Nilpotence in Finitary Linear Groups

B. A. F. WEHRFRITZ

1. Results and Examples

Throughout this paper F denotes a field and V a vector space over F. The *finitary general linear group* $\mathrm{FGL}(V) = \mathrm{F\,Aut}_F V$ over V is the subgroup of $\mathrm{Aut}_F V$ of F-automorphisms g of V such that $[V, g] = V(g-1)$ is finite-dimensional over F. A *finitary linear group* is a subgroup of $\mathrm{FGL}(V)$ for some F and V. We can always choose F as large as we please, algebraically closed for example; for if E is an extension field of F then there is an obvious embedding of $\mathrm{F\,Aut}_F V$ into $\mathrm{F\,Aut}_E(E \otimes_F V)$.

There has been much interest of late in the group-theoretic structure of finitary linear groups, with works of Hall [4] and Meierfrankenfeld, Phillips, and Puglisi [5] especially of note. In particular, [5] analyses the solubility structure of such groups. Here we carry out an analogous exercise for nilpotence, except that we are interested not just in nilpotent and locally nilpotent groups, but also in the various canonical locally nilpotent and hypercentral normal subgroups and the four canonical Engel sets of an *arbitrary* finitary linear group.

The following theorem summarizes our main positive conclusions. Our notation, which we explain in detail immediately after the statement of the theorem, is standard (with the exception of the introduction of $\eta_2(G)$), following [9] or [10] for example.

THEOREM. *Let F be a field of characteristic $p \geq 0$, V a vector space over F, and G any subgroup of* $\mathrm{FGL}(V)$.

(a) $L(G) = \eta(G) = \sigma(G) = \eta_2(G) = \langle M \triangleleft G : M = \zeta_{\omega 2}(M) \rangle = \langle M \triangleleft G : \exists \alpha < \omega 2,\ M = \zeta_\alpha(M) \rangle$.

(b) $\bar{L}(G) = \bar{\sigma}(G) = \eta_1 \eta(G) = \eta_1(G)$.

(c) $R(G) = \rho(G) \geq \zeta(G)$. *For each finite subset X of G there is a normal subgroup K of G with $K \supseteq X$ and $R(G) \cap K \leq \zeta_{\omega 2}(K)$.*

(d) $\bar{R}(G) = \bar{\rho}(G) \geq \zeta_\omega(G)$. *For each finite subset X of G there is a normal subgroup K of G with $K \supseteq X$ and $\bar{R}(G) \cap K \leq \zeta_\omega(K)$.*

(e) *Modulo its unipotent radical, G has central height at most $\omega 2$.*

Received November 18, 1991. Revision received April 3, 1992.
Michigan Math. J. 40 (1993).

(f) *If* $p>0$ *then* $[R(G), G]$ *and* $R(G)/(R(G)\cap\zeta_1(G))$ *are locally finite groups. If the unipotent radical of* $R(G)$ *is trivial then* $[R(G), G]$ *and* $R(G)/(R(G)\cap\zeta_1(G))$ *are locally finite* p'*-groups.*

NOTATION. Commutators are all left-normed. Let G be any group.

$L(G)$ denotes the set of left Engel elements of G,
$\bar{L}(G)$ the set of bounded left Engel elements of G;
$R(G)$ denotes the set of right Engel elements of G,
$\bar{R}(G)$ the set of bounded right Engel elements of G;
$\eta(G)$ is the Hirsch–Plotkin radical of G,
$\eta_1(G)$ the Fitting subgroup of G, and
$\eta_2(G)$ the subgroup of G generated by all normal subgroups of G that are hypercentral groups (see [4, Sect. 2.1] for the only published discussion of this subgroup known to me);
$\sigma(G)$ denotes the Gruenberg radical of $G=\{x\in G: \langle x\rangle \operatorname{asc} G\}$,
$\bar{\sigma}(G)$ the Baer radical of $G=\{x\in G: \langle x\rangle \operatorname{sn} G\}$;
$\rho(G)=\{x\in G: \forall g\in G, g\in\sigma(\langle g, x^G\rangle)\}$,
$\bar{\rho}(G)=\{x\in G: (\exists k\in \mathbf{N})(\forall g\in G), \langle g\rangle$ is subnormal in $\langle g, x^G\rangle$ in k steps$\}$;
$\zeta(G)$ is the hypercentre of G,
$\zeta_\alpha(G)$ the αth term of the upper central series of G, and so
$\zeta_1(G)$ denotes the centre of G.

For the general theory of these objects, with the exception of $\eta_2(G)$, see [8]. Note that while the first four above are usually only subsets of G, the rest are always subgroups of G. This remark is occasionally important in the proofs that follow. For an arbitrary group G, the following are true (see [8, vol. 2, p. 63]):

$$L(G)\supseteq\eta(G)\geq\sigma(G)\geq\eta_2(G), \qquad R(G)\supseteq\rho(G)\geq\zeta(G);$$

$$\bar{L}(G)\supseteq\bar{\sigma}(G)\geq\eta_1\eta(G)\geq\eta_1(G), \qquad \bar{R}(G)\supseteq\bar{\rho}(G)\geq\zeta_\omega(G).$$

(The point of introducing $\eta_1\eta(G)$ is that in certain skew linear situations it is possible to identify $\eta_1\eta(G)$ even when one cannot identify $\eta_1(G)$; for example, see [9, 3.5.7] and [11, Thm. C].)

If G is a subgroup of some $\mathrm{FGL}(V)$ then $u(G)$ denotes the unipotent radical of G, that is, the unique maximal unipotent normal subgroup of G. This always exists (see [5, Thm. B(v)]. Further, there is an obvious completely reducible faithful finitary action of $G/u(G)$ on the direct sum of the factors in a composition series of V as FG-module. That this gives a well-defined $G/u(G)$-action follows from [5, Thm. B(iv)]; the fact that it is faithful comes from [5, Thm. B(iii)].

EXAMPLES. Except for the equalities mentioned in the Theorem, the canonical subgroups listed above are distinct in general for finitary groups. In most cases this is already the case for linear groups. The group G below is assumed to be finitary linear. The numbering mirrors that of the Theorem.

(a) *In general, $\eta(G)$ does not equal $\eta_1(G)$, $\eta_1\eta(G)$, or $\zeta\eta(G)$.* First, $\eta(G)$ and $\eta_1(G)$ can differ even in the linear case. The simplest example is probably the infinite locally dihedral 2-group G. This G is isomorphic to a linear group of degree 2 over any algebraically closed field of characteristic not 2 and satisfies

$$\zeta_\omega(G)=\eta_1\eta(G)=\eta_1(G)<G=\eta(G)=\zeta_{\omega+1}(G).$$

The reader should have no difficulty in constructing a similar example that is isomorphic to a linear group of characteristic 2 and degree 3 (but not 2).

For any totally ordered set Λ the McLain group $\mathrm{M}(\Lambda,F)$ is by definition a unipotent finitary linear group over F (see [8, vol. 2, p. 14]). For a suitable infinite Λ the group $G=\mathrm{M}(\Lambda,F)$ is characteristically simple [8, vol. 2, p. 15] and so

$$G=\eta(G)>\zeta\eta(G)=\zeta(G)=\langle 1\rangle.$$

A highly non-unipotent example of this kind is given by a suitable q-subgroup, for $q\neq p$ a prime, of an infinite finitary symmetric group embedded into $\mathrm{FGL}(V)$ via permutations of a fixed basis of V. For an example, the wreath power (see [8, vol. 2, p. 18]) over $\mathbf{N}$ of the cyclic group of order q would do.

(b) *In general, $\eta_1(G)$ and $\zeta\eta_1(G)$ differ.* A characteristically simple McLain group is again a suitable example. Incidentally, $G=\eta_1(G)$ for every unipotent finitary linear group (see [5, Thm. B(vi)]). Also, any locally nilpotent finitary permutation group is a Fitting group [6, Thm. 3], so the preceding non-unipotent example would also do.

(c), (d) *In general $\rho(G)>\zeta(G)$ and $\bar{\rho}(G)>\zeta_\omega(G)$.* The same examples work here, too. In the statements of (c) and (d) in the Theorem, the normality of K is important. Without this restriction on K the results become very simple.

(e) *There exist finitary linear groups of arbitrary central height. There exist hypercentral finitary linear d-groups of central height α for all ordinals α satisfying $0\leq\alpha\leq\omega 2$. There is no bound on the central height of a hypercentral unipotent finitary linear group.* We start with the third claim. It is easy to construct unipotent groups of unbounded central height. It is less easy to make them also hypercentral. With care the techniques below would probably show the existence of hypercentral unipotent finitary linear groups of arbitrary central height.

Let V be a vector space over the field F and set $V^*=\mathrm{Hom}_F(V,F)$. Let H be a hypercentral unipotent subgroup of $\mathrm{FGL}(V)$. Then H acts faithfully, finitarily, and unipotently on V^* with the natural left action. If $z\in\zeta_1(H)\backslash\langle 1\rangle$ then $(1-z)V^*=[z,V^*]$ is a finite-dimensional FH-submodule of V^*. A simple induction shows that V^* (and for that matter V) is H-hypercentral.

Set $W=F\oplus V$. Then $W^*=\mathrm{Hom}_F(W,F)\cong F\oplus V^*$ in the obvious way. Let H act on W via the trivial action on F and the given action on V. Then H becomes a hypercentral unipotent subgroup of $\mathrm{FGL}(W)$. The natural action of H on W^* is then given by the trivial action on F and the natural action on V^*. In particular, $H \leq \mathrm{FGL}(W^*)$.

$U = 1_W + V^*$ is an abelian unipotent subgroup of $\mathrm{FGL}(W)$. Here, if $u = 1_W + \phi \in U$, $a \in F$, and $v \in V$, then $u: a + v \mapsto (a + v\phi) + v$. If also $b \in F$ and $v^* \in V^*$ then the action of u on W^* is given by $u: b + v^* \mapsto b + (v^* + b\phi)$. Hence $(u-1)V^* = F\phi \leq V^*$ and so $U \leq \mathrm{FGL}(W^*)$. If $h \in H$ then $huh^{-1} = 1_W + h\phi \in U$. Hence H normalizes U and $U \cong_H V^*$.

Set $G = \langle H, U \rangle$. Then G is a hypercentral unipotent subgroup of $\mathrm{FGL}(W)$ and of $\mathrm{FGL}(W^*)$. Just as with V^* and H above, so W^* is G-hypercentral. Clearly

$$\text{central height } (G) \geq G\text{-central height } (U) \ldots \qquad (*1)$$
$$\geq H\text{-central height } (V^*).$$

We claim that G-central height $(W^*) = H$-central height $(V^*) + 1$.

Suppose $X < V^* < F \oplus V^* = W^*$ is an FG-submodule of W^*. In the notation above, assume that $b + v^* \in W^*$ is fixed by U modulo X. Then for all $\phi \in V^*$ we have

$$b\phi = (u-1)(b + v^*) \in X < V^*.$$

Hence $b = 0$. Conversely, U centralizes $V^* < W^*$. Let $\{\zeta_\alpha(V^*: H)\}$ denote the upper H-central series of V^* and similarly define $\{\zeta_\alpha(W^*: G)\}$. The above shows that $\zeta_\alpha(W^*: G) = \zeta_\alpha(V^*: H)$ for all $\alpha \leq \gamma$, the H-central height of V^*. Therefore

$$\zeta_\gamma(W^*: G) = V^* < W^* = \zeta_{\gamma+1}(W^*: G)$$

and W^* has G-central height $\gamma + 1$ as claimed.

Suppose λ is a limit ordinal and assume that for each $\mu < \lambda$ there is a vector space V_μ over F and a hypercentral unipotent subgroup H_μ of $\mathrm{FGL}(V_\mu)$ such that $H_\mu \leq \mathrm{FGL}(V_\mu^*)$ and V_μ^* has H_μ-central height μ. Let $V = \bigoplus_{\mu<\lambda} V_\mu$. The direct product $H = \times_{\mu<\lambda} H_\mu$ acts as a faithful hypercentral unipotent finitary linear group on both V and V^*, and $V^* \cong \prod_{\mu<\lambda} V_\mu^*$. Clearly the central height of H is the supremum of the central heights of the H_μ. A simple calculation, depending on H being only the direct product, shows that

$$\zeta_\alpha(V^*: H) = \prod_{\mu<\lambda} \zeta_\alpha(V_\mu^*: H_\mu)$$

for all nonlimit ordinals α. However, if $\alpha \leq \lambda$ is a limit ordinal then

$$\zeta_\alpha(V^*: H) < \prod_{\mu<\lambda} \zeta_\alpha(V_\mu^*: H_\mu).$$

In particular, V^* has H-central height of exactly $\lambda + 1$.

We have now shown the following: given a field F and a nonlimit ordinal α, there exists a vector space V over F and a hypercentral unipotent subgroup H of $\mathrm{FGL}(V) \cap \mathrm{FGL}(V^*)$ such that V^* has H-central height α. Then $(*1)$ yields that $\mathrm{FGL}(F \oplus V)$ contains a hypercentral unipotent subgroup G with central height at least α.

Continue the notation above and let σ be any ordinal. Choose V and H as before with V^* of H-central height $\alpha \geq \sigma$. If E is an extension field of F of sufficient cardinality then there is a free subgroup L of $\mathrm{GL}(2, E)$ and a homomorphism ϕ of L onto H. Set $A = \zeta_\sigma(U:G)$ and

$$K = \langle \mathrm{diag}(x\phi \cdot a, x) : x \in L \text{ and } a \in A \rangle.$$

Then K is a finitary linear group over E with central height exactly σ. The hypercentre of K is $\{\mathrm{diag}(a, 1) : a \in A\}$, $K/\zeta(K) \cong L$, and $K/C_K(\zeta(K)) \cong H$.

Finally, there exist hypercentral linear d-groups of given characteristic and central height α for all ordinals α satisfying $0 \leq \alpha < \omega 2$ and $\alpha \neq \omega$ (see [10, 8.3 and its proof]). The two ordinals ω and $\omega 2$ can be achieved in the finitary case by taking suitable direct products of linear d-groups.

(f) *Even in the linear case, the groups* $[R(G), G]$ *and* $R(G)/(R(G) \cap \zeta_1(G))$ *need not be periodic if* $p = 0$ *and need not be* p'*-groups if* $u(G) \neq \langle 1 \rangle$. (Note that if $p = 0$ then by a $0'$-group we mean a periodic group.)

Let $N \leq J$ be normal subgroups of the subgroup G of $\mathrm{FGL}(V)$. Suppose that $N\mathbf{e}J$ (resp. $N\mathbf{e}\,|\,J$); that is, assume that for all x in N and y in J there is an integer k (resp. an integer k independent of the choice of y in J) such that $[x, {}_k y] = [x, y, y, \ldots, y] = 1$ (k y's). It seems possible that

$$\text{for each finite subset } X \text{ of } G \text{ there is a normal subgroup } K \text{ of } G \text{ with } K \supseteq X \text{ and } N \cap K \leq \zeta_{\omega 2}(J \cap K) \text{ (resp. } \zeta_\omega(J \cap K)). \tag{$*2$}$$

If (∗2) is true then the left and right Engel cases of the Theorem could be handled simultaneously, $N = J$ being the left Engel case and $J = G$ being the right Engel case. Although we have proved special cases of (∗2), the full result has eluded us. The problem seems to reside entirely within the unipotent elements.

2. Generalities

Consider a finitely generated subgroup H of $\mathrm{FGL}(V)$. There exist finite-dimensional subspaces U of V with $[V, H] \leq U$ and $V = U + C_V(H)$. Then $V = U \oplus C$ for some subspace C of $C_V(H)$, and relative to this direct decomposition we have

$$h = \mathrm{diag}(h|_U, 1_C) \quad \text{for all } h \in H. \tag{$*3$}$$

H acts faithfully on U and so U defines a Zariski topology on H. This topology on H is independent of the choice of U by (∗3) and so we call it *the* Zariski topology on H. This holds for any subgroup H of $\mathrm{FGL}(V)$, finitely generated or not, for which such a finite-dimensional subspace U exists.

If K is any subgroup of H then $[V, K] \leq [V, H]$ and $C_V(K) \geq C_V(H)$. Thus U also defines the Zariski topology on K, and the Zariski topology on H induces that topology on K. Now $H \leq \mathrm{GL}(U) \leq \mathrm{FGL}(V)$ via (∗3). Let $\bar{H}$ be the Zariski closure of H in $\mathrm{GL}(U)$, regarded via (∗3) as a subgroup of FGL(V). Then $\bar{H}$ is independent of the choice of U and we call $\bar{H}$ the closure of H in FGL(V).

Now suppose that G is any subgroup of FGL(V). Set $G^+ = \bigcup_Y (G \cap Y)^o$, where Y ranges over the finitely generated subgroups of FGL(V) and H^o denotes the connected component of the identity in the Zariski topology on $H = G \cap Y$. If $X \leq Y$ then $(G \cap X)^o \leq (G \cap Y)^o$ and the following is immediate.

2.1. *G^+ is a normal subgroup of G such that G/G^+ is locally finite.*

2.2. *If G is a locally soluble subgroup of* FGL(V) *then* $(G^+)'$ *is unipotent.*

In particular, G is locally nilpotent by abelian by locally finite (cf. [5, Thm. A(vi)]).

Proof. If Y is a finitely generated subgroup of FGL(V), then $G \cap Y$ is soluble [10, 3.8] and $(G \cap Y)^o$ is triangularizable by the Lie–Kolchin theorem [10, 5.8]. Consequently $((G \cap Y)^o)'$ is unipotent. The claim follows. ☐

Suppose that the field F is algebraically closed. An element g of FGL(V) is called diagonalizable or a d-element if V is a direct sum of 1-dimensional $\langle g \rangle$-invariant subspaces. This happens if and only if g is diagonalizable in the usual sense on any finite-dimensional subspace U with $[V, g] \leq U$ and $V = U + C_V(g)$. Let $g \in$ FGL(V) and pick such a subspace U. There is a Jordan decomposition of $g|_U$ in GL(U). Hence via (*3) there is a diagonalizable element g_d of FGL(V) and a unipotent element g_u of FGL(V) with $g = g_d g_u = g_u g_d$. Since Jordan decomposition is unique in the finite-dimensional case, the same applies here; that is, g_d and g_u are uniquely determined by g and the preceding equality. In particular they do not depend upon the choice of U. Of course we call $g = g_d g_u$ the *Jordan decomposition* of g in FGL(V).

2.3. *Let G be a locally nilpotent subgroup of* FGL(V). *Then $g \mapsto g_d$ and $g \mapsto g_u$ are homomorphisms of G onto subgroups G_d and G_u of* FGL(V). *Also* $[G_d, G_u] = \langle 1 \rangle$ *and* $GG_d = GG_u = G_d \times G_u$.

Proof. This follows easily from the linear case; see [10, 7.14f and 7.11]. (Note that G_d in [10] denotes not the G_d of 2.3 above but $G \cap G_d$ in our notation; a similar remark applies to G_u.) ☐

2.4. *Let G be any subgroup of* FGL(V). *Set $K = \zeta(G)$ and $\bar{G} = K_d G = K_u G$. Then $\zeta(\bar{G}) = K_d \times K_u$ and $\zeta_i(G) = G \cap \zeta_i(\bar{G})$ for all $i \leq \omega$.*

Proof. Modify the proof of [9, 3.1.8]. ☐

Using 2.3 and 2.4 we split our problems into consideration of unipotent groups, the G_u and K_u, and d-groups, that is, groups consisting only of diagonalizable elements, the G_d and K_d. Unipotent finitary linear groups are unitriangularizable in the obvious sense, like their linear compatriots (see [5, Thm. B(iii)]). With d-groups more care is needed. For example, the linear case would suggest that abelian d-groups should be diagonalizable. However this is not the case, as the following easy example shows.

2.5. Let $\bar{V}$ be the Cartesian product of the 1-dimensional subspaces Fv_i for $i=1,2,\ldots$, set $v_o=(v_1,v_2,\ldots,v_i,\ldots)$ and let $V=\bigoplus_{i\geq 0}Fv_i$ be the F-subspace of $\bar{V}$ spanned by the v_i for $i\geq 0$. Suppose $F\neq \mathrm{GF}(2)$ and pick $\alpha\in F$ with $\alpha\neq 0,1$. Let $a_i\in \mathrm{End}_F(\bar{V})$ be given by

$$a_i\colon (\alpha_j v_j)_{j\geq 1}\mapsto (\beta_j v_j)_{j\geq 1}$$

where $\beta_j=\alpha_j$ if $i\neq j$ and $\beta_i=\alpha\alpha_i$. Then $v_i a_i=\alpha v_i$ and $v_j a_i=v_j$ for $i,j\geq 1$ with $i\neq j$, and $v_o a_i=v_o+(\alpha-1)v_i\in V$. Thus V is an $F\langle a_i\rangle$-module. Also, a_i acts diagonally and finitarily on V; specifically, $v_o-v_i, v_j\colon j\geq 1$ is an a_i-basis of V.

Set $A=\langle a_i\colon i\geq 1\rangle$. *Then A is an abelian d-subgroup of* $\mathrm{FGL}(V)$ *but A is not diagonalizable.* In a linear group, commuting d-elements generate a diagonalizable group [10, 7.1]. Clearly A is an abelian subgroup of $\mathrm{FGL}(V)$ generated by d-elements, so A is a d-group. (This is also easy to see by writing down suitable bases of V.) Suppose A is diagonalizable. Then V contains an A-invariant 1-dimensional subspace, say Fv, with $v\notin\bigoplus_{j\geq 1}Fv_j$. Let $v=(\alpha_j v_j)\in\bar{V}$. There exist $k\neq l$ with $\alpha_k\neq 0\neq\alpha_l$. Then $va_k-v=\alpha_k(\alpha-1)v_k\notin Fv$. This contradicts the invariance of Fv. The proof is complete. □

3. Unipotent Subgroups

3.1. *Let F be an algebraically closed field and G a (Zariski) closed subgroup of* $\mathrm{GL}(n,F)$. *Suppose U is a maximal unipotent subgroup of G and N is a closed normal subgroup of G containing every diagonalizable element of G. Then $G=UN$.*

Proof. The Jordan decomposition of the elements of G takes place in G; that is, if $g\in G$ then $g_d, g_u\in G$ (see [10, 7.3]). Thus G/N is isomorphic to a unipotent linear group [10, 6.4 and 6.6] and as such is nilpotent. Suppose $UN<G$. Then there is a normal subgroup L of G with $UN\leq L<G$. Let $g\in G\backslash L$. The maximal unipotent subgroups of G are all conjugate [1, 1.3; 7, 4.6]. Hence there exists $x\in G$ with $g_u\in U^x$. Then

$$g=g_u g_d\in U^x N=(UN)^x\leq L^x=L,$$

a contradiction. The result follows. □

3.2. *Let G be a group such that every countable subgroup of G is a Fitting group. Then G is a Fitting group.*

Proof. Certainly G is locally nilpotent. Let $g\in G$. It suffices to prove that $\langle g^G\rangle$ is nilpotent, so assume otherwise. Then for each $i\geq 1$ there is a finite subset X_i of G such that $G_i = \langle g^x : x \in X_i \rangle$ is nilpotent of class exceeding i. Set $X = \langle g, X_i : i \geq 1 \rangle$. Then X is countable, so X is by hypothesis a Fitting group. Therefore $\langle g^X \rangle$ is nilpotent, say of class at most c. Then so is G_c, which is a contradiction. □

3.3. *Let N be a unipotent normal subgroup of the subgroup G of* FGL(V) *such that $N\mathfrak{e}G$. Then:*

(a) *$G/C_G(N)$ is a Fitting group; and*
(b) *for every finite subset X of G there is a normal subgroup K of G such that $K \supseteq X$ and $N \cap K$ lies in some finitely suffixed term of the upper central series of K.*

Proof. We may assume that F is algebraically closed. Let P be the (Zariski) closure in FGL(V) of a finitely generated subgroup of G (see Section 2) and let M be the closure of $N \cap P$ in P. Then M is a unipotent normal subgroup of P with $[M, {}_nP] = \langle 1 \rangle$ for some positive integer n; see [10, 4.13, 5.9, 5.10 and 1.21]. If $x \in P$ then $\langle M, x \rangle$ is nilpotent; $\langle M, x \rangle \leq \langle M, x \rangle_d \times \langle M, x \rangle_u$ by 2.3, and so x_d centralizes M.

Let Q be the closure of a finitely generated subgroup of $P \cap G$. The previous paragraph shows that $\langle y_d : y \in Q \rangle \leq C_Q(M)$. Hence

$$\langle y_d : y \in Q \rangle \leq \bigcap_{P, P \geq Q} C_Q(N \cap P) = C_Q(N).$$

Therefore $Q = U \cdot C_Q(N)$ by 3.1 for any maximal unipotent subgroup U of Q.

Now suppose that H is any countable subgroup of G. Then H is a union of a chain $\{H_i\}_{i \geq 1}$ of finitely generated subgroups H_i. Let Q_i denote the closure of H_i in FGL(V). For each i pick inductively a maximal unipotent subgroup U_i of Q_i with $U_i \leq U_{i+1}$ for each i and set $U = \bigcup_{i \geq 1} U_i$. Then U is a unipotent subgroup of FGL(V) and in particular is a Fitting group [5, Thm. B(vi)]. Also, if $Q = \bigcup_i Q_i$ then

$$H \leq Q = \bigcup_i U_i \cdot C_{Q_i}(N) = U \cdot C_Q(N).$$

Thus $H \cdot C_G(N)/C_G(N)$ is isomorphic to a section of the Fitting group U and therefore is a Fitting group. Part (a) now follows from 3.2.

Suppose $H \supseteq X$, where $H \leq Q = U \cdot C_Q(N)$ is as above, and let J denote the normal closure of $N \cap Q$ in Q. From [10, 5.10] it follows that $C_Q(N) \leq C_Q(J)$. Also, U is a Fitting group containing J. Hence there is a nilpotent normal subgroup W of U with $L = W \cdot C_Q(N) \supseteq X$ and $J \cap C_Q(N) \leq W$. Clearly L is normal in Q. Suppose W is nilpotent of class c. Then

$$[N \cap L, {}_cL] \leq [J \cap L, {}_cL] \leq J \cap L \cap C_Q(N) \leq J \cap W.$$

Thus

$$[N \cap L, {}_{2c}L] \leq [J \cap W, {}_cW \cdot C_Q(N)] \leq [W, {}_cW] = \langle 1 \rangle.$$

Therefore $N \cap H \cap L \leq \zeta_{2c}(H \cap L)$.

Consider $K=\langle X^G\rangle$ and suppose that $[N\cap K, {}_iK]\neq\langle 1\rangle$ for $i=1,2,\ldots$. Then for each $i\geq 1$ there exists $x_i\in N\cap K$ and $y_{i1},\ldots,y_{ii}\in K$ with $[x_i, y_{i1},\ldots,y_{ii}]\neq 1$. Choose a countable subset Y of G with $\langle X^Y\rangle\supseteq\{x_i, y_{ij}: 1\leq j\leq i<\infty\}$. Then $H=\langle X, Y\rangle$ is countable. By the above there is a normal subgroup $H\cap L$ of H and a positive integer $m=2c$ such that $H\cap L\supseteq X$ and $[N\cap H\cap L, {}_mH\cap L]=\langle 1\rangle$. But then $\langle X^Y\rangle\leq\langle X^H\rangle\leq H\cap L$, $x_m\in N\cap H\cap L$, and the $y_{mi}\in H\cap L$. Consequently $[x_m, y_{m1},\ldots,y_{mm}]=1$. This contradiction proves that $N\cap K\leq\zeta_i(K)$ for some positive integer i. Part (b) is now proved. □

3.4. *Let G be any subgroup of* FGL(V). *If X is any finite subset of G then there exists a normal subgroup K of G with* $K\supseteq X$ *and* $u(K)$ *nilpotent.*

The case $|X|=1$ is Theorem B(vi) of [5]; we modify the proof there.

Proof. Let $\{(\Lambda_\alpha, V_\alpha): \alpha\in\Omega\}$ be a composition series of V as FG-module (see [8, vol. 1] for definition). Now $[V,\langle X\rangle]$ is finite-dimensional, so there exists a finite subset $\alpha(0),\ldots,\alpha(r)$ of Ω such that $V_{\alpha(0)}=V$, $V_{\alpha(i)}\cap[V,\langle X\rangle]\leq\Lambda_{\alpha(i+1)}$, and $\Lambda_{\alpha(r)}=\{0\}$. Then $[V_{\alpha(i)},\langle X\rangle]\leq\Lambda_{\alpha(i+1)}$.

Set $K=\langle X^G\rangle$ and $U=u(K)$. Trivially K is a normal subgroup of G containing X. Further, $[V_{\alpha(i)}, K]\leq\Lambda_{\alpha(i+1)}$ since the series is G-invariant. Also U is normal in G, so $[\Lambda_\alpha, U]\leq V_\alpha$ for every α in Ω by [5, Thm. B(iv)]. Consequently U stabilizes the series

$$V=V_{\alpha(0)}\geq\Lambda_{\alpha(1)}\geq V_{\alpha(1)}\geq\cdots\geq V_{\alpha(r-1)}\geq\Lambda_{\alpha(r)}=\{0\},$$

and therefore U is nilpotent (of class at most $2r-2$). □

4. *d*-Subgroups

4.1. *Let G be a group such that* $N=\langle L(G), R(G)\rangle$ *is generated by soluble normal subgroups of G. Then*

$$L(G)=\eta(G)=\sigma(G),\quad \bar{L}(G)=\bar{\sigma}(G),\quad R(G)=\rho(G),\quad \bar{R}(G)=\bar{\rho}(G).$$

In particular, the four Engel sets are normal subgroups of G.

This slightly generalizes Theorem 1.5 of Gruenberg's paper [3]; the conclusion there being the same as 4.1 above, but the hypothesis being that N is soluble. In 4.1 the group N is certainly locally soluble. It would be interesting if 4.1 held whenever N is just locally soluble. Certainly $L(G)=\eta(G)$ and $R(G)\leq\eta(G)$ are always subgroups under this hypothesis.

Proof. Since N is locally soluble (and hence locally nilpotent by [2, Thm. 4]), $L(G)=\eta(G)$ and $R(G)$ are subgroups of G by [2, Lemma 14]. Let $a\in L(G)$. Then $\langle a^G\rangle\leq N$ is soluble by hypothesis. Consequently

$$a\in L(\langle a^G\rangle)=\sigma(\langle a^G\rangle)\leq\sigma(G)$$

by [2, Thm. 4] and it follows that $L(G)=\sigma(G)$. A similar argument yields that $\bar{L}(G)=\bar{\sigma}(G)$ and further $\bar{R}(G)=\bar{\rho}(G)$ is an immediate consequence of [3, Thm. 1.6].

Let $a\in R(G)$. Since we have shown that $R(G)$ is a normal subgroup of G, we have $\langle a^G\rangle \le R(G)$. Consequently $\langle a^G\rangle \mathbf{e} \langle x, a^G\rangle$ for any $x\in G$. But $\langle x, a^G\rangle = \langle x\rangle\langle a^G\rangle$, so $\langle x, a^G\rangle$ is a soluble Engel group; [2, Thm. 4] again yields that

$$x\in L(\langle x, a^G\rangle)=\sigma(\langle x, a^G\rangle).$$

Thus $\langle x\rangle$ is an ascendent subgroup of $\langle x, a^G\rangle$ and $a\in\rho(G)$. Therefore $R(G)=\rho(G)$. □

4.2. *Let G be any subgroup of* FGL(V). *Then* $N=\langle L(G), R(G)\rangle$ *is generated by soluble normal subgroups of G. In particular, the conclusions of 4.1 hold.*

Proof. Since G is locally linear, the linear case yields that N is locally nilpotent and in particular locally soluble. Let $x\in N$. Then $\langle x^G\rangle\le N$ is locally soluble and hence soluble by Proposition 1 of [5]. The proof is complete. □

Neumann [6, p. 563] divides transitive finitary permutation groups into two kinds, which he calls almost primitive and totally imprimitive. We refer the reader to [6] for details. Note that in 4.3(b) below the group J acts finitarily on Ω as a set since it acts finitarily on V as a vector space.

4.3. *Let* $N\le J$ *be normal subgroups of the subgroup G of* FGL(V) *such that* $N\mathbf{e}J$ *and* $u(N)=\langle 1\rangle$. *Then:*

(a) $[N, J]$ *and* $N/(N\cap\zeta_1(J))$ *are locally finite p′-groups for* $p=\operatorname{char} F\ge 0$. *Also* $N^+\le\zeta_1(J)$.

(b) *Suppose* $V=\bigoplus_{\omega\in\Omega}V_\omega$ *is a system of imprimitivity for J in V, where* Ω *is infinite and J acts transitively and almost primitively on* Ω. *Then* $N=\langle 1\rangle$.

(c) *Assume* $u(G)=\langle 1\rangle$. *For every finite subset X of G there is a normal subgroup K of G with* $K\supseteq X$, $R(J\cap K)=\zeta_{\omega 2}(J\cap K)$, *and* $\bar{R}(J\cap K)=\zeta_\omega(J\cap K)$.

(d) *For every finite subset X of G there exists a normal subgroup K of G with* $K\supseteq X$ *and* $N\cap K\le\zeta_{\omega 2}(J\cap K)$. *If in fact* $N\mathbf{e}|J$ *then* $N\cap K\le\zeta_\omega(J\cap K)$.

(e) *Assume* $u(G)=\langle 1\rangle$. *For every finite subset X of G there is a normal subgroup K of G with* $K\supseteq X$ *and*

$$L(K)=\eta(K)=\zeta_{\omega 2}(\eta(K)),\qquad R(K)=\zeta_{\omega 2}(K);$$

$$\bar{L}(K)=\eta_1(K)\ \textit{is nilpotent},\qquad \bar{R}(K)=\zeta_\omega(K).$$

Proof. (a) By 4.2 the group N is locally nilpotent, and so (see 2.3) $u(N)$ is exactly the set of unipotent elements of N and $u(N_1)=\langle 1\rangle$ for every subgroup

N_1 of N. Clearly $[N,J]=\bigcup_X[N\cap X,X]$, where X ranges over the finitely generated subgroups of J. Each $[N\cap X,X]$ is a locally finite p'-group by the linear case [9, 3.4.5]. Therefore $[N,J]$ is a locally finite p'-group.

Now consider finitely generated subgroups $X\le Y$ of FGL(V). By the linear case $(N\cap Y)^o\le\zeta_1(J\cap Y)$ and $(N\cap Y)/(N\cap Y\cap\zeta_1(J\cap Y))$ is a (finite) p'-group [9, 3.4.5]. Now

$$(N\cap X)^o\le \bigcap_{Y\ge X}(N\cap Y)^o\le \bigcap_{Y\ge X}(N\cap Y\cap\zeta_1(J\cap Y))=N\cap X\cap\zeta_1(J).$$

First, this shows that $N^+=\bigcup_X(N\cap X)^o\le\zeta_1(J)$. Second, it implies that $(N\cap X)/(N\cap X\cap\zeta_1(J))$ is finite and residually a p'-group. Therefore it is a finite p'-group and so $N/(N\cap\zeta_1(J))$ is a locally finite p'-group.

(b) By factoring Ω by a suitable congruence we may assume that J acts primitively on Ω. Set $K=\bigcap_{\omega\in\Omega}N_J(\omega)$. Then J/K is FSym(Ω) or Alt(Ω) (see [6, 2.3]). By 4.2 the group N is locally nilpotent, so certainly $N\le K$. By replacing J by a subgroup of index 2 if necessary we may assume that $J/K\cong$ Alt(Ω).

If an element of $\zeta_1(J)$ acts nontrivially on some V_ω then by transitivity it would act nontrivially on every V_ω. But J is finitary and Ω is infinite. Therefore $\zeta_1(J)=\langle 1\rangle$. In particular part (a) implies that N is locally finite. Also, each V_ω is finite-dimensional since J is finitary and transitive on Ω and $|\Omega|>1$. Apply the linear case [10, 8.15] to the action of K on V_ω. Then $N \bmod C_K(V_\omega)\le\zeta(K \bmod C_K(V_\omega))$. By finitariness K embeds into $\times_\Omega K/C_K(V_\omega)$ and so $N\le\zeta(K)$. Hence, assuming that $N\ne\langle 1\rangle$, we can pick $x\in N\cap\zeta_1(K)$ of prime order, q say.

Pick $yK\in J/K\cong$ Alt(Ω) of order $r>1$ and prime to q. Consider $Y=\langle x,y\rangle$. By the linear case again $x\in N\cap Y\le\zeta(Y)$ and so Y is nilpotent. Clearly $\langle x^Y\rangle\le\zeta_1(K)$, the latter being normal in J, so $\langle x^Y\rangle$ is a finite (recall Y is finitely generated nilpotent) q-group centralized by $y^r\in K$. But Y stabilizes a series in $\langle x^Y\rangle$, so $Y/C_Y(x^Y)$ is also a q-group. Therefore $y\in C_Y(x^Y)$ and hence $[x,y]=1$. We have now proved that

$$C_J(x)/K\ge\langle z\in J/K\colon (|z|,q)=1\rangle.$$

The latter is J/K since $J/K\cong$ Alt(Ω) and Ω is infinite. Therefore $x\in\zeta_1(J)$. But we saw earlier that $\zeta_1(J)=\langle 1\rangle$. This contradiction completes the proof of (b).

(c) Since $u(G)=\langle 1\rangle$ we may assume that G is completely reducible. Then finitariness enables us to assume that G is actually irreducible. Set $N=R(J)$. If $N=\langle 1\rangle$ or if V is finite-dimensional, choose $K=G$ for every choice of the subset X. Henceforth assume that neither of these degenerate cases hold.

Let X be any finite subset of G and x a nontrivial element of N. Set $Y=\langle x,X\rangle$. Now $H=\langle x^G\rangle\le N$ is soluble by 4.2 and [5, Prop. 1]. Moreover, V is a direct sum of finite-dimensional irreducible FH-submodules by [5, Prop. 3(ii)] and a version of Clifford's theorem, which, in our situation, follows from [5, Prop. 3(i)]. Now G permutes transitively the (nonzero) homogeneous components of V, say the V_ω for $\omega\in\Omega$, as FH-module. In particular the V_ω are finite-dimensional, for if $V=V_\omega$ then $H=\langle 1\rangle$ or $\dim V<\infty$. Then Ω is infinite, G acts on Ω via its permutation of the V_ω, and G acts on Ω as a transitive group of finitary permutations.

Suppose G acts almost primitively on Ω. If J also acts almost primitively (and transitively) on Ω then $N=\langle 1\rangle$ by part (b), which we have assumed is not the case. Hence by [6, Thm. 2.3] there is a G-congruence on Ω whose blocks are all finite and fixed by J. In this case set $K=G$. Trivially $K\supseteq X$. By finitariness and the linear case $R(J)=\zeta_{\omega 2}(J)$ and $\bar{R}(J)=\zeta_\omega(J)$.

We are left with the case where G acts totally imprimitively on Ω. Since Y is finitely generated, $\dim[V,Y]$ is finite and so $[V,Y]\leq\bigoplus_\Lambda V_\omega$ for some finite subset Λ of Ω. By [6, Thm. 2.4(i)] there is a G-congruence $\mathbf{q}$ on Ω with all its blocks finite such that Λ lies in a block of $\mathbf{q}$. Let K be the kernel of the action of G on the set of $\mathbf{q}$-blocks. Certainly K is a normal subgroup of G. Also, V is a direct sum of finite-dimensional FK-modules, so by finitariness and the linear case again $R(J\cap K)=\zeta_{\omega 2}(J\cap K)$ and $\bar{R}(J\cap K)=\zeta_\omega(J\cap K)$. Finally, Y permutes the V_ω since G does and Y acts trivially on $V/[V,Y]$ and hence also on $Y/\bigoplus_\Lambda V_\omega$. Therefore $V_\omega y=V_\omega$ for all ω in $\Omega\backslash\Lambda$ and all y in Y. It follows that Y normalizes each block of $\mathbf{q}$ and consequently $K\supseteq Y$. But $Y\supseteq X$. Therefore 4.3(c) follows.

(d) Apply part (c) to $G/u(G)$ and use $N\cap u(G)=\langle 1\rangle$.

(e) Since X is finite, we may assume that G is irreducible (cf. the proof of (c)). Consider the proof of (c) with $J=\eta(G)$. If $N=\langle 1\rangle$ then trivially $K=G$ suffices. In cases where K is chosen as a subdirect product of linear groups permuted transitively by G, the linear case yields that K has the properties required by (e). This leaves the case where G acts almost primitively on Ω. Here $\eta(G)$ is a subdirect product of linear groups permuted transitively by G. Apply part (c) again, but this time with $J=G$. For the so-constructed K the sets $R(K)$ and $\bar{R}(K)$ are as required. But $\eta(G)\supseteq L(K)\cup\bar{L}(K)$, so the linear case again shows that $L(K)$ and $\bar{L}(K)$ also satisfy the requirements of (e). □

5. Proof of the Theorem

5.1. *Let N be a normal subgroup of the subgroup G of* FGL(V) *with $N\mathbf{e}G$, and let X be a finite subset of G. Then there exists a normal subgroup K of G with $K\supseteq X$ and $N\cap K\leq\zeta_{\omega 2}(K)$. If in fact $N\mathbf{e}\,|\,G$ then we can choose such a K with $N\cap K\leq\zeta_\omega(K)$.*

Proof. By 3.3(b) there is a normal subgroup L of G and a positive integer l such that $L\supseteq X$ and $u(N)\cap L\leq\zeta_l(L)$. Apply 4.3(d) to the standard completely reducible faithful representation of $G/u(G)$. Hence there is a normal subgroup M of G with $M\supseteq u(G)\cup X$ and

$$(N\cdot u(G)\cap M)/u(G)\leq\zeta_\alpha(M/u(G)),$$

where α either is $\omega 2$ or, if $N\mathbf{e}\,|\,G$, is ω. But

$$(N\cdot u(G)\cap M)/u(G)\cong_G (N\cap M)/u(N).$$

Set $K=L\cap M$. Then K is a normal subgroup of G with $K\supseteq X$ and $N\cap K$ of K-central height at most $l+\alpha=\alpha$ in both cases. That is, $N\cap K\leq \zeta_\alpha(K)$, and the proof is complete. □

5.2. *Let G be a subgroup of* FGL(V).

(a) *If G is irreducible and* $\dim V=\infty$, *then* $\zeta_1(G)=\langle 1\rangle$.

(b) $G/u(G)$ *has central height at most* $\omega 2$.

(c) *The central height of G is bounded by the maximum of* $\omega 2$ *and the G-central height of* $\zeta(G)_u\cong \zeta(G)/(\zeta(G)\cap\zeta(G)_d)$.

Proof. (a) Let $z\in\zeta_1(G)\backslash\langle 1\rangle$. Then $[V,z]$ is a finite-dimensional FG-submodule of V. Hence if $\zeta_1(G)\neq\langle 1\rangle$ and G is irreducible then $\dim V<\infty$.

(b) $G/u(G)$ is isomorphic to a completely reducible group. By part (a) and the linear case, each irreducible constituent of this group has central height less than $\omega 2$. By finitariness it follows that $G/u(G)$ has central height at most $\omega 2$.

(c) Set $K=\zeta(G)$ and $\bar{G}=K_d\cdot G$. Then $K_d\cdot K_u=\zeta(\bar{G})$ by 2.4. Now, by 5.2(b) the $\bar{G}$-central height of K_d is at most $\omega 2$. But $K_d\leq\zeta(\bar{G})$, so the G-central height of K_d is at most the $\bar{G}$-central height of K_d and so is at most $\omega 2$. Thus the central height of G is the G-central height of K, which is at most the G-central height of $K_d\times K_u$, which is at most the maximum of $\omega 2$ and the G-central height of K_u. □

Note that in the preceding proof the central height of G bounds the G-central height of K_u. Thus part (c) implies that either G has central height at most $\omega 2$ or the central height of G is equal to the G-central height of $\zeta(G)_u$.

5.3. The proof of part (a) of the theorem proceeds as follows. By 4.2 we have $L(G)=\eta(G)=\sigma(G)$. Let $N=\eta(G)_d\times\eta(G)_u$ and $\bar{G}=GN$. Then $N\leq\eta(\bar{G})$ by 2.3. Apply 4.3(d) to $\bar{G}/u(\bar{G})$ with $J=N_d\cdot u(\bar{G})/u(\bar{G})$. Thus N_d is covered by normal subgroups M of $\bar{G}$ with $M=\zeta_{\omega 2}(M)$. By Theorem B(vi) of [5], we have $N_u\leq\eta_1(\bar{G})$. Hence if $x\in\eta(G)$ then there is a normal subgroup M of $\bar{G}$ in N such that $x=x_d x_u\in M$ and $M=\zeta_{\omega 2}(M)$. It follows that $\eta(G)\leq\langle M\lhd G\colon M=\zeta_{\omega 2}(M)\rangle\leq\eta_2(G)\leq\eta(G)$. The result now follows. □

5.4. For the proof of part (b) of the theorem, note that by 4.2 we have $\bar{L}(G)=\bar{\sigma}(G)$. Repeat the proof of 5.3 with $N=\bar{L}(G)_d\times\bar{L}(G)_u$. We obtain that $N_u\leq\eta_1(\bar{G})$, while $N_d\leq\bar{\sigma}(\bar{G})$. If $x\in N_d$ then $\langle x^{\bar{G}}\rangle$ is soluble (4.2) and hence is isomorphic to a subdirect product of irreducible linear groups [5, Prop. 3(ii)]. Each of the latter is nilpotent, so $\langle x^{\bar{G}}\rangle=\zeta_\omega(\langle x^{\bar{G}}\rangle)$. But each $\zeta_i(\langle x^{\bar{G}}\rangle)$ is normal in $\bar{G}$. Hence $\langle x^{\bar{G}}\rangle=\zeta_i(\langle x^{\bar{G}}\rangle)$ for some $i<\omega$ and $\langle x^{\bar{G}}\rangle$ is nilpotent. It follows that $N_d\leq\eta_1(\bar{G})$ and hence that

$$\bar{L}(G)\leq\eta_1(\bar{G})\cap G\leq\eta_1(G)\leq\bar{L}(G).$$ □

5.5. The proof of parts (c), (d), (e), and (f) of the theorem is as follows. That $R(G)=\rho(G)\geq\zeta(G)$ and $\bar{R}(G)=\bar{\rho}(G)\geq\zeta_\omega(G)$ come from 4.2. The remainder of parts (c) and (d) is given by 5.1. Part (e) is simply 5.2 (b). For Part (f), set $\bar{G}=G\cdot R(G)_d$. The local linearity of G yields $R(G)_d \mathbf{e} \bar{G}$, so by 4.3(a) the groups $[R(G)_d, \bar{G}]$ and $R(G)_d/(R(G)_d \cap \zeta_1(\bar{G}))$ are locally finite p'-groups. If $p>0$ then $R(G)_u$ is a locally finite p-group [5, Thm. B(i)] and so $[R(G), G]$ and $R(G)/(R(G)\cap\zeta_1(G))$ are locally finite groups. Alternatively, if $uR(G)=\langle 1\rangle$ then these two groups are locally finite p'-groups directly by 4.3(a). □

References

[1] A. Borel, *Linear algebraic groups,* Benjamin, New York, 1969.

[2] K. W. Gruenberg, *The Engel elements of a soluble group,* Illinois J. Math 3 (1959), 151–168.

[3] ———, *The upper central series in soluble groups,* Illinois J. Math. 5 (1961), 436–466.

[4] P. Hall, *The Frattini subgroups of finitely generated groups,* Proc. London Math. Soc. (3) 11 (1961), 327–352.

[5] U. Meierfrankenfeld, R. E. Phillips, and O. Puglisi, *Locally solvable finitary linear groups,* J. London Math. Soc. 47 (1993), 31–40.

[6] P. M. Neumann, *The lawlessness of groups of finitary permutations,* Arch. Math. (Basel) 26 (1975), 561–566.

[7] V. P. Platonov, *The theory of algebraic linear groups and periodic groups,* Izv. Akad. Nauk SSSR Ser. Mat. 30 (1966), 573–620; translation in Amer. Math. Soc. Transl. Ser. 2, 69, pp. 61–110, Amer. Math. Soc., Providence, RI, 1968.

[8] D. J. S. Robinson, *Finiteness Conditions and Generalized Soluble Groups* (2 vols.), Springer, Berlin, 1972.

[9] M. Shirvani and B. A. F. Wehrfritz, *Skew linear groups,* Cambridge Univ. Press, Cambridge, 1986.

[10] B. A. F. Wehrfritz, *Infinite linear groups,* Springer, Berlin, 1973.

[11] ———, *The upper central series of some matrix groups,* Fund. Math. 135 (1990), 111–126.

School of Mathematical Sciences
Queen Mary and Westfield College
London E1 4NS
England