Operators Defined on Projective and Natural Tensor Products

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Introduction

This paper studies the behaviour of operators defined on the projective and natural tensor product of an l_p space and an arbitrary Banach space X. The main result is the following (Theorem 1): If E and F are Banach spaces, E not containing l_1, and all operators from E into l_p and all operators from E into F are compact, then all operators from E into $l_p \hat{\otimes}_\pi X$ are compact. Two applications of the techniques involved in the proof of this result are considered: a study of the tensor stability (with respect to the projective and natural tensor product with an l_p-space) of the scale of operator ideals formed by the p-converging operators for $1 \le p \le +\infty$, and a vector-valued version of Pitt's theorem.

Background

Throughout the paper p^* denotes the dual number of p. We base our approach to the properties of natural and projective tensor products on the use of the representations of those spaces as sequence spaces. A sequence (x_n) in a Banach space X is said to be *weakly p-summable* ($p \ge 1$) if there is a $C > 0$ such that, for each (ξ_n) in l_{p^*},

$$w_p(\{x_n\})_n = \sup_k \left\{ \left\| \sum_{n=1}^{k} \xi_n x_n \right\| : \|(\xi_n)\|_{l_{p^*}} \le 1 \right\} < +\infty$$

(here, if $p = 1$ then c_0 plays the role of l_∞); it is said to be *absolutely p-summable* when $p \ge 1$ if

$$s_p(\{x_n\}_n) = \left[\sum_{n=1}^{+\infty} \|x_n\|^p \right]^{1/p} < +\infty$$

(if $p = +\infty$ then the l_p norm must be replaced by the sup norm); it is said to be *strongly p-summable* for $p \ge 1$ if

$$\sigma_p(\{x_n\}_n) = \sup \left\{ \left| \sum_{n=1}^{+\infty} f_n(x_n) \right| : w_{p^*}(\{f_n\}) \le 1, (f_n) \in X^* \right\} < +\infty.$$

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Following [2] and [3], we shall denote by $l_p(X)$, $l_p[X]$, and $l_p\langle X\rangle$ (respectively) the spaces of weakly p-summable, absolutely p-summable, and strongly p-summable sequences of X, endowed with their natural topologies induced by the norms w_p, s_p, and σ_p, respectively. The following two isometries are well known: $l_p(X)=L(l_{p^*},X)$ for $1<p<+\infty$, and $l_1(X)=L(c_0,X)$ (see [3]). The symbols π and ϵ shall denote the projective and injective norms on the space $l_p\otimes X$. The symbol Δ_p denotes the norm induced by s_p over $l_p\otimes X$; the topology induced by s_p is termed the *natural* topology. We shall denote by $l_p\hat{\otimes}_\epsilon X$, $l_p\hat{\otimes}_\pi X$, and $l_p\hat{\otimes}_{\Delta_p} X=l_p[X]$ the completion of $l_p\otimes X$ with respect to ϵ, π, and Δ_p, respectively. The closed subspace of $l_p\langle X\rangle$ formed by those sequences which are the limit of their finite sections will be denoted by $l_{p,0}\langle X\rangle$. It is easy to see that $l_{p,0}\langle X\rangle=l_p\hat{\otimes}_\pi X$ if $1\le p<\infty$.

We shall consider the following operator ideals: The ideal W of *weakly compact* operators; the ideal U of *unconditionally converging* operators—that is, those sending weakly 1-summable sequences into unconditionally summable sequences; the ideal K of compact operators; and the ideal B of *completely continuous* operators—that is, those sending weakly convergent sequences into convergent ones.

DEFINITION. We say that an operator $T\in L(X,Y)$ is *p-converging* for $1\le p<+\infty$ if it transforms weakly p-summable sequences of X into norm null sequences of Y. We shall use C_p to denote the ideal of p-converging operators.

The classes C_p form injective, nonsurjective closed operator ideals. It is clear that $C_1=U$ and, with the convention that the weakly ∞-summable are the weakly null sequences, that $C_\infty=B$. A characterization of p-converging operators is contained in the following proposition of [1].

PROPOSITION 0. *Let X be a Banach space, and let $1\le p<+\infty$. If $p>1$, the operator* Id(X) *belongs to C_p if and only if all operators from l_{p^*} into X are compact. If $p=1$ then* Id(X) *belongs to C_1 if and only if all operators from c_0 into X are compact.*

The result known as Pitt's theorem, $L(l_p,l_q)=K(l_p,l_q)$ if and only if $p>q$, can therefore be written as follows: If $1\le p,r<\infty$ then Id$(l_p)\in C_r$ if and only if $r<p^*$. This must be taken into account for the hypotheses of Corollary 3.

Main Results

We begin our study of C_p operators in projective and natural tensor products with a technical result of independent interest.

THEOREM 1. *Let E and F be Banach spaces, E not containing l_1. Let $1\le p<+\infty$. If all operators from E into l_p are compact and all operators from E*

into F are compact, then all operators from E into $l_p\hat{\otimes}_\pi F$ (resp. $l_p\hat{\otimes}_{\Delta_p} F$) are compact.

Proof. Let $A: E\to l_p\hat{\otimes}_\pi F$ be an operator and let (x_n) be a weakly null sequence in E. By Rosenthal's l_1 theorem, it is sufficient to verify that (Ax_n) is norm null. If it is not norm null, one can assume that $\|Ax_n\|\geq\epsilon$ for some $\epsilon>0$ and all $n\in\mathbf{N}$. Since $l_p\hat{\otimes}_\pi F=l_{p,0}\langle F\rangle$, Ax_n can be identified with some sequence (y_j^n). If the image of A is contained in some finite product of copies of X, the proof is finished because of the hypothesis $L(E,F)=K(E,F)$. If not, it is then possible to proceed inductively to obtain sequences of naturals (n_j) and (k_j) so that

$$\|(0,0,\dots,y_{k_j+1}^{n_{j+1}},\dots,y_{k_{j+1}}^{n_{j+1}},0,0,\dots)\|>\epsilon/2.$$

Let I_j be the set $\{k_j+1,\dots,k_{j+1}\}$. We shall use $P_j: l_p\hat{\otimes}_\pi F\to l_p\hat{\otimes}_\pi F$ and $Q_j: l_p\to l_p$ to denote the projections over the indices of I_j. For each index j there is an element z_j in $(l_p\hat{\otimes}_\pi F)^*=L(l_p,F^*)$ with $\|z_j\|\leq 1$ such that

$$|\langle P_jAx_{n_j},z_j\rangle|>\epsilon/2.$$

This implies that

$$|\langle P_jAx_{n_j},z_jQ_j\rangle|=\left|\left\langle\sum_{i\in I_j}e_i\otimes y_i^{n_j},z_jQ_j\right\rangle\right|=\left|\sum_{i\in I_j}\langle z_jQ_j(e_i),y_i^{n_j}\rangle\right|$$
$$=\left|\sum_{i\in I_j}\langle z_j(e_i),y_i^{n_j}\rangle\right|=|\langle P_jAx_{n_j},z_j\rangle|>\frac{\epsilon}{2}.$$

A continuous operator $B: E\to l_p$ is defined by $Bx=(\langle P_jAx,z_jQ_j\rangle)_j$. This operator is well-defined; for if $Ax=(y_j)$ then

$$\left(\sum_j|\langle P_jAx,z_jQ_j\rangle|^p\right)^{1/p}$$
$$=\sup_{\|\eta\|_{p^*}\leq 1}\left|\sum_j\eta_j\langle P_jAx,z_jQ_j\rangle\right|=\sup_{\|\eta\|_{p^*}\leq 1}\left|\sum_j\eta_j\left\langle\sum_{i\in I_j}e_i\otimes y_i,z_jQ\right\rangle\right|$$
$$\leq\sup_{\|\eta\|_{p^*}\leq 1}\left|\sum_j\left\langle\sum_{i\in I_j}e_i\otimes y_i,\eta_jz_jQ_j\right\rangle\right|=\sup_{\|\eta\|_{p^*}\leq 1}\left|\sum_j\left\langle\sum_{i\in I_j}e_i\otimes y_i,\sum_k\eta_kz_kQ_k\right\rangle\right|$$
$$=\sup_{\|\eta\|_{p^*}\leq 1}\left|\left\langle\sum_ie_i\otimes y_i,\sum_j\eta_jz_jQ_j\right\rangle\right|\leq\|Ax\|\sup_{\|\eta\|_{p^*}\leq 1}\left\|\sum_j\eta_jz_jQ_j\right\|.$$

This last expression is finite, since, if s belongs to the unit ball of l_p and $p>1$, then

$$\left\|\sum_j\eta_jz_jQ_j(s)\right\|_{F^*}=\sup_{\|f\|_F\leq 1}\left|\left\langle\sum_j\eta_jz_jQ_j(s),f\right\rangle\right|\leq\sup_{\|f\|_F\leq 1}\sum_j|\langle\eta_jz_jQ_j(s),f\rangle|$$
$$\leq\sup_{\|f\|_F\leq 1}\left(\sum_j|\eta_j|^{p^*}\right)^{1/p^*}\left(\sum_j|\langle z_jQ_j(s),f\rangle|^p\right)^{1/p}$$
$$\leq\left(\sum_j|\eta_j|^{p^*}\right)^{1/p^*}\left(\sum_j\left(\sum_{i\in I_j}|s_i|^p\right)\|z_j\|^p\right)^{1/p}\leq 1,$$

from which one deduces that $\|Bx\| \leq \|Ax\|$. If $p=1$, the proof is analogous. Therefore B is continuous. By the hypothesis, B must be compact, and hence $\lim_{i\to\infty} Bx_{n_i}=0$. This is in contradiction with the fact that, for every $i \in \mathbf{N}$,

$$\|Bx_{n_i}\| = \|(\langle P_j A x_{n_i}, z_j Q_j\rangle)_j\|_{l_p} \geq |\langle P_i A x_{n_i}, z_i Q_i\rangle| \geq \epsilon/2,$$

and the theorem is proved. □

The proof for the natural product is essentially the same.

The following extension of Pitt's lemma (case $X, Y = \mathbf{R}$ or $\mathbf{C}$) can be established.

THEOREM 2. *Assume that X and Y are Banach spaces, and that X and Y^* do not contain l_1. Let $1<q<p<\infty$. If $L(l_p, l_q) = K(L_p, l_q)$, $L(l_p, Y) = K(l_p, Y)$, $L(X, l_q) = K(X, l_q)$, $L(X, Y) = K(X, Y)$, and $L(Y^*, X^*) = K(Y^*, X^*)$, then*

$$L(l_p \hat{\otimes}_\epsilon X, l_q \hat{\otimes}_\pi Y) = K(l_p \hat{\otimes}_\epsilon X, l_q \hat{\otimes}_\pi Y)$$

and

$$L(l_p[X], l_q[Y]) = K(l_p[X], l_q[Y]).$$

Proof. By a result of Samuel [7, Thm. 3], l_1 is not contained in $l_p \hat{\otimes}_\epsilon X$. That l_1 is not contained in $l_p[X]$ is a consequence of a result of Pisier [5]. By Schauder's theorem and Theorem 1, the conclusion follows. □

REMARKS. (i) Except for $L(Y^*, X^*) = K(Y^*, X^*)$, all the conditions of the hypothesis are also necessary.

(ii) The ideal C_1 is, in general, not tensor stable with respect to projective or injective products; see [6] for the projective case, and there is the trivial example $l_2 \hat{\otimes}_\epsilon l_2 = K(l_2, l_2)$ for the injective case. However, when one considers tensor products with an l_p-space, the ideals C_p are to some extent tensor stable, as follows from Theorem 1 and Proposition 0. This yields the next corollary.

COROLLARY 3. *Let $1 \leq p, r < \infty$. If the operators* $\mathrm{Id}(l_p) \in C_r$ *(i.e. $r < p^*$) and* $\mathrm{Id}(X) \in C_r$, *then* $\mathrm{Id}(l_p \hat{\otimes}_\pi X) \in C_r$ *and* $\mathrm{Id}(l_p \hat{\otimes}_{\Delta_p} X) \in C_r$.

Since $\mathrm{Id}(X) \in C_1$ if and only if X does not contain an isomorphic copy of c_0, we obtain the following.

COROLLARY 4. *For $1 \leq p < +\infty$, $l_p \hat{\otimes}_\pi X$ contains a copy of c_0 if and only if X does.*

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