Invariant Subspaces in VMOA and BMOA

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1. Introduction

In this article we describe completely the invariant subspaces of the operator S of multiplication by the coordinate function z on the Banach spaces BMOA and VMOA. BMOA stands for analytic functions of bounded mean oscillation on the unit circle T and VMOA is its subspace consisting of functions of vanishing mean oscillation. Due to the nonseparability of BMOA, the nature of the invariant subspaces is somewhat like their nature in the case of H^∞. Just as in the case of H^∞, we describe those invariant subspaces of BMOA that are closed in the weak-star topology when BMOA is treated as the dual of the Hardy space H^1. On VMOA, the invariant subspaces that are characterized are closed in the norm topology. Inner functions play a central role in both cases. For precise statements we refer to Section 2, Theorem A and Theorem C.

We also prove that the maximal ideals of the Banach algebra QA which correspond to fibers are dense in VMOA. The same kind of density result holds in the weak-star topology of BMOA for the maximal ideals of H^∞ that correspond to fibers. See Section 2, Theorem B and Theorem D. Quite interestingly, outer functions in both spaces turn out to be cyclic vectors for the operator S. For BMOA the cyclicity is in the context of the weak-star topology.

In the remainder of this section we outline very briefly those parts of the theory of Hardy spaces and BMOA that will be needed in the rest of this paper. Section 2 contains the precise statements of the main results. Section 3 contains preliminary results. Sections 4, 5, 6, and 7 are the proofs of the main results, namely, Theorems A, B, C, and D (respectively).

T will denote the unit circle in the complex plane and D its interior. L^p and H^p will denote the familiar Lebesgue and Hardy spaces on T, with $\|f\|_p$ as the L^p norm of f. It is well-known that each element f of H^p can be looked upon as an analytic function in D satisfying a certain growth condition. Of great importance is the fact that any analytic $f(z)$ in H^p can be uniquely factorized as $f(z)=B(z)S(z)O(z)$, where $B(z)S(z)$ is the inner factor of $f(z)$ and $O(z)$ is its outer factor. For further details we refer to [3] or [5].

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THE SPACES BMOA AND VMOA. Let f be in L^1 and let $I(f)$ denote $|I|^{-1}\int_I f$, where I is a subarc of T. We say f is of bounded mean oscillation and write $f \in \mathrm{BMO}$ if

$$\|f\|_* = \sup_I(|f - I(f)|) < \infty. \tag{1.1}$$

BMO is a Banach space under the norm given by

$$\|f\| = \|f\|_* + |f(0)|.$$

The space of functions of vanishing mean oscillation (i.e. VMO) is the closure in BMO of the continuous functions on T. $\mathrm{BMOA} = \mathrm{BMO} \cap H^1$ and $\mathrm{VMOA} = \mathrm{VMO} \cap H^1$. BMOA is contained in H^p for every finite p.

FEFFERMAN'S THEOREM. *BMOA is the dual of H^1. For each f in BMOA, its action as a linear functional is given by*

$$\lim_{r \to 1} \int_T \bar{f}(re^{i\theta}) g(re^{i\theta})\, d\theta \quad \textit{for } g \in H^1. \tag{1.2}$$

H^1 is the dual of VMOA, and its action is given by (1.2). For further details on these spaces refer to [5].

2. Statement of Main Results

Given f in BMOA (VMOA), it is easy to see that $e^{i\theta} f$ is in BMOA (VMOA). Denoting by S the operator of multiplication by the coordinate function $e^{i\theta}$, we have the following theorem.

THEOREM A. *Let M be a proper nontrivial closed subspace of VMOA invariant under the action of the operator S. Then there is a unique inner function I and a unique and dense subspace N of VMOA such that $M = I(z)N$. Further, $N =$ VMOA if and only if $I(z)$ is a finite Blaschke product.*

Note: For details of the fact that both kinds of subspaces exist (i.e., $N =$ VMOA and N dense in VMOA), we refer to Remark 4.1 at the end of the proof of Theorem A in Section 4. We also show there that every inner function $I(z)$ arises this way and that no inner function $I(z)$ can give rise to two different spaces N.

There are some interesting questions connected with the Banach algebra $QA = H^\infty \cap \mathrm{VMOA}$; see [2]. As an application of Theorem A, and adopting the terminology of [2], we have the next theorem.

THEOREM B. *Let M be any maximal ideal in a fiber m_λ of the algebra QA. Then M is dense in VMOA.*

Due to the nonseparability of BMOA, it is to be expected that there cannot be an easy characterization of the invariant subspaces of S on BMOA. For instance VMOA, $B(z)$BMOA ($B(z)$ is a finite Blaschke product), and the closure in BMOA of H^∞ are all proper nontrivial invariant subspaces. However, as in the H^∞ case, the situation is retrieved by discussing weak-star closed subspaces of BMOA. For such spaces we have a complete characterization given in Theorem C.

THEOREM C. *Let M be a weak-star closed subspace of BMOA invariant under S. Then corresponding to this M there exists a unique inner function $I(z)$ and a unique subspace N of BMOA such that N is weak-star dense in BMOA and $M=I(z)N$. Further, $N=$ BMOA if and only if $I(z)$ is a finite Blaschke product.*

THEOREM D. *Let M be any maximal ideal in a fiber m_λ on H^∞. Then M is weak-star dense in BMOA.*

3. Preliminary Results

Let S^* be the operator on H^1 that sends $f\to e^{-i\theta}(f-f(0))$. It is easy to see that S^* is the adjoint of S on VMOA and that $\|S^{*n}\|\leq n+1$.

THEOREM 3.1. *Let M be a closed subspace of H^1 invariant under S^*. Then there is a unique inner function I such that $M=I\bar{H}_0^1\cap H^1$.*

Proof. Let $f(z)=\sum_0^\infty \alpha_n z^n$ be a fixed but arbitrarily chosen element of M. Let $k(e^{i\theta})=\sup|f(re^{i\theta})|$, where the supremum is taken over $0\leq r<1$. Then, by a theorem of Hardy and Littlewood [3, Thm. 1.9], k belongs to L^1. Putting $\varphi=\exp(-|k|+i|\tilde{k}|)$ (where $\tilde{}$ denotes the conjugate), we find that φk is in L^∞. Further, letting $h_r(e^{i\theta})$ stand for $h(re^{i\theta})$ for h in H^1, we can see by the dominated convergence theorem that

$$\|\bar{\varphi}f_r-\bar{\varphi}f\|_2\to 0 \quad \text{as } r\to 1$$

and hence

$$\|(I-P)\bar{\varphi}f_r-(I-P)\bar{\varphi}f\|\to 0 \quad \text{as } r\to 1, \tag{3.1}$$

where P is the analytic projection, so that $I-P$ is the co-analytic projection.

If we let $\varphi(z)=\sum_0^\infty \beta_n z^n$, then (3.1) can now be rewritten as

$$\left\|\sum_{n=1}^{\infty}\left(\sum_{k=0}^{\infty}\alpha_k\bar{\beta}_{k+n}r^k\right)e^{-in\theta}-(I-P)\bar{\varphi}f\right\|_2\to 0 \quad \text{as } r\to 1, \tag{3.2}$$

and hence

$$\left\|\sum_{n=1}^{\infty}\left(\sum_{k=0}^{\infty}\alpha_k\bar{\beta}_{k+n}r^k\right)r^n e^{-in\theta}-(I-P)\bar{\varphi}f\right\|_2\to 0 \quad \text{as } r\to 1, \tag{3.3}$$

Thus, using the fact (deduced by the dominated convergence theorem) that

$$\|\bar{\varphi}_r f-\bar{\varphi}f\|_1\to 0 \quad \text{as } r\to 1, \tag{3.4}$$

we conclude that

$$\begin{aligned} P(\bar{\varphi} f) &= \bar{\varphi} f - (I-P)\bar{\varphi} f \\ &= \lim_{r\to 1} \bar{\varphi}_r f - \lim_{r\to 1} \sum_{n=1}^{\infty}\left(\sum_{k=0}^{\infty} \alpha_k \bar{\beta}_{k+n} r^k\right) e^{-in\theta} \\ &= \lim_{r\to 1}\left(\bar{\varphi}_r f - \sum_{n=1}^{\infty}\left(\sum_{k=0}^{\infty} \alpha_k \bar{\beta}_{k+n} r^k\right) r^n e^{-in\theta}\right) \end{aligned} \tag{3.5}$$

in the norm of L^1 by (3.2) and (3.3). But for each fixed r, $0<r<1$, we find that

$$\bar{\varphi}_r f - \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} \alpha_k \bar{\beta}_{k+n} r^k\right) r^n e^{-in\theta} = \sum_{n=0}^{\infty} \beta_n r^n S^{*n} f. \tag{3.6}$$

This is easily established by using the fact that the series on the right converges in L^1 for each fixed r because $\|S^{*n}\| \le n+1$; for each such r, the Fourier coefficients of the expression on the right-hand side of (3.6) coincide with the corresponding Fourier coefficients of the expression on the left-hand side of (3.6). Hence, from (3.5) and (3.6) we find that

$$\left\| P(\bar{\varphi} f) - \sum_{n=0}^{\infty} \beta_n r^n S^{*n} f \right\|_1 \to 0 \quad \text{as } r \to 1.$$

But since $\sum_{n=0}^{\infty} \beta_n r^n S^{*n} f$ is in M for each r (as f is in M), we conclude that $P(\bar{\varphi} f)$ is in M. Now $\bar{\varphi} f$ is in L^∞. So $P(\bar{\varphi} f)$ is in H^2, in fact, in BMOA. Hence $M \cap H^2 \neq \{0\}$.

Fix a p, $0<p<1$. In view of Beurling's theorem – which says that on H^2 the invariant subspaces of S are of the form IH^2, where I is an inner function (see [4]) – the invariant subspaces of S^* on H^2 (the adjoint of S) have the form $I\bar{H}_0^2 \cap H^2$. Hence, using the fact that $M \cap H^2$ is obviously S^* invariant and closed in H^2, we observe that $M \cap H^2 = I\bar{H}_0^2 \cap H^2$. Thus $I\bar{H}_0^1 \cap H^1$ is contained in M, being the closure of $M \cap H^2$ in M. On the other hand, let f be any function in M and let $\varphi_n = \exp(-[(|k| + i|\tilde{k}|)/n])$ for every natural n, where k is as chosen in the beginning of the proof. Then it follows by the same arguments as before that $P(\bar{\varphi}_n f)$ is in $M \cap H^2$ for each n; since $\varphi_n \to 1$ a.e., it thus follows that $P(\bar{\varphi}_n f) \to P(f) = f$ a.e. Further, in view of the inequality $\|\tilde{g}\|_p \le \alpha \|g\|$ (due to Kolmogoroff's theorem on the weak (1,1) nature of the conjugation operator [4, p. 115]) and the fact that $2P(g) = g + i\tilde{g} + g(0)$ for any g in L^2, we conclude by simple reasoning that $P(\bar{\varphi}_n f)$; converges to $P(f) = f$ in L^p. But $P(\bar{\varphi}_n f)$ is always in $IH_0^2 \cap H^2$, so f lies in $I\bar{H}_0^p \cap H^p$ and hence $f = I\bar{h}$ for some h in H^p. But I is inner and so of modulus 1 a.e. on T. This means that h is in H^1, from which it is easy to conclude that f is in $I\bar{H}_0^1 \cap H^1$. Since f is an arbitrary element of M we see that $M \subset I\bar{H}_0^1 \cap H^1$. We have already shown the reverse inclusion, so $M = I\bar{H}_0^1 \cap H^1$. ❐

LEMMA 3.2. *Let $f = IO$ be an element of BMOA (VMOA), where I is the inner part of f and O its outer factor. Let I_1 be any inner function that divides I. Then $\bar{I}_1 f$ is in BMOA (VMOA).*

Proof. See [2].

LEMMA 3.3. *The only inner functions in VMOA are the finite Blaschke products.*

Proof. See [2]. □

LEMMA 3.4. *Let I be an inner function that multiples BMOA. Then I is a finite Blaschke product.*

Proof. This is a theorem of Stegenga in [6]. □

LEMMA 3.5. *If I is a finite Blaschke product then I multiplies BMOA.*

Proof. This is a simple and well-known consequence of the duality relationship between H^1 and BMOA. □

LEMMA 3.6. *Let M be an invariant subspace in VMOA. Further, assume that the following conditions are satisfied:*

(i) *There is a finite Blaschke product B in M.*
(ii) *B divides the inner part of every f in M.*

Then $M = B(z)VMOA$.

Proof. Let $H = \{B(z)\,p(z): p$ is a polynomial in VMOA$\}$. Since B is in M, we see that H is in M. In view of Lemma 3.2 and the VMOA-H^1 duality, it follows that the operator $f \to P(\bar{B}f)$ is bounded on VMOA. Hence Bp_n is convergent in VMOA if and only if $\{p_n\}$ is convergent in VMOA. Thus the closure of H in VMOA is $B(z)$VMOA. This closure is contained in M because M is closed. On the other hand, by (ii) above and by Lemma 3.2, if f is in M then $f = Bg$ (where g is in VMOA), so that f is in $B(z)$VMOA. This proves the result. □

4. Proof of Theorem A

By the duality relation (1.3), $A(M)$, the annihilator of M, is a closed subspace of H^1 which is invariant under S^*. Thus $A(M) = I\bar{H}_0^1 \cap H^1$ by Theorem 3.1. Now VMOA is contained in H^2 and so M is orthogonal to $A(M) \cap H^2$, because every element of $A(M) \cap H^2$ is in the annihilator $A(M)$ and this action as a linear functional coincides with the inner product of H^2. In other words, $M \subset IH^2$. This means that I divides the inner part of every function in M. Without loss of generality, let I stand for the highest common factor of the inner parts of all functions in M. Then there is a set N in H^2 such that $M = IN$. By Lemma 3.2, N is contained in VMOA. It is also evident that N is a vector subspace which is invariant under the action of S.

Claim 1: If N is closed then $M = B(z)VMOA$, where B is a finite Blaschke product. So suppose N is closed. We show that $M =$ VMOA. If $N \neq$ VMOA then by the same arguments used for M we conclude that $N = I_1 N_1$, where I_1 is a nontrivial inner function and $N_1 \subset$ VMOA. But then II_1 will divide the inner part of every function in M, contradicting the choice of I as the highest common factor of the inner parts of all functions in M. Thus if N is closed then it is equal to VMOA. Since 1 is in VMOA we conclude that I is in VMOA. But then, by Lemma 3.3, I is a finite Blaschke product.

Claim 2: If N is not closed then $M = I(z)N$, where I is not a finite Blaschke factor and N is dense in VMOA: Suppose N is not closed and let N_1 be its closure in VMOA. We shall show that $N_1 =$ VMOA. If this is not so then N_1 is a proper invariant subspace, so that (by the same reasoning as used on M) we deduce that there is a nontrivial inner function I_1 which divides the inner part of every function in N_1. This contradicts the choice of I, thereby showing that N is dense in VMOA. Now, we have $M = I(z)N$. We still must show that I is not a finite Blaschke product. So let I be a finite Blaschke product. Since N is dense, there exists a sequence $\{f_n\}$ in N such that $f_n \to 1$ in VMOA. By Lemma 3.5, multiplication by I is a bounded operator on BMOA; hence $If_n \to I$. Since If_n is in M we conclude that I is in M. Also, by its very choice, I satisfies the conditions of Lemma 3.6 and so $M = I(z)$VMOA. But this means $N =$ VMOA, contradicting the fact that N is not closed. Hence I is not a finite Blaschke product.

The uniqueness of N, I, and B is easy to see. This completes the proof of Theorem A. □

REMARK 4.1. It is easy to see that both kinds of subspaces, $B(z)$VMOA and $I(z)N$ (N dense in VMOA), exist. That the first kind exists follows from Lemma 3.2 and Lemma 3.4. To see that the second kind exists, let $S(z)$ be a singular inner function and let f be any function in A, the Banach algebra of analytic functions on D which are continuous on T, such that $f(z)$ vanishes on the closed support of the measure determining $S(z)$. Then $S(z)f(z)$ is in A and hence in VMOA. Now let M be the closure in VMOA of $\{S(z)f(z)p(z) : p(z)$ is a polynomial in $A\}$. Obviously, M is an invariant subspace of the second type.

We next show that every inner function arises as given in the theorem. Let I be any inner function. By a theorem of Wolff [7], there is an outer function h in QA such that Ih is in QA. Now let M be the closure in VMOA of $\{Ihp : p$ is a polynomial in VMOA). Then it is not difficult to see that $M = IN$, where N is dense in VMOA.

Finally, we show that the same inner function cannot give rise to two different dense subspaces N and L. That is for a given inner I, if IN and IL are two closed invariant subspaces of VMOA, where N and L are dense in VMOA, then $N = L$. To see this, we first observe that $\operatorname{Ann}(IN) = I\bar{H}_0^1 \cap H^1$. This is proved as follows: Clearly $I\bar{H}_0^1 \cap H^1 \subset \operatorname{Ann}(IN)$. On the other hand, by Theorem 3.1 $\operatorname{Ann}(IN) = I_1 \bar{H}_0^1 \cap H^1$ for some inner I_1 so that $I\bar{H}_0^1 \cap H^1 \subset I_1 \bar{H}_0^1 \cap H^1$. This means $IH^2 \supset I_1 H^2$. Also, since $I_1 \bar{H}_0^1 \cap H^1$ annihilates IN, we get that $I_1 \bar{H}_0^2 \cap H^2$ annihilates IH^2 because N, being dense in VMOA, is dense in H^2. So $IH^2 \subset I_1 H^2$. Hence $IH^2 = I_1 H^2$ so that $I = I_1$.

Then, by the preceding paragraph, we have $\mathrm{Ann}(IN) = I\bar{H}_0^1 \cap H^1 = \mathrm{Ann}(IL)$ because N and L are dense in VMOA. This forces $IN = IL$ by the Hahn–Banach theorem, so $N = L$.

COROLLARY 4.2. *Let g be an outer function in VMOA. Then g is a cyclic vector for S. That is, given $\epsilon > 0$, for any h in VMOA there exists a polynomial p such that $\|h - gp\| < \epsilon$.*

Proof. Let $N = \{gp : p$ is an analytic polynomial$\}$. Let M be the closure of N in VMOA. M is an invariant subspace of VMOA, and since g is in M, the highest common factor of the inner parts of all elements of M is 1. Thus $M =$ VMOA, that is, N is dense in VMOA. □

5. Proof of Theorem B

In view of Corollary 4.2, the proof is quite easy. Let N be a maximal ideal corresponding to the fiber m_λ. Then $z - \lambda$ is an outer function in N and hence $z - \lambda$ belongs to the closure of N in VMOA, which we call M. As M is obviously an invariant subspace, we conclude that $(z - \lambda)p(z)$ is in M for every analytic polynomial $p(z)$. Thus, by Corollary 4.2, $M =$ VMOA. This means that N is dense in VMOA. □

6. Proof of Theorem C

Remarks preceding the Proof of Theorem C. We wish to observe here that the BMOA analogue in the weak-star topology of Lemma 3.6 is valid because $\{Bf_n\}$ is convergent in the weak-star topology of BMOA, where B is a finite Blaschke product and $\{f_n\}$ is a sequence. This is a result of Lemma 3.5, which states that if B is a finite Blaschke product then $f \to P(\bar{B}f)$ is bounded on H^1.

Proof of Theorem C. In the proof of this theorem we shall make use of Lemma 3.4 (which is a theorem of Stegenga [6]), instead of Lemma 3.3 as in the proof of Theorem A. We also need the following observations: If M is a weak-star closed subspace of BMOA, then there is a norm closed subspace N of H^1 such that the annihilator of N is M. Further, if M is invariant under S then it is easy to see, by the duality relation (1.3), that N is invariant under S^*.

Because of the remarks preceding this proof and the above observations, the proof of Theorem C is almost identical to the proof of Theorem A. □

REMARK 6.1. We observe here that both kinds of subspaces described in Theorem C exist. The proof is based on the same ideas as contained in Remark 4.1, which show that every inner function I arises as in the statement of the theorem and that an inner function I cannot give rise to more than one weak-star dense subspace N.

COROLLARY 6.2. *Let f be an outer function in BMOA. Then the weak-star closure in BMOA of* $\{fp\colon p$ is an analytic polynomial$\}$ *is BMOA.*

Proof. The proof is obvious in view of the proof of Corollary 4.2. □

7. Proof of Theorem D

This result is obvious once we observe that the outer function $(z-\lambda)$ is in every maximal ideal of the fiber m_λ in H^∞. Proceeding along the lines of the proof of Theorem B, the result follows by Corollary 6.2.

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