that L_n is a finite proper complex. We will be comparing three circle packings for L_n, namely: $\tilde{Q}_n$ itself; the circle packing $Q_n\subset P_n$ corresponding to L_n; and an appropriate maximal packing P_{L_n} for L_n.

The statement that f maps infinitesimal circles to infinitesimal circles reflects the fact that given any σ, $0<\sigma\ll 1$, there exists an arbitrarily small r, $0<r<(1-|z_0|)/2$, such that one of the following holds: If $f'(z_0)=0$, then

$$|f(z)-w_0|<\sigma r^{3/2}\quad \text{for } |z-z_0|<r. \tag{2}$$

If $f'(z_0)\neq 0$, then f is univalent on $\Delta(z_0,2r)$ and satisfies

$$(1-2\sigma)r|f'(z_0)|<|f(z)-w_0|<(1+\sigma)r|f'(z_0)|\quad \text{for } (1-\sigma)r<|z-z_0|<r. \tag{3}$$

Fix σ and r. The hypotheses tell us that radii of circles of $\tilde{Q}_n$ are going to zero uniformly on $\Delta(z_0,2r)$, so we may choose n sufficiently large that

$$\Delta(z_0,(1-\sigma)r)\subset \operatorname{carr}(\tilde{Q}_n)\quad\text{and}\quad \tilde{Q}_n\subset\Delta(z_0,r). \tag{4}$$

Let v denote any vertex of L_n whose circle in $\tilde{Q}_n$ has center z lying in $\Delta(z_0,\sigma r)$. Write ρ_1 for its radius. Let P_{L_n} denote the maximal packing for L_n which has the circle for v centered at the origin, and let ρ_a be its radius. By (4) and Lemma 1, scaled by the factor r, we have

$$(1-2\sigma)r<\frac{\rho_1}{\rho_a}<(1+\sigma)r. \tag{5}$$

Finally, let ρ_2 be the radius of the circle for v in Q_n.

Our task is to compare $F^{\#}_{P_n}(z)$ to $|f'(z_0)|$. Start with the case $f'(z_0)=0$. Since the F_{P_n} converge uniformly to f on $\Delta(z_0,2r)$, given $\eta>0$, (2) implies that for n large, Q_n has diameter less than $\eta+2\sigma r^{3/2}$. The DSL implies

$$F^{\#}_{Q_n}(0)=\frac{\rho_2}{\rho_a}<\eta+2\sigma r^{3/2},$$

which along with (5) implies

$$F^{\#}_{P_n}(z) = \frac{\rho_2}{\rho_1} = \frac{(\rho_2/\rho_a)}{(\rho_1/\rho_a)} < \frac{\eta + 2\sigma r^{3/2}}{(1-2\sigma)r}, \quad z \in \Delta(z_0, \sigma r). \tag{6}$$

This can be made arbitrarily small by choosing η, σ, and r small.

Next, assume $|f'(z_0)| \neq 0$. We have chosen n so large that f is univalent on $\Delta(z_0, 2r)$. From the uniform convergence of the F_{P_n} to f, we may also assume that $|F_{P_n}(z) - f(z)| < \sigma$, $z \in \Delta(z_0, 2r)$, and that Q_n is a univalent packing. From (3) we have

$$\Delta(w_0, (1-3\sigma)r|f'(z_0)|) \subset \operatorname{carr}(Q_n) \text{ and } Q_n \subset \Delta(w_0, (1+2\sigma)r|f'(z_0)|). \tag{7}$$

Also, for large n,

$$F_{P_n}(\Delta(z_0, \sigma r)) \subset \Delta(w_0, 2\sigma r|f'(z_0)|). \tag{8}$$

In particular, $|F_{P_n}(z) - w_0| < 2\sigma r|f'(z_0)|$, and with (7) and Lemma 1,

$$(1-5\sigma)r|f'(z_0)| < \frac{\rho_2}{\rho_a} < (1+4\sigma)r|f'(z_0)|. \tag{9}$$

Combining (5) and (9) gives

$$\frac{(1-5\sigma)r|f'(z_0)|}{(1+\sigma)r} < F^{\#}_{P_n}(z) = \frac{\rho_2}{\rho_1} < \frac{(1+4\sigma)r|f'(z_0)|}{(1-2\sigma)r}.$$

Therefore,

$$|F^{\#}_{P_n}(z) - |f'(z_0)|| < \frac{6\sigma}{1-2\sigma}|f'(z_0)|, \quad z \in \Delta(z_0, \sigma r). \tag{10}$$

Again, we can make this small by choosing σ small.

To conclude the proof, choose σ, r, and η so that the right-hand sides of (6) and (10) are less than the given ϵ. In the compact set E, there are at most finitely many points where f' vanishes, and one can choose neighborhoods of those points so that (6) holds for large n. Then, in the remainder of E, one may choose r independent of z_0 to meet the conditions of (3), giving uniform neighborhoods where (10) holds for large n. The reader may verify that this establishes uniformity and completes the proof. □

COMMENTS. The convergence of the ratio functions is actually a local phenomenon, and the fact that we use maximal packings in the statement is simply a convenience. We have defined circle packing maps more generally, so one could, for instance, replace the P_{K_n} here by circle packings that exhaust some open set Ω other than $\mathbf{D}$.

Proof of Theorem 2. This is completely elementary when the hypotheses are properly interpreted. The given circle packing P for K corresponds with a radius function R for which the labeled complex $K(R)$ is a packing. (The hyperbolic structure induced on K by P is described in Section 2, but see [BS2, Sec. 2] for additional detail.) The hypothesis on completeness in Theorem 2 refers to the completeness of K in this induced metric. If K is infinite without boundary, the hypothesis simply means that P is univalent and its carrier fills $\mathbf{D}$. Indeed, the classical theorem of Killing and Hopf implies that $K(R)$ is isometrically isomorphic to $\mathbf{D}$ with the Poincaré metric. If K has boundary vertices, the hypothesis implies that the corresponding circles of P must be horocycles. In particular, the boundary vertices of K are ideal boundary points in the induced metric while the boundary edges of K are complete geodesics.

In any case, the hypothesis on completeness implies that P properly covers the disc $\{|w|<1-\epsilon\}$ for any $\epsilon>0$. By the DDL and DSL, respectively, the ratio function associated with F_P satisfies the two inequalities

$$1-\epsilon \leq F_P^{\#}(0) \leq 1.$$

Since this holds for $\epsilon>0$, we conclude that $F_P^{\#}(0)=1$. In particular, the circles at the origin in P and P_K have the same Euclidean and hence the same hyperbolic radii. By Proposition 1(i), this implies that the hyperbolic radius functions for P and P_K are identical. Since a hyperbolic circle packing is determined, up to normalizations, by its radius function, one easily concludes that P is a rotated image of P_K, and Theorem 2 is established. □

6. Branched Circle Packings

Let K be a complex, either hyperbolic or parabolic. Recall that a branch point of multiplicity $n \geq 2$ (order $n-1$) for a circle packing P refers to a circle whose angle sum is $2\pi n$, so its neighbors wrap n times around it. At the associated vertex v of K, the simplicial map s_P is locally n-to-1. Write $\mathrm{br}(P)$ for the set of (necessarily interior) vertices of K associated with the branch points of P. In this paper, branch sets always reflect multiplicities; that is, if v is associated with a branch point of multiplicity n, then v occurs $n-1$ times in $\mathrm{br}(P)$.

It should be evident that the combinatoric properties of K restrict branching; even locally, for example, the flower of the branch point of multiplicity n must have at least $2n+1$ petals. Global necessary and sufficient conditions are established in [D1, Sec. 2, Thm. 2; D2, Thm. 4.1] (see also [Bo] and [Ga]). We incorporate these in a definition.

DEFINITION. A set $\beta=\{v_1, v_2, \ldots, v_k\}$ of interior vertices of K, perhaps with repetitions, is called a *branch structure* for K if every simple closed edge-path γ in K has at least $2m+3$ edges, where m is the number of points of β enclosed by γ (counting multiplicities).

We will stick to finite numbers of branch points, and in this case several fundamental results on branched packings were established by Dubejko. In particular, for β finite, $\beta=\mathrm{br}(P)$ for some circle packing P for K if and only if β is a branch structure. Moreover, if any such packing lies in $\mathbf{D}$, then there exists an essentially unique extremal one in $\mathbf{D}$, which we will denote by $P_{K,\beta}$. It is natural to call this the *maximal branched packing* for K associated with β, since it generalizes the essential features of P_K.

We are interested only in those cases in which $P_{K,\beta}$ lies in $\mathbf{D}$. It is necessary that K be hyperbolic, but we do not know in general whether this is sufficient. However, for our purposes it is enough to observe that $P_{K,\beta}$ will lie in $\mathbf{D}$ if K is finite or if K has some packing in $\mathbf{D}$ with branch set containing β. When $P_{K,\beta}$ lies in $\mathbf{D}$, write $R_{K,\beta}$ for its hyperbolic radius function. The metric this induces on K is again complete, with constant curvature -1 except at the branch points; boundary vertices have infinite radius. We need to extend our terminology to accommodate branch points. Thus, given R, we say the labeled complex $K(R)$ is a β-packing (resp. β-subpacking, β-superpacking) if the angle sum at each interior vertex v of K is equal (resp. greater than or equal, less than or equal) to $2\pi n_v$, where $n_v - 1$ is the number of times v occurs in β.

The theory of branched circle packings in case $P_{K,\beta} \subset \mathbf{D}$ is largely a generalization of the unbranched theory (which is just the special case $\beta = \emptyset$). We will gather the fundamental results we need in the following generalization of Proposition 1.

PROPOSITION 2. *Let K be a complex and β a finite branch structure so that $P_{K,\beta}$ lies in $\mathbf{D}$, and let $R_{K,\beta}$ be the hyperbolic radius function for $P_{K,\beta}$.*

- (i) *If $K(R)$ is a β-subpacking then $R \le R_{K,\beta}$; moreover, equality at any interior vertex implies $R \equiv R_{K,\beta}$.*
- (ii) *Let L be a subcomplex of K which is itself a proper complex, and let α be a branch structure for L with the property that $\alpha \subseteq \beta$ (counting multiplicities). If R is the restriction of $R_{K,\beta}$ to the vertices of L, then $R \le R_{L,\alpha}$. Equality for any interior vertex of L implies $L \equiv K$, $\alpha \equiv \beta$, and $R \equiv R_{L,\alpha}$.*

Assuming that K is finite:

- (iii) *Given any function g defined on K_{bdy} and taking values in $(0,\infty]$, there exists a unique radius function R that agrees with g at boundary vertices and for which $K(R)$ is a β-packing.*
- (iv) *Assume that $K(R_1)$ is a β-superpacking, that $K(R_2)$ is a β-subpacking, and that $R_1 \ge R_2$ on K_{bdy}. Then $R_1 \ge R_2$ on all vertices of K, and equality at any interior vertex implies $R_1 \equiv R_2$.*

Proof. Part (i) explains the extremal nature of the maximal branched packing $P_{K,\beta}$. Its proof is bound up with the existence of $P_{K,\beta}$ in [D1, Sec. 2; D2, Sec. 3] and is quite involved, so we will not go into it here. The proofs of the other parts are straightforward generalizations of the techniques used in the unbranched setting, basically relying on (i) and the Perron methods of [BS2] and [Bo]. We leave the details to the interested reader. □

These kinds of "monotonicity" results are much more useful if one thinks of them in an informal, dynamic way. Let us consider finite circle packings in $\mathbf{D}$. They are uniquely determined up to Möbius transformations by this "data": (1) the underlying complex, (2) the boundary radii, and (3) the branch structure. Therefore, if we start with a circle packing P for K, we change its data, and we find the packing Q for the new data, then we should have some feel for how the circles change in the transition from P to Q. The monotonicity results may be summarized informally as follows: *The radii will increase if one: (a) decreases the number of branch points, (b) discards part of the complex, and/or (c) increases boundary radii.* These changes are often quite dramatic if you work with computer-generated images of packings: for instance, when a branch point is removed (i.e., some circle is told that it is no longer a branch point), then that circle must grow so that its neighbors no longer wrap more than once around it. This increase in size leads to increases in each of the neighbors (so that they can maintain their proper angle sums), which in turn forces increases in yet other circles, and so forth and so on. Ultimately, there must be a general increase in hyperbolic radii to accommodate removal of the branch point. The whole process is quite intuitive after some mental experiments. (See [St], where these dynamics are associated with Markov processes.)

7. Extending the Discrete Results

Let P be a circle packing for K and suppose $\beta \subseteq \text{br}(P)$. In addition to the usual maximal packing P_K, we have a new circle packing available for comparison: the maximal branched packing $P_{K,\beta}$. This provides the setting for our extensions of the Schwarz and distortion lemmas.

As before, F_P will denote the circle packing map from P_K to P. Write $F_{K,\beta}$ for the circle packing map from P_K to $P_{K,\beta}$; that is, $F_{K,\beta}\colon \text{carr}(P_K) \to \text{carr}(P_{K,\beta})$. (We use this notation in preference to the notation $F_{P_{K,\beta}}$ for typographic reasons.) Assume our usual normalization, so the circles corresponding with some interior vertex v_0 of K are always centered at the origin. The proofs of these generalizations of our earlier lemmas proceed precisely as before; one need only substitute Proposition 2 for Proposition 1. Indeed, this realization is what motivated the expanded statements. We leave the verifications to the reader. Note also that the original versions correspond to the case $\beta = \emptyset$, since then $P_{K,\beta} = P_K$, $F_{K,\beta}$ is the identity map, and $F^{\#}_{K,\beta}(0) = 1$.

DISCRETE SCHWARZ LEMMA (Branched). *Let $P \subset \mathbf{D}$ be a circle packing for K. If β is a finite subset of* br(P), *counting multiplicities, then $P_{K,\beta}$ lies in $\mathbf{D}$. Assume that the associated circle packing maps F_P and $F_{K,\beta}$ satisfy $F_P(0) = 0 = F_{K,\beta}(0)$. Then $F^{\#}_P(0) \le F^{\#}_{K,\beta}(0)$, with equality iff $\beta \equiv$* br(P) *and P is a rotated copy of $P_{K,\beta}$, that is, $P = \lambda P_{K,\beta}$ for some unimodular constant λ.*

This lemma states that—among all circle packings for K that lie in $\mathbf{D}$, have branch structures containing β, and have the circle c_0 associated with v_0 centered at the origin—c_0 is largest for $P_{K,\beta}$. This follows immediately from Proposition 2(i), since the labeled complex $K(R)$ associated with P is a β-subpacking.

For the DDL, we need to revisit the notion that P "properly covers" $r\mathbf{D}$: Recall that the circle for v_0 is centered at the origin, and we may assume that it lies in $r\mathbf{D}$. Let V be the set of vertices of K whose circles intersect $\{|w|=r\}$. Let G be the edge-path connected component of vertices of $K^{(0)}\backslash V$ containing v_0, and let L be the subcomplex of K formed by the union of the stars of the vertices of G. One can verify (this uses the convexity of $r\mathbf{D}$) that L is a proper complex, and in analogy with analytic functions we will refer to L as the *component* of $F_P^{-1}(r\mathbf{D})$ containing 0. The statement that P properly covers $r\mathbf{D}$ is equivalent to the condition that L be finite and have its boundary vertices in V.

DISCRETE DISTORTION LEMMA (Branched). *Let P be a circle packing for a hyperbolic complex K, and assume that the circle packing map F_P satisfies $F_P(0)=0$. If P properly covers the disc $\{|w|<r\}$ for $r>0$, if β denotes the branch points of P interior to the component of $F_P^{-1}(r\mathbf{D})$, and if $P_{K,\beta}\subset\mathbf{D}$, then $F_P^{\#}(0)\geq rF_{K,\beta}^{\#}(0)$, with equality iff $P=(\lambda r)P_{K,\beta}$ for some unimodular constant λ.*

8. Extending the Classical Results

The foregoing discrete results immediately raise the issue of extending the classical lemmas, so that they too account for branch points. Once one recognizes that the finite Blaschke products are the classical models for the functions $F_{K,\beta}$, the discrete statements easily convert to the classical ones given in the introduction. Here we will prove the classical versions.

Finite Blaschke products have long played an important role in classical function theory on the disc (see [Co; Du]). They are typically represented as finite products of Blaschke factors (Möbius transformations of the disc), and along with more general "inner" functions traditionally arise as factors of other functions. They occur here in a different role, however. Particularly appropriate is this geometric characterization: *The finite Blaschke products are precisely the proper mappings of* $\mathbf{D}$ *onto* $\mathbf{D}$. Thus, an n-fold Blaschke product b assumes every value in the unit disc n times, counting multiplicities. Moreover, b is analytic across the unit circle and maps the unit circle n times about itself in the positive direction. Direct computation or elementary reasoning with the argument principle shows that b' has $n-1$ zeros in $\mathbf{D}$; that is, b has $n-1$ branch points. As we will see, among n-fold Blaschke products, b is uniquely determined up to a conformal automorphism of $\mathbf{D}$ by its branch set.

Compare, now, the properties of our discrete versions $F_{K,\beta}$ (see [D1, Sec. 3]): When K is an infinite complex of bounded degree without boundary, then given a finite branch structure β for K containing $n-1$ points, counting multiplicities, and having $P_{K,\beta}\subset\mathbf{D}$, one can then prove that $F_{K,\beta}$ is an n-fold proper mapping of $\mathbf{D}$ $(=\text{carr}(P_K))$ onto $\mathbf{D}$. In particular, $\text{carr}(P_{K,\beta})$, when endowed with the conformal structure induced by its immersion in $\mathbf{D}$, is precisely the image Riemann surface of some n-fold finite Blaschke product b (though, of course, the point mapping $F_{K,\beta}$ from the disc is piecewise affine while b is analytic). So, for example, $F_{K,\beta}$ would have precisely n zeros. We do not require the infinite case, and when K is finite, $\text{carr}(P_K)$ and $\text{carr}(P_{K,\beta})$ are proper subsets of $\mathbf{D}$. Nonetheless, $F_{K,\beta}$ is as close to proper as is possible; for instance, the hyperbolic metric induced on K by $P_{K,\beta}$ is complete (with boundary vertices at the ideal boundary and with geodesic boundary edges). The chain of boundary horocycles of P_K is carried to the chain of boundary horocycles of $P_{K,\beta}$, which wrap n times around the inside of the unit circle. An illustration of a branched packing is given in Figure 3.

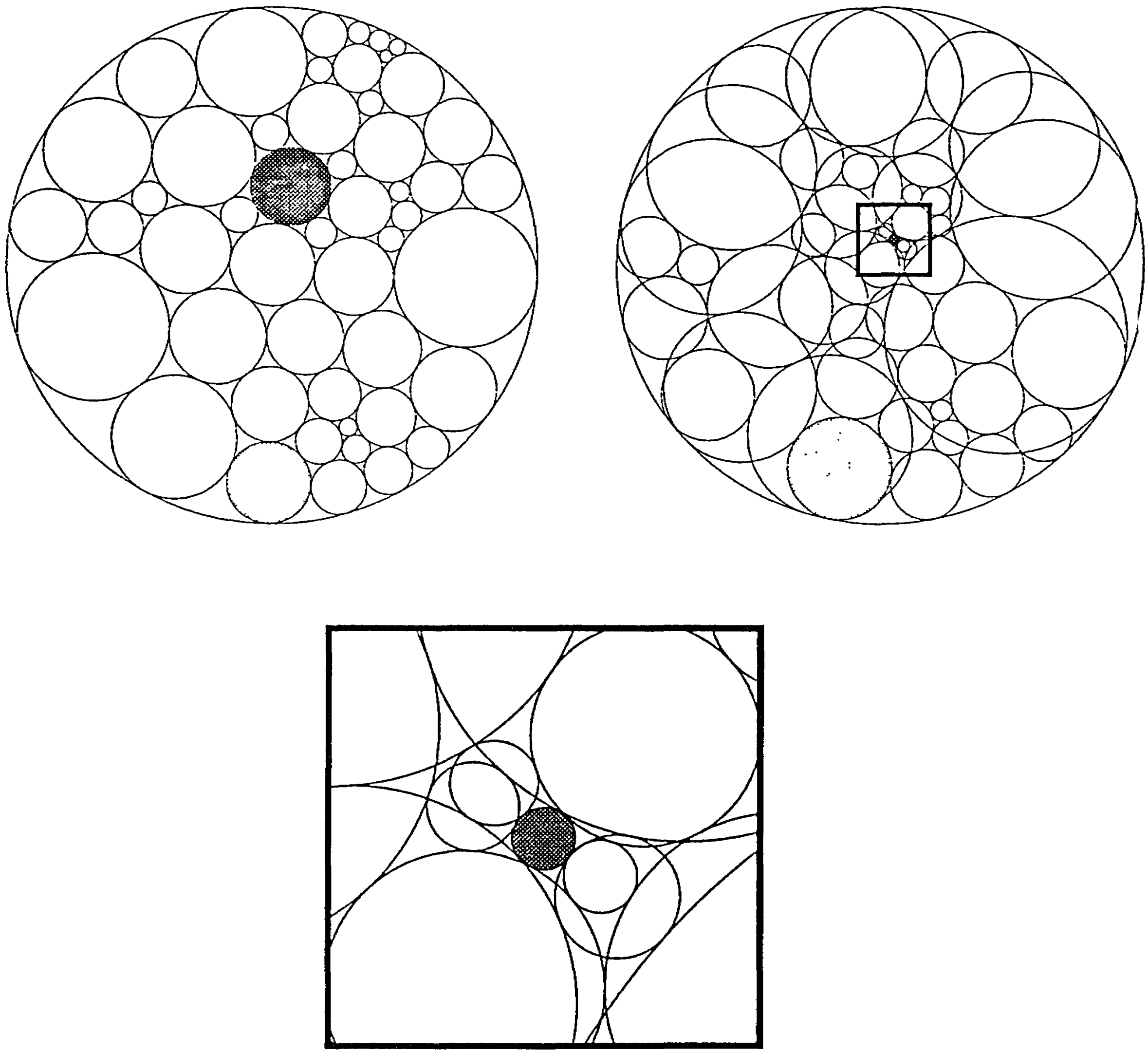

Figure 3 A branched circle packing

On the left is P_K and on the right is $P_{K,\beta}$; in each packing, the circles corresponding with the branch vertex and with a designated boundary vertex have been shaded for reference. Despite this and the fact that β has only a single branch point here, the picture is rather difficult to interpret since the circles overlap to form two "sheets". The enlargement shows the circle where the branching occurs.

Even without the approximation results to be discussed shortly, when β is finite and $P_{K,\beta} \subset \mathbf{D}$, we feel that the properties of the functions $F_{K,\beta}$ parallel those of finite Blaschke products so closely that we will refer to these functions (and Möbius transformations of them) as *discrete finite Blaschke products.*

The inequalities in the Schwarz and distortion lemmas of Section 1 will be established by approximating the analytic functions involved with circle packing maps. The approximations rely on results established in [D1, Sec. 4; D2, Sec. 5], which we will need to review here. The proofs of the cases involving equality will be deferred until the end of the section, since they rely on classical techniques.

Although greater generality is possible, it will be convenient to work with hexagonal packings. Let P_H denote the regular hexagonal packing of $\mathbf{C}$ in which each circle has radius 1, a circle c_0 is centered at the origin, and a neighboring circle is centered on the positive real axis—this is the familiar "penny-packing" in $\mathbf{C}$. Write H for the underlying infinite, constant 6-degree complex, and write v_0 for the vertex associated with c_0. H is parabolic, and P_H is its maximal packing. For $t > 0$, one can scale P_H by applying the map $z \mapsto tz$ to obtain a hexagonal packing, which we denote by tP_H, consisting of circles of radius t. For $0 < t < 1/2$, define P^t to be the circle packing consisting of those circles from tP_H that lie in $\bar{\mathbf{D}}$. Write H^t for the underlying complex, which is a finite proper complex containing v_0, and write $\tilde{P}^t$ for its maximal packing.

A few comments are in order regarding $\tilde{P}^t$. First, normalizations: P^t is unambiguously defined, but $\tilde{P}^t$ is determined only up to automorphisms of $\mathbf{D}$. We assume that the circle for v_0 is placed at the origin and that the circles centered on the positive real axis in P^t correspond with circles centered on the positive real axis in $\tilde{P}^t$; that this is possible can be verified by noting the symmetries of H^t. Next we observe that the circle packings P^t and $\tilde{P}^t$ are nearly identical for small t; indeed, the P^t exhaust $\mathbf{D}$ as $t \to 0$, so with our normalization the mappings F_{P^t} converge uniformly on compact subsets of $\mathbf{D}$ to the identity function by the theorem of Rodin and Sullivan [RS].

We begin our preparations by approximating finite Blaschke products, then polynomials, and finally, arbitrary analytic functions on the disc. It will be a standing assumption that all the circle packings have the circle for v_0 centered at the origin, but rotational and scalar normalizations will depend on circumstances.

Finite Blaschke Products. We build our approximants geometrically rather than analytically, by specifying where we want the branch points. Start with a fixed set $S = \{z_1, \dots, z_{n-1}\}$ of points in $\mathbf{D}$, possibly with repetitions, which will be the intended branch set. For $t > 0$, embed H^t as $\operatorname{carr}(\tilde{P}^t)$ and choose a set $M^t = \{z_1^t, \dots, z_{n-1}^t\}$ of circle centers approximating S. Let β^t denote the corresponding set of vertices of H^t. The approximation here means that

$$\lim_{t\to 0} z_j^t = z_j, \quad 1 \le j \le n-1.$$

We abuse notation by writing $\beta^t \to S$, even though it is actually the associated centers that converge to S. When t is sufficiently small, β^t may be chosen to be a branch structure for H^t; meeting the combinatoric conditions is easy, since the points of S are fixed while the hexagonal mesh of centers of $\tilde{P}^t$ becomes increasingly fine with decreasing t. (Note that since an interior vertex of H^t has only six neighbors, it can be a branch point of order at most 1; thus each of the sets M^t has $n-1$ distinct points, though S may have repeated points.) The n-fold discrete Blaschke product associated with H^t and β^t is the function F_{H^t,β^t}; we will simplify the notation to B^t in this setting. There is some ambiguity about the rotational normalization here, since B^t is determined only up to rotations. It is difficult in the case of branched packings to identify any natural normalization, so we will simply specify that some designated neighbor v_1 of v_0 has its circle centered on the positive real axis. We then circumvent any problems in our statement by appealing to diagonalization and subsequences.

THEOREM 3 [D1, Sec. 4, Thms. 7.2, 7.3]. *Let b be an n-fold finite Blaschke product with $b(0) = 0$. There exists a sequence $\{B_j\}$ of n-fold discrete finite Blaschke products, each B_j associated with H^{t_j} for some $t_j > 0$, such that $\{B_j\}$ converges uniformly on compacta (of* **D**) *to b as $j \to \infty$. In particular, $t_j \to 0$ and $\text{br}(B_j) \to \text{br}(b)$ as $j \to \infty$.*

Conversely, suppose that $\{B_j\}$ is a sequence of n-fold discrete finite Blaschke products, each associated with H^{t_j} for some t_j, and that $\text{br}(B_j) \to S$ and $t_j \to 0$ as $j \to \infty$, where S is a set of $n-1$ points of **D**, *counting repetitions. Then there exists a subsequence $\{B_{j_n}\}$ and a finite Blaschke product b with $\text{br}(b) = S$ so that B_{j_n} converges uniformly on compacta to b.*

POLYNOMIALS. Given an infinite circle packing P for H, one may define the circle packing map $F_{P_H,P}: \text{carr}(P_H) \to \text{carr}(P)$ in the usual way. It is clear from our earlier precedents that this map should be thought of as a discrete entire function. Because H is infinite, it is in fact quite difficult to construct associated circle packings P. Until the following theorem was established in [D2], the only examples (other than P_H) were the "Doyle" spirals studied in [BDS], which are discrete versions of the exponential function. Here, however, our interest is in polynomials.

THEOREM 4 [D2, Lemma 5.2, Thm. 6.1]. *Let β be a branch structure for H containing $n-1$ points (necessarily distinct). Then there exists a branched circle packing $P_{H,\beta}$ for H, with $\text{br}(P_{H,\beta}) = \beta$, whose carrier is a proper n-fold branched covering surface of* **C**. *$P_{H,\beta}$ is unique up to similarities of* **C**.

The circle packing map from P_H to $P_{H,\beta}$ will be denoted by E_β. Again, there is ambiguity here which requires some normalization. In addition to our usual assumption that the circles for v_0 in P_H and $P_{H,\beta}$ are centered at the origin, assume they both have radius 1. Thus, $E_\beta(0)=0$ and $E_\beta^\#(0)=1$. For rotational normalization, require that the circle for v_1 be centered on the positive real axis.

The numerous parallels between these maps and polynomials, as developed in [D2, Sec. 5, 6], justify describing them as *discrete polynomials*. Note, for instance, that E_β is an n-fold proper mapping of $\mathbf{C}$ to itself, a property that characterizes n-degree polynomials among entire functions. Of course, we may modify these circle packing maps by applying Möbius transformations to domain and range packings. We find that the class of discrete polynomials is then sufficiently rich to approximate classical polynomials.

An n-degree (classical) polynomial p is determined up to complex affine mappings of $\mathbf{C}$ by its $n-1$ branch points, so we'll start agan with a given branch set $S=\{z_1,\ldots,z_{n-1}\}$, no longer restricted to the unit disc. As before, for $t>0$, embed H as $\operatorname{carr}(tP_H)$, and choose a set M^t of $n-1$ circle centers approximating S so that the corresponding set of vertices β^t of H form a branch set. Theorem 4 yields an n-degree discrete polynomial E_{β^t}. This isn't quite the one we want, however. Define the discrete polynomial E^t by

$$E^t(z)=tE_{\beta^t}(tz), \quad z\in\mathbf{C}.$$

Note that the domain packing for E^t is tP_H (so the pattern of circles in the domain packing becomes finer as t decreases) and the range packing is tP_{H,β^t}; this definition ensures that $E^t(0)=0$, that $(E^t)^\#(0)=1$, and that the branch points of E^t are precisely the points of M^t.

THEOREM 5 [D2, Thm. 5.3]. *If p is an n-degree polynomial, then there exists a sequence $\{E_j\}$ of n-degree discrete polynomials that converges uniformly on compacta of the plane to p. In particular, if $p(0)=0$, one may take E_j to be of the form $s_jE^{t_j}$ for appropriate parameters $t_j>0$, branch sets β^{t_j}, and complex scalars s_j, where $t_j\to 0$, $\operatorname{br}(E_j)\to\operatorname{br}(p)$, and $|s_j|\to |p'(0)|$ as $j\to\infty$.*

We could formulate a converse, as was done in Theorem 3, but we will not need that here.

ANALYTIC FUNCTIONS ON **D**. Our next objective is to show that analytic functions on $\mathbf{D}$ can be approximated uniformly on compacta by circle packing maps—again, maps associated with the complexes H^t. If f is analytic on $\mathbf{D}$ with $f(0)=0$, it is well known that f can be approximated uniformly on compacta of $\mathbf{D}$ by polynomials p with $p(0)=0$. To approximate f, therefore, it suffices to approximate p, and by the previous theorem this can be done with discrete polynomials. Suppose sE^t is one of the discrete polynomials involved. Its domain packing is tP_H, its range packing is stP_{H,β^t}, and it has branch structure β^t. The basic idea is to restrict sE^t to P^t, which the reader will recall consists of the circles of tP_H lying in $\mathbf{D}$, and has complex H^t. Let Q^t denote the corresponding circles of stP_{H,β^t}. The circle packing map from P^t to Q^t clearly approximates p on $\mathbf{D}$. However, there are technicalities to address.

(1) We want the maximal packing $\tilde{P}^t$ rather than P^t to be the domain packing for our approximations. To that end, replace the restricted map $sE^t\,|_{P^t}$ by the map F_{H^t,Q^t} from $\tilde{P}^t$ to Q^t. This function, which we will abbreviate to F^t, can be written as a composition

$$F^t(\cdot) = (sE^t)(F_{P^t}(\cdot)).$$

We have previously observed that the functions F_{P^t} converge uniformly on compacta of $\mathbf{D}$ to the identity function, so the functions F^t approximate p uniformly on compacta of $\mathbf{D}$.

(2) The branch structure of F^t is the intersection of β^t with the set of interior vertices of P^t. If p has m branch points in $\mathbf{D}$, we may choose the branch sets β^t of the approximating discrete polynomials in such a way that, for sufficiently small t, F^t will also have m branch points in $\mathbf{D}$.

(3) Suppose $|f(z)| < 1$, $z \in \mathbf{D}$. We may assume the same condition for the approximating polynomial p. Suppose $|p'(z)| < M < \infty$, $z \in \bar{\mathbf{D}}$. By Theorem 5, the ratio functions $(sE^t)^\#$ are bounded by, say, $2M$ for sufficiently small t, implying that the circles of the packings Q^t have radii bounded by $2tM$. In particular, there exists $\epsilon(t) > 0$ which goes to zero with t so that the circles of $(1-\epsilon(t))Q^t$ will lie in $\mathbf{D}$. Therefore, in the discussion above, replace F^t by $(1-\epsilon(t))F^t$; we still have uniform convergence to p on compacta, but now the image packings lie in the hyperbolic plane.

We may summarize as follows.

THEOREM 6. *Let f be an analytic function on $\mathbf{D}$ with $f(0) = 0$. There exist a sequence of parameters $\{t_j\}$ converging to zero and circle packings $\{P_j\}$ for H^{t_j} whose associated functions F_{P_j} converge uniformly on compacta of $\mathbf{D}$ to f. If the range of f lies in $\mathbf{D}$, the P_j may be chosen to lie in $\mathbf{D}$.*

We are now in position to prove the inequalities of the classical results stated in the introduction; we comment on the cases of equality later.

Proof of the Schwarz Inequality. Let f and b be the functions hypothesized, with $S = \mathrm{br}(b) \subseteq \mathrm{br}(f)$. Choose a sequence $\{P_j\}$ of circle packings as guaranteed by Theorem 6. The functions F_{P_j} are open, continuous, light-interior mappings, so one can apply the argument principle, just as with analytic functions, to conclude that the F_{P_j} have branch sets α_j in H^{t_j} satisfying $\alpha_j \to \mathrm{br}(f)$. Identify subsets $\beta_j \subset \alpha_j$ of size $n-1$ so that $\beta_j \to S$, and for each j let B_j be the discrete finite Blaschke product associated with H^{t_j} and having branch set β_j. By Theorem 3, the sequence $\{B_j\}$ converges uniformly on compacta to λb for some unimodular constant λ.

For each t_j we have two packings of H^{t_j}, namely, the packing P_j with branch set α_j and the maximal branched packing B_j with branch set $\beta_j \subseteq \alpha_j$. By Proposition 2, $P_j^{\#}(0) \le B_j^{\#}(0)$. In conjunction with Theorem 1, we have

$$|f'(0)| = \lim_j P_j^{\#}(0) \le \lim_j B_j^{\#}(0) = |b'(0)|. \qquad \square$$

Proof of Distortion Inequality. This can be proven directly from the discrete version via approximation. However, as in the unbranched case, there is an easy argument based on the Schwarz lemma and subordination. Assume without loss of generality that $r = 1$. Let ω be a one-to-one conformal mapping of $\mathbf{D}$ onto Ω with $\omega(0) = 0$, and define $f_1 \equiv f \circ \omega$ and $b_1 \equiv b \circ \omega$. Both of these are self-maps of $\mathbf{D}$, both fix 0, and both have the same branch set. Since f_1 is a proper map of $\mathbf{D}$ onto $\mathbf{D}$, it is an n-fold finite Blaschke product for some $n > 0$. By the branched Schwarz lemma, $|b_1'(0)| \le |f_1'(0)|$ and we are done. $\square$

We conclude the proofs by appealing to Nehari's paper for the case of equality in the (branched) Schwarz lemma. Following this through the preceding subordination argument then settles the case of equality in the distortion lemma. (Alternately, one can rely on classical interpolation results of Omya; see [Co, p. 71].)

This leaves us with the following interesting open issue: The classical and discrete statements are precisely parallel regarding cases of equality. Can one obtain sufficient quantitative information about the discrete inequalities to prove the classical cases of equality? Quantitative estimates are needed to carry one through the approximation step.

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