Global Integrability of the Jacobian and Quasiconformal Maps

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1. Introduction

Here we present alternate proofs to certain results arrived at in Astala and Koskela's recent article, "Quasiconformal mappings and global integrability of the derivative" [AK]. In addition, we examine how these new ideas shed light on some of the questions raised therein on the geometry of Gehring domains.

Denote the Jacobian matrix of f at x by $F(x)$ and its determinant by $J(x,f)$. Define

$$|f'(x)| = \sup_{h \in \mathbf{R}^n, |h|=1} |F(x)h|. \tag{1.1}$$

Let D and D' be domains in $\mathbf{R}^n$, $n \geq 2$. A homeomorphism $f\colon D \to D'$ is said to be *K-quasiconformal* if $f \in W^1_{n,\text{loc}}(D)$ and

$$|f'(x)|^n \leq KJ(x,f) \text{ a.e. in } D. \tag{1.2}$$

Local integrability results of the following type are well known for quasiconformal maps [Ge]. If $f\colon D \to D'$ is K-quasiconformal and E is any compact set in D, then there exists an exponent $p = p(n,K) > 1$ such that

$$\int_E (J(x,f))^p \, dm \leq M < \infty. \tag{1.3}$$

Here M depends on E and f.

In order to understand corresponding global integrability results, we need the following definitions.

DEFINITION 1.4. The *quasihyperbolic distance* between x and y in D is given by

$$k_D(x,y) = \inf_\gamma \int_\gamma \frac{1}{d(z,\partial D)} \, ds,$$

where γ is any rectifiable curve in D joining x to y. Here ∂D denotes the boundary of D and $d(z,\partial D)$ stands for the distance from z to the boundary of D.

Received December 2, 1991. Revision received March 27, 1992.

The author was supported by Grant #DMS-9004251 from the U.S. National Science Foundation.

Michigan Math. J. 40 (1993).

DEFINITION 1.5. Let $D \subset \mathbf{R}^n$ be a proper subdomain. We say that D satisfies a *quasihyperbolic boundary condition* if there are constants c_1 and c_2 such that

$$k_D(x, x_0) \le c_1 \log \frac{d(x_0, \partial D)}{d(x, \partial D)} + c_2 \tag{1.6}$$

for some fixed point x_0 in D and all x in D.

Astala and Koskela [AK] determined geometric properties of D and D' which characterize the higher integrability of $J(x, f)$ or equivalently $|f'(x)|^n$. Their results follow here.

THEOREM 1.7 [AK, Thm. 1.2]. *Let $D' \subset \mathbf{R}^n$ be a domain satisfying a quasihyperbolic boundary condition. If $D \subset \mathbf{R}^n$ and $f: D \to D'$ is K-quasiconformal, then*

$$\int_D |f'(x)|^p \, dm < \infty$$

for some $p = p(K, n) > n$.

THEOREM 1.8 [AK, Thm. 1.3]. *Let $D \subset \mathbf{R}^n$ satisfy a quasihyperbolic boundary condition and let $f: D \to D'$ be quasiconformal. Then $|f'(x)| \in L^p(D)$ for some $p > n$ if and only if D' satisfies a quasihyperbolic boundary condition.*

A crucial step in their argument [AK, Lemma 4.3, Thm. 4.4] involves establishing the equivalence of this problem with the local Lipschitz function problem studied by Gehring and Martio [GM]. The above theorems then follow from the results in [GM].

Here we present proofs of the above theorems which proceed using BMO estimates on $J(x, f)$ and take advantage of the integrability of $e^{ck_D(x, x_0)}$ on domains satisfying a quasihyperbolic boundary condition.

Astala and Koskela also defined a new class of domains, named Gehring domains, which arise naturally in this global integrability problem. In this paper we reformulate their definition and study the geometric properties of these domains. In particular we extend their result that domains with a quasihyperbolic boundary condition are Gehring domains, and provide examples of domains that are not Gehring domains.

2. Global Integrability Theorems

We begin with a list of notation, terminology, and preliminary lemmas. Throughout this paper D and D' denote domains in $\mathbf{R}^n$ with $n \ge 2$, and $B(x, r)$ stands for the ball of radius r centered at x.

Let $u: D \to \mathbf{R}^n$ be a locally integrable function. We say that u is of *bounded mean oscillation* in D, $u \in \mathrm{BMO}(D)$, if $\|u\|_* < \infty$, where

$$\|u\|_* = \sup_{B \subset D} |B|^{-1} \int_B |u(x) - u_B| \, dx.$$

Here the supremum is taken over all open balls $B \subset D$, $|B|$ is the Lebesgue measure of B, and u_B denotes the average of u over B, namely

$$u_B = |B|^{-1} \int_B u(x)\, dx.$$

In order to obtain estimates on the integrability of the Jacobian, we make use of the average derivative a_f, introduced in [AG], along with the relationship established between a_f and $J(x,f)$ in [AK].

DEFINITION 2.1. Let f be a quasiconformal map in a proper subdomain $D \subset \mathbf{R}^n$. Set $B(x) = B(x, d(x, \partial D)/2)$. We define the *average derivative* a_f as

$$a_f(x) = \exp\left(\frac{1}{n|B(x)|} \int_{B(x)} \log J(y,f)\, dm(y)\right), \quad x \in D.$$

THEOREM 2.2 [AK, Thm. 3.4]. *There exists an $\epsilon = \epsilon(n,K) > 0$ such that, whenever f is K-quasiconformal in D,*

$$c_1 \int_D (a_f(x))^p\, dm \le \int_D (J(x,f))^{p/n}\, dm \le c_2 \int_D (a_f(x))^p\, dm$$

holds for all p such that $-\epsilon < p/n < 1+\epsilon$. Here the constants c_1 and c_2 depend only on n, K, and p.

Denote by D' the image of D under the K-quasiconformal map f. Since the geometry of D' plays the decisive role in the integrability results, we calculate the integrals in terms of D' and f^{-1}. By the change-of-variables formula we have

$$\int_D J(x,f)^{1+\alpha}\, dm = \int_{D'} J(x,f^{-1})^{-\alpha}\, dm. \tag{2.3}$$

Additionally, since Theorem 2.2 can be applied to f^{-1} and D', we need only show that the integral

$$\int_{D'} (a_{f^{-1}}(x))^{-\delta}\, dm < \infty \quad \text{for some value } \delta = \delta(K), \text{ with } 0 < \delta < \epsilon n, \tag{2.4}$$

to establish the higher integrability of f.

For the ensuing estimates, we use the simpler notation Ω for our domain in $\mathbf{R}^n$ and g for our quasiconformal map, replacing D' and f^{-1} respectively.

We have the following lemma concerning $a_g(x)$.

LEMMA 2.5. *Let g be a K-quasiconformal map defined on Ω, let x_0 be a fixed point in Ω, and let $\delta > 0$. Then the estimate*

$$(a_g(x))^{-\delta} \le b(a_g(x_0))^{-\delta} \exp\left(\frac{c\delta}{n} k_\Omega(x, x_0)\right)$$

holds for all $x \in \Omega$. Here b and c depend only on K and n.

Proof. We set $\log J(x,g) = u(x)$ and $B(x) = B(x, d(x, \partial D)/2)$. With this notation we see

$$a_g(x) = a_g(x_0)\exp\left(\frac{1}{n}(u_{B(x)} - u_{B(x_0)})\right). \tag{2.6}$$

Thus

$$(a_g(x))^{-\delta} \le (a_g(x_0))^{-\delta}\exp\left(\frac{\delta}{n}|u_{B(x)} - u_{B(x_0)}|\right). \tag{2.7}$$

By a well-known result of Reimann [Rm], $u(x) = \log J(x, g) \in \mathrm{BMO}(\Omega)$. We now utilize an estimate [S1, Lemma 2.11, with $\tau = 2$] which exists for comparing averages over balls of BMO functions. We have

$$|u_{B(x)} - u_{B(x_0)}| \le c(k_\Omega(x, x_0) + 1), \quad c = c(K, n). \tag{2.8}$$

Using this in (2.7) completes the proof. □

Smith and Stegenga have established the following characterization of domains with a quasihyperbolic boundary condition.

THEOREM 2.9 [SS, Thm. A]. *Let Ω be a proper subdomain of $\mathbf{R}^n$, and let $x_0 \in \Omega$. The following are equivalent:*

(2.10) *Ω satisfies a quasihyperbolic boundary condition.*

(2.11) *There is a $\tau > 0$ such that*

$$\int_\Omega \exp(\tau k_\Omega(x, x_0))\, dm < \infty.$$

Lemma 2.5 and Theorem 2.9 lead to a direct proof of Theorem 1.7 in [AK], restated here as follows.

THEOREM 2.12. *Let D' satisfy a quasihyperbolic boundary condition. If $D \subset \mathbf{R}^n$ and $f: D \to D'$ is K-quasiconformal, then*

$$\int_D (J(x, f))^{1+\alpha}\, dm < \infty \quad \textit{for some } \alpha = \alpha(K, n) > 0.$$

Proof. Combining (2.3), (2.4), Lemma 2.5, and Theorem 2.9, we need only choose any $\alpha < \min(\epsilon, \tau/c)$. Here ϵ is as in Theorem 2.2, c arises from (2.8), and τ is as in Theorem 2.9. □

We now provide an alternative proof of Theorem 1.8.

THEOREM 2.13. *Let $D \subset \mathbf{R}^n$ satisfy a quasihyperbolic boundary condition, and let $f: D \to D'$ be quasiconformal. Then $J(x, f) \in L^{1+\epsilon}(D)$ for some $\epsilon > 0$ if and only if D' satisfies a quasihyperbolic boundary condition.*

Proof. Assume $J(x, f) \in L^{1+\epsilon}(D)$. By Theorem 2.9, there exists a $\tau > 0$ such that

$$\int_D \exp(\tau k_D(x, x_0))\, dm < \infty. \tag{2.14}$$

Gehring and Osgood [GO] provide the following useful distortion estimate on the quasihyperbolic metric:

$$k_{D'}(y, y_0) \le c(k_D(x, x_0) + 1), \tag{2.15}$$

where $y = f(x)$ and $y_0 = f(x_0)$.

Applying this estimate in conjunction with Hölder's inequality, we have

$$\begin{aligned}
&\int_{D'} \exp(dk_{D'}(y, y_0))\, dm \\
&\quad = \int_D \exp(dk_{D'}(f(x), f(x_0)))\, J(x, f)\, dm \\
&\quad \le \int_D \exp(dc(k_D(x, x_0) + 1))\, J(x, f)\, dm \\
&\quad \le \left(\int_D (\exp(dc(k_D(x, x_0) + 1)))^{(1+\epsilon)/\epsilon}\, dm\right)^{\epsilon/(1+\epsilon)} \left(\int_D J(x, f)^{1+\epsilon}\, dm\right)^{1/(1+\epsilon)}.
\end{aligned}$$

The second integral is bounded by hypothesis. Moreover, if we take $d < \tau\epsilon/c(1+\epsilon)$, then condition (2.14) guarantees that the first integral will also be bounded. Thus, by Theorem 2.9, D' satisfies a quasihyperbolic boundary condition. □

3. Gehring Domains

We begin with the definition of Gehring domains found in [AK] and immediately follow with an equivalent definition which we use in this paper.

Definition 3.1 [AK]. We say that a domain $\Omega \subset \mathbf{R}^n$ is a *Gehring domain* if, for all $K \ge 1$, there is a number $p = p(K) > n$ such that

$$\int_D |f'(x)|^p\, dm < \infty$$

for each domain D and each K-quasiconformal map $f: D \to \Omega$.

In order to focus on the role of Ω alone, we eliminate the domain D in our reformulation. This involves only a change of variables as in (2.3).

Proposition 3.2. *The domain $\Omega \subset \mathbf{R}^n$ is a Gehring domain if and only if, for all $K \ge 1$, there exists a number $\alpha = \alpha(K) > 0$ such that*

$$\int_\Omega (J_g(x))^{-\alpha}\, dm < \infty$$

for all K-quasiconformal maps g defined on Ω.

Astala and Koskela's proofs for Theorems 1.7 and 1.8 rely on estimating the Minkowski dimension of the boundary of the image domain D'. In the case where $\dim_M(\partial D') < n$, they establish the equivalence of the higher integrability problem with the local Lipschitz problem examined by Gehring and

Martio [GM]. (Note that Hanson and Koskela [HaK] have recently shown that some such type of additional condition on $\partial D'$ is necessary in order to guarantee this equivalence.) Furthermore, since $\dim_M(\partial D') < n$ whenever $D' \subset \mathbf{R}^n$ satisfies a quasihyperbolic boundary condition [SS], Astala and Koskela then apply the results in [GM].

In light of Theorem 1.7 or 2.12, it is an immediate consequence that domains which satisfy a quasihyperbolic boundary condition are Gehring domains. The sufficient conditions in [AK] describing Gehring domains Ω include the constraint that $\dim_M(\partial\Omega) < n$. Astala and Koskela do however provide examples [AK, (2.5)] of Gehring domains Ω with both the Hausdorff dimension, $\dim_H(\partial\Omega) = n$, and $\dim_M(\partial\Omega) = n$.

The following observations lead to a sufficient condition for Ω to be a Gehring domain. Note that certain sets E when removed from a domain Ω may alter the Minkowski dimension; in other words, $\dim_M(\partial(\Omega\setminus E))$ need not equal $\dim_M(\partial\Omega)$. However, removal of these same sets E may not affect the question of higher integrability.

DEFINITION 3.3. Let D be a domain in $\mathbf{R}^n$ and $E \subset D$ a set closed relative to D. We say that E is a *removable set* if every K-quasiconformal map f: $D\setminus E \to \mathbf{R}^n$ can be extended to a K'-quasiconformal map $\tilde{f}: D \to \mathbf{R}^n \cup \{\infty\}$, where $K' = K'(K)$.

THEOREM 3.4. *Let E be a removable set. If Ω satisfies a quasihyperbolic boundary condition then $\Omega\setminus E$ is also a Gehring domain.*

Proof. Let g be any K-quasiconformal map defined on $\Omega\setminus E$. Denote by $\tilde{g}$ the K'-quasiconformal extension of g to Ω. Suppose first that $\tilde{g}: \Omega \to \mathbf{R}^n$, and let $\alpha = \alpha(K')$ be the constant for Ω given by 3.2. Then

$$\int_{\Omega\setminus E} (J(x,g))^{-\alpha}\, dm \le \int_{\Omega} (J(x,\tilde{g}))^{-\alpha}\, dm < \infty$$

since Ω is a Gehring domain.

If $\tilde{g}(z) = \infty$ for some $z \in \Omega$, we define $B = B(z, \frac{1}{2}d(z,\partial\Omega)) \equiv B(z,r)$. We establish first that $\Omega\setminus B$ also satisfies a quasihyperbolic boundary condition. Let z_0 be a point on $\partial B(z, \frac{3}{2}r)$. Since Ω satisfies a quasihyperbolic boundary condition, there exist constants a and b such that

$$k_\Omega(z_0, y) \le a \log \frac{d(z_0,\partial\Omega)}{d(y,\partial\Omega)} + b \tag{3.5}$$

for all points y in Ω. Observe that for any point $z_1 \in \partial B(z, \frac{3}{2}r)$ we have

$$k_\Omega(z_0, z_1) \le 3\pi \quad \text{and} \quad k_{\Omega\setminus B}(z_0, z_1) \le 3\pi. \tag{3.6}$$

Consider any point y in $\Omega\setminus B$. Suppose first that $y \in \Omega\setminus B(z, \frac{3}{2}r)$. Let γ be the quasihyperbolic geodesic in Ω joining z_0 and y. Denote by z_1 the last intersection point of γ and $\partial B(z, \frac{3}{2}r)$. Then

$$k_{\Omega\setminus B}(z_0, y) \le k_{\Omega\setminus B}(z_0, z_1) + k_{\Omega\setminus B}(z_1, y). \tag{3.7}$$

Note also that

$$d(w, \partial(\Omega\setminus B)) \le d(w, \partial\Omega) \le 7d(w, \partial(\Omega\setminus B)) \quad \text{for any } w \in \Omega\setminus B(z, \tfrac{3}{2}r),$$

so that

$$k_{\Omega\setminus B}(z_1, y) \le 7k_{\Omega}(z_1, y). \tag{3.8}$$

The triangle inequality in Ω states that

$$k_{\Omega}(z_1, y) \le k_{\Omega}(z_0, y) + k_{\Omega}(z_0, z_1). \tag{3.9}$$

Combining all of the above inequalities, we see that there exist constants a_1 and b_1 such that

$$k_{\Omega\setminus B}(z_0, y) \le a_1 \log \frac{d(z_0, \partial(\Omega\setminus B))}{d(y, \partial(\Omega\setminus B))} + b_1.$$

For any point $y \in (\Omega\setminus B) \cap B(z, \frac{3}{2}r)$, we can use a path γ from z_0 to z_1 along $\partial(B(z, \frac{3}{2}r))$ followed by the radial path from z_1 to y. Here z_1 is that point on $\partial(B(z, \frac{3}{2}r))$ which lies on the ray from z through y. Direct computation then yields

$$k_{\Omega\setminus B}(z_0, y) \le a_2 \log \frac{d(z_0, \partial(\Omega\setminus B))}{d(y, \partial(\Omega\setminus B))} + b_2.$$

We conclude that $\Omega\setminus B$ satisfies a quasihyperbolic boundary condition,

$$k_{\Omega\setminus B}(z_0, y) \le A \log \frac{d(z_0, \partial(\Omega\setminus B))}{d(y, \partial(\Omega\setminus B))} + B,$$

with $A = A(a, b)$ and $B = B(a, b)$. We let $\beta_1 = \beta_1(K')$ be the constant for $\Omega\setminus B$ given by (3.2).

Astala and Koskela proved that $B - \{z\}$ is a Gehring domain. Let $\beta_2 = \beta_2(K')$ be the associated constant for $B - \{z\}$. Finally, set $\beta = \min(\beta_1, \beta_2)$. We then have

$$\begin{aligned}\int_{\Omega\setminus E} (J(x, g))^{-\beta}\, dm &\le \int_{\Omega} (J(x, \tilde{g}))^{-\beta}\, dm = \int_{\Omega - \{z\}} (J(x, \tilde{g}))^{-\beta}\, dm \\ &= \int_{\Omega\setminus B} (J(x, \tilde{g}))^{-\beta}\, dm + \int_{B - \{z\}} (J(x, \tilde{g}))^{-\beta}\, dm < \infty. \qquad \square\end{aligned}$$

Numerous necessary and sufficient conditions exist to describe removable sets. Here we list a few examples and mention the further references of [As], [AS], [HeK], [V1], and [V2]. Väisälä showed that if the $(n-1)$-dimensional Hausdorff measure of E is zero then E is removable, and if E is removable then the topological dimension of E satisfies $\dim E \le n-2$. Aseev [As] generated examples of Cantor sets with positive $(n-1)$-dimensional Hausdorff measure which are removable, and presented capacity estimates for removable sets. More recently, Herron and Koskela [HeK] determined various relationships between capacity domains, extension domains, and uniform domains, thereby producing further examples of removable sets. Example (2.5) of [AK] provides a model for generating removable sets of Hausdorff dimension n.

The proof of Lemma 2.5 leads to further information on the geometric aspects of Gehring domains. In particular, Lemma 2.5 provides insight into which types of quasiconformal test maps prove useful in determining if Ω is a Gehring domain.

THEOREM 3.10. *Consider any domain $\Omega \subset \mathbf{R}^n$ which can essentially be written as a union of cubes. That is, $\Omega = \bigcup (Q_i \cup B_i)$, $i = 1, 2, \ldots$, where each cube Q_i is open, $\partial Q_i \cap \partial Q_{i+1} \neq \emptyset$ for all i, and we let $B_i = (\partial Q_i \cap \partial Q_{i+1})^0$. Furthermore these cubes are pairwise disjoint, and each cube Q_i is centered at y_i on the x_1-axis and has sides parallel to the coordinate axes. There exists a K-quasiconformal map g, $K > 1$, satisfying*

$$K^{-in} \leq J(x, g) \leq K^{(-i+1)n} \tag{3.11}$$

for all $x \in Q_i$.

We make the following remarks to motivate this result before proceeding with the construction. First observe that if there exist constants c_1 and c_2 such that $c_1 \leq l(Q_i)/l(Q_{i+1}) \leq c_2$ for $i = 1, 2, 3, \ldots$, then

$$c_3 i \leq k_\Omega(y_1, y_i) \leq c_4 i, \quad i = 2, 3, \ldots .$$

In that case our map g will satisfy

$$u_{B(y_1)} - u_{B(y_i)} \geq c_5 k_\Omega(y_1, y_i), \quad i = 1, 2, 3, \ldots,$$

where $u(x) = \log J(x, g)$. This is the desired property of the map g in the following examples: to force the estimate in (2.8) to be sharp on the set of centers $\{y_i\}$. Also note that we obtain the following simple criterion from Proposition 3.2.

COROLLARY 3.12. *Let $\Omega \subset \mathbf{R}^n$ be as in Theorem 3.10. If*

$$\sum_i K^{i\alpha n} m(Q_i) = \infty \quad \textit{for all } \alpha > 0,$$

then Ω is not a Gehring domain.

Proof of Theorem 3.10. Let each cube Q_i be centered at $y_i = (c_i, 0, 0, \ldots, 0)$. Here we provide the precise details for the construction of g in the case $n = 2$. Note that with appropriate dimensional adjustments this map g can be constructed for any $n \geq 2$.

We modify the construction in [S2] to produce our map g. Decompose each cube Q_i into four regions $T_{i,1}$, $T_{i,2}$, $T_{i,3}$ and $T_{i,4}$. Here

$$T_{i,4} = \{x = (x_1, x_2) \in Q_i : x_1 \geq c_i\},$$

$$T_{i,2} = \{x = (x_1, x_2) \in Q_i : x_1 \leq c_i \text{ and } x_2 \geq c_i - x_1\},$$

$$T_{i,3} = \{x = (x_1, x_2) \in Q_i : x_1 \leq c_i \text{ and } x_2 \leq x_1 - c_i\},$$

and

$$T_{i,1} = Q_i \setminus \left(\bigcup_{j=2}^{4} T_{i,j} \right).$$

Define g in a piecewise manner on each cube Q_i as a radial stretching map with respect to y_i followed by a suitable translation. On Q_1, let $g(y_1) = y_1$, and let the radial stretching factor be 1 in $T_{1,1}$ (i.e., $g(x) = x$ in $T_{1,1}$), and let the radial stretching factor be K^{-1} in $T_{1,4}$. Note that any point in $T_{1,2}$ is on a ray emanating from y_1 of the form $x_1 + bx_2 = c_1$ for some $b \in [0,1]$. Let the radial stretching factor along this ray be $(b(K-1)+1)/K$. Similarly, each point in $T_{1,3}$ is on a ray with equation $x_1 - bx_2 = c_1$, $b \in [0,1]$. We let the stretching factor along this ray be $(b(K-1)+1)/K$.

We continue to define g inductively. Assume that g has been defined on $Q_1, Q_2, \ldots, Q_{i-1}$. Define g on Q_i as follows. On $T_{i,1}$ let g be that radial stretching map with constant stretching factor $K^{-(i-1)}$ followed by a suitable translation such that $g(\partial T_{i,1}) = g(\partial T_{i-1,4})$ on $\partial T_{i,1} \cap \partial T_{i-1,4}$. Note that this uniquely determines $g(y_i)$. Now in $T_{i,4}$ let the radial stretching factor be K^{-i}; in $T_{i,2}$ and $T_{i,3}$ we let the radial stretching factor be $K^{-i}(b(K-1)+1)$, where b is defined in a way analogous to that above. This completes the construction of g, and one can readily verify that g is K-quasiconformal. □

This construction enables us to prove that certain domains are Gehring domains if and only if they satisfy a quasihyperbolic boundary condition. Lemma 3.13 and Theorem 3.19 provide one such scenario.

LEMMA 3.13. *Let $\Omega \subset \mathbf{R}^n$ be a domain as in Theorem 3.10, with the additional constraint that there exists a constant c such that $1/c \le l(Q_i)/l(Q_{i+1}) \le c$, $i = 1, 2, 3, \ldots$. The following two conditions are equivalent:*

(3.14) *Ω satisfies a quasihyperbolic boundary condition.*

(3.15) *There exist constants $r, s > 0$ such that $l(Q_i) \le r l(Q_1) e^{-is}$ for $i = 2, 3, \ldots$.*

Proof. For this domain Ω we have the estimate

$$c_1 i \le k_\Omega(y_1, y_i) \le c_2 i, \quad i = 2, 3, \ldots, \tag{3.16}$$

with $c_1 = 1/\sqrt{n}$ and $c_2 = 2 + \log c$. If we assume (3.14) then, for some constants a and b,

$$k_\Omega(y_1, y_i) \le a \log \frac{d(y_1, \partial\Omega)}{d(y_i, \partial\Omega)} + b, \quad i = 2, 3, \ldots.$$

Combining this with (3.16) yields (3.15) with $r = e^{b/a}$ and $s = c_1/a$.

For the other implication we note first that (3.16) together with (3.15) yields

$$k_\Omega(y_1, y_i) \le c_2 i \le a \log \frac{d(y_1, \partial\Omega)}{d(y_i, \partial\Omega)} + b, \quad i = 2, 3, \ldots, \tag{3.17}$$

for $a = c_2/s$ and $b = (c_2 \log r)/s$. Moreover, for any point $z \in Q_i$, we can compute that

$$k_\Omega(y_i, z) \le p \log \frac{d(y_i, \partial\Omega)}{d(z, \partial\Omega)} + q, \tag{3.18}$$

where p and q depend only on n and c. Estimate (3.18) follows from Lemma 3.11 in [GM], since each of the domains $Q_1 \cup B_1 \cup Q_2$ and $Q_{i-1} \cup B_{i-1} \cup Q_i \cup B_i \cup Q_{i+1}$ for $i > 1$ are John domains with constants independent of i and depending only on n and c.

Finally, (3.17) and (3.18) together yield that for any point $z \in \Omega$,

$$k_\Omega(y_1, z) \le a_3 \log \frac{d(y_1, \partial\Omega)}{d(z, \partial\Omega)} + b_3,$$

with $a_3 = \max(a, p)$ and $b_3 = b + q$. □

THEOREM 3.19. *Let $\Omega \subset \mathbf{R}^n$ be a domain as in Lemma 3.13. Then Ω is a Gehring domain if and only if Ω satisfies a quasihyperbolic boundary condition.*

Proof. We need only show that if Ω does not satisfy a quasihyperbolic boundary condition then Ω is not a Gehring domain. This follows directly from Lemma 3.13 and Corollary 3.12. □

REMARK 3.20. Lemma 3.13 and Theorem 3.19 can be generalized to characterize certain domains constructed by adjoining consecutive rectangular box regions. For example, suppose $\Omega = \bigcup (R_i \cup B_i)$, where each open box R_i is centered at $y_i = (c_i, 0, \ldots, 0)$, these boxes are pairwise disjoint, and $B_i = (\partial R_i \cap \partial R_{i+1})^0 \ne \emptyset$. Here R_i is of the form $(c_i - (l_i/2), c_i + (l_i/2)) \times Q_i$, with Q_i an $(n-1)$-dimensional cube centered at the origin and $l(Q_i) = h_i$. In addition, we assume for some constant c that $1/c \le h_i/h_{i+1} \le c$ for $i = 1, 2, \ldots$ and that $l_i/h_i \ge 1$ and is nondecreasing. Under these conditions, Ω satisfies a quasihyperbolic boundary condition if and only if condition (3.15) holds and there exists a constant M such that $l_i/h_i \le M$ for all i. Moreover, by chopping up each R_i into essentially l_i/h_i cubes and using the ideas of Theorem 3.10, we can show that Ω is a Gehring domain if and only if it satisfies a quasihyperbolic boundary condition.

Example 3.21 presents the computations for a specific non-Gehring domain. Note that Definition 3.1 directly implies that Ω must satisfy $|\Omega| < \infty$ in order to be a Gehring domain. We also check this condition to verify that Example 3.21 is not trivial.

EXAMPLE 3.21. Let $\Omega = \bigcup Q_i \subset \mathbf{R}^n$, where each Q_i is centered at $y_i = (c_i, 0, 0, \ldots, 0)$. Fix any constant $p > 1$, set $c_1 = 1/2$, and let

$$c_i = \frac{(i)^{-p/n}}{2} + \sum_{j=1}^{i-1} (j)^{-p/n}.$$

Furthermore, let the side length of Q_i be $l(Q_i) = (i)^{-p/n}$. Thus

$$m(\Omega) = \sum_1^\infty \frac{1}{i^p} < \infty.$$

Moreover, if we consider the map g from Theorem 3.10, we see that

$$\int_{\Omega} J(x,g)^{-\alpha}\,dm \geq \sum_{1}^{\infty} \frac{K^{\alpha(i-1)n}}{i^p} = \infty,$$

since $K > 1$. Thus Ω is not a Gehring domain. □

Note that a modification of the map g in Theorem 3.10 also rules out the possibility of certain outward directed spires in Gehring domains. We conclude with the following corollary.

COROLLARY 3.22. *Let S be the infinite spire of revolution in $\mathbf{R}^n$ given by*

$$S = \left\{(x_1, x_2, \dots, x_n) : \sum_{2}^{n} (x_i)^2 < g(x_1)^2,\ 0 \leq x_1 < \infty\right\},$$

where $g(x) \geq 0$ and $g'(x) < 0$ for $0 \leq x < \infty$. Suppose that $m(S) < \infty$; that is, suppose

$$\int_0^\infty g(x)^{n-1}\,dx < \infty.$$

If there exists a constant $K > 0$ such that

$$\int_0^\infty K^{\alpha x} g(x)^{n-1}\,dx = \infty$$

for all $\alpha > 0$, then S cannot be part of a Gehring domain.

ACKNOWLEDGMENT. The author would like to thank Pekka Koskela for his careful reading of the manuscript and helpful comments.

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